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Vacuum fluctuations of a quantized scalar field in a Robertson-Walker universe*

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The vacuum expectation value of the energy-momentum tensor of a quantized scalar field is calculated in a Robertson-Walker universe. The resulting divergences are regularized by averaging over an appropriate mass spectrum, which fulfills the same regularization conditions as needed for regularization of the vacuum energy-momentum tensor in Minkowski space. Up to higher orders in time derivatives and inverse radius of universe, there result three contributions to the vacuum energy-momentum density: a cosmological term, a term proportional to the Einstein tensor, and a term derivable by variation of a gravitational Lagrangian containing quadratic expressions of the curvature quantities R, $R_{\mu\nu}$, $R_{\mu\nu\rho\sigma}$. These contributions are estimated with the following result: There is no direct evidence that the collapse of a Robertson-Walker universe is averted at some realistic dimensions of the universe, or that gravitation is described in a natural way by the elastic properties of the vacuum of some particles. On the other hand, there is given some evidence that quantum corrections of the type shown must be taken into account in general relativity and are important at least for highly collapsed states of the early universe.

I. INTRODUCTION

Since the very beginning of quantum electrodynamics, it has been known that the concept of particle vacua (like the Dirac sea) leads to physically observable effects of quantization. The pure presence of the electron vacuum in an unquantized electromagnetic background causes higher-order corrections to the classical Lagrangian of the free electromagnetic field. Well-known effects such as Delbrück scattering of light by light are manifestations of the nonlinear properties characterizing the electron vacuum. Its influence on an unquantized electromagnetic field is most easily depicted by writing down an effective Lagrangian which can be deduced, as shown by Weisskopf,[1] in a highly transparent manner from the shifted energy of the Dirac sea in the presence of an electromagnetic background. The same result was obtained by Schwinger[2] by a somewhat more formal method, calculating the vacuum current induced by the electromagnetic field by means of a Green's-function procedure. Both procedures include renormalization and lead to finite, nonlinear vacuum polarization effects, which are important in strong electromagnetic fields.

The influence of particle vacua on classical field equations is not restricted to electrodynamics, but also has to be discussed for the other classical field, namely the gravitational field. Apart from the general interest in this question, there are two aspects accentuating its discussion: On the one hand it was suggested by Sakharov[3] that gravitation is an effect of "metric elasticity," in analogy to the elasticity of solids, which is based on the properties of its microscopic constituents; the gravitational constant is assumed to be an elasticity constant describing the resistance of particle vacua against deformations of space-time. On the other hand, there are several powerful theorems[4] showing that under very general conditions the occurrence of a collapsed initial or final state is an inevitable consequence of the Einstein equations of general relativity. If Einstein's equations are modified because of the presence of particle vacua[5] we have to investigate whether and how these modifications are able to prevent the development of singular states of matter and the universe.[6]

But if we now look for the direct influence of particle vacua on general relativity, we have first of all to study the vacuum expectation value of the particles' energy-momentum tensor. This vacuum expectation value, now coupled to the gravitational field by Einstein's equations, is thrown away by the usual normal-ordering procedure of quantum field theory, renormalizing the zero point of energy by a divergent and "unobservable" constant. In general relativity, however, we have to calculate the full vacuum expectation value of the energy-momentum tensor of the quantized fields describing the particles, and as a second step this divergent quantity has to be regularized in a manner consistent with the corresponding procedure in Minkowski space-time.

Before doing so, we follow up the history of calculating the vacuum energy-momentum tensor of a quantized field in presence of gravitational interaction. The first investigation on this subject was done by Utiyama and DeWitt,[7] who studied the energy-momentum tensor of a quantized scalar field in the presence of the corresponding gravitational background. Here—and also in an appropriate generalization of Schwinger's calculations[2] of the vacuum quantities given by DeWitt[8]—space-time is assumed to be asymptotically flat at large spacelike and timelike distances.

To avoid this assumption, which seems foreign to the spirit of general relativity, we have to solve the field equations in the background of the given, not necessarily asymptotically flat metric. These solutions are used to calculate the vacuum expectation value of the field's energy-momentum tensor. The resulting formally divergent quantities are renormalized by Parker and Fulling[9] using a momentum-cutoff method. The divergences are, as also shown by Utiyama and DeWitt[7] and by DeWitt,[8] of three types:

(a) a cosmological term proportional to $g_{\mu\nu}$,

(b) an Einstein term proportional to the Einstein tensor $G_{\mu\nu}$, and

(c) terms which are proportional to a tensor $\tilde{G}_{\mu\nu}$ obtained by variation from a Lagrangian quadratic in the curvature quantities R, $R_{\mu\nu}$, and $R_{\mu\nu\rho\sigma}$.

Some finite terms in the vacuum energy-momentum density are not identified by Parker and Fulling[9] as tensor quantities, and owing to the cutoff procedure there also some troubles in identifying the covariant behavior of the different divergences (a), (b), and (c).

In this paper the vacuum energy-momentum tensor of a minimally coupled massive quantized scalar field on the background of a given metric is calculated using the solutions of the scalar field equations (Sec. II). The resulting energy-momentum density is regularized by means of an appropriate mass spectrum for an Einstein universe (static and closed Robertson-Walker metric) (Sec. III). The finite regularized energy-momentum tensor shows all terms of the types (a), (b), and (c) discussed before. Also, finite terms of higher order in the curvature are present and can be given in a closed form for the static case. The regularization conditions for the mass spectrum are the same as those used by Zel'dovich[10] to regularize vacuum energy and pressure for flat space. In this manner our regularization procedure separates the pure curvature effects acting on the vacuum. The magnitude of the vacuum energy-momentum tensor obtained depends on the regulator mass spectrum $\rho(m^2)$, which is not determined by the regularization procedure but only restricted to fulfil the given regularization conditions. These regulator conditions also imply that the regulator mass spectrum $\rho(m^2)$ must be negative for some values of m^2, which is not the case for realistic bosons but could be expected for a theory containing fermions and bosons (in flat space the vacuum energy of free fermions becomes negative infinite[11]). Our results are generalized then to a closed nonstatic Robertson-Walker metric using the same regularization conditions for the mass spectrum as in flat space and in the Einstein case (Sec. IV). The field equations are now solved by a refined WKB procedure[12] expanding the eigenfunctions up to fourth-order time derivatives of the nonstatic metric. Once more we obtain at least a cosmological, an Einstein, and a quadratic-curvature term for the vacuum expectation value of the energy-momentum density. The results allow an estimate of the relative order of magnitude of this vacuum corrections to general relativity (Sec. V).

II. VACUUM EXPECTATION VALUE OF THE ENERGY-MOMENTUM TENSOR

Starting from the Lagrangian of a massive scalar field which is minimally coupled to a given metric $g_{\mu\nu}$ by

$$\mathcal{L} = \tfrac{1}{2}\sqrt{-g}\,(g_{\mu\nu}\partial^\mu\Phi\partial^\nu\Phi - m^2\Phi^2)\,, \tag{2.1}$$

the energy-momentum tensor for the field gets the form

$$\sqrt{-g}\,T_{\mu\nu} = \tfrac{1}{2}\sqrt{-g}\,(\partial_\mu\Phi\partial_\nu\Phi + \partial_\nu\Phi\partial_\mu\Phi)\sqrt{-g} - g_{\mu\nu}\mathcal{L}\,. \tag{2.2}$$

We investigate its vacuum expectation value for a closed Robertson-Walker universe with a metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - S^2(t)(1+\tfrac{1}{4}r^2)^{-2}d\vec{x}^2\,. \tag{2.3}$$

The equation of motion of the scalar field Φ reads

$$\ddot{\Phi} + 3\dot{S}\dot{\Phi} - S^{-2}\tilde{\Delta}\Phi + m^2\Phi = 0\,, \tag{2.4}$$

where $\tilde{\Delta}$ means the three-dimensional covariant Laplacian. Now we separate the scalar field operator into its time-dependent and space-dependent parts by setting

$$\Phi(\vec{x},t) = \sum_{lmn}[A_{lmn}Y_{lmn}(\vec{x})\psi_l(t) + \text{H.c.}]\,. \tag{2.5}$$

Y_{lmn} are four-dimensional spherical harmonics, eigenfunctions of $\tilde{\Delta}$. A^+_{lmn} and A^-_{lmn} are particle creation and destruction operators acting on the vacuum, which is assumed to be stable. The $l(l+1)$-fold eigenfunctions Y_{lmn} of $\tilde{\Delta}$ correspond to the degenerate eigenvalues $-l(l+2)$, which lead to time-dependent field equations of the form

$$\partial_\tau{}^2\psi_l(\tau) + S^6(\tau)\omega_l{}^2(\tau)\psi_l(\tau) = 0\,, \tag{2.6}$$

where $\omega_l{}^2 = l(l+2)S^{-2} + m^2$ and $\partial_\tau = S^3\partial_t$, $\partial_t = \partial_0$.

The vacuum expectation value of the energy-momentum tensor (2.2) can be rewritten in terms of the eigenfunctions ψ_l as

$$\langle T^0{}_0\rangle_0 = \sum_{l=0}^{\infty}(l+1)^2[|\partial_0\psi_l|^2+\omega_l{}^2|\psi_l|^2] \;, \qquad (2.7)$$

$$\langle T^i{}_i\rangle_0 = \sum_{l=0}^{\infty}(l+1)^2\left[|\partial_0\psi_l|^2-\left(\frac{l(l+2)}{3S^2}+m^2\right)|\psi_l|^2\right], \qquad (2.8)$$

$$\langle T^i{}_k\rangle_0 = 0, \quad i\neq k \;. \qquad (2.9)$$

III. EINSTEIN UNIVERSE

For a closed and static Robertson-Walker metric the field equation (2.4) reduces to

$$\ddot{\Phi} - S^{-2}\tilde{\Delta}\Phi + m^2\Phi = 0 \;, \qquad (3.1)$$

which is solved by the ansatz

$$\Phi_{lmn} \propto (2S^3E_l)^{-1/2}\exp(-iE_lt)Y_{lmn}(\vec{x}) \;. \qquad (3.2)$$

The energy eigenvalues E_l are given by

$$E_l = \left(\frac{l(l+2)}{S^2}+m^2\right)^{1/2} \;. \qquad (3.3)$$

Inserting the time-dependent part of solution (3.2) into expressions (2.7)–(2.9) for the vacuum energy-momentum density, we obtain

$$\begin{aligned}\epsilon &= \langle T^0{}_0\rangle_0 \\ &= (2\pi^2S^3)^{-1}\sum_{l=0}^{\infty}(l+1)^2E_l \\ &= (2\pi^2S^3)^{-1}\sum_{l=0}^{\infty}(l+1)^2[l(l+2)S^{-2}+m^2]^{1/2} \qquad (3.4)\end{aligned}$$

and

$$\begin{aligned}p &= -\langle T^i{}_i\rangle_0 \\ &= \tfrac{1}{3}(2\pi^2S^3)^{-1}\sum_{l=0}^{\infty}(l+1)^2(E_l{}^2-m^2)S^{-2}/E_l \\ &= \tfrac{1}{3}(2\pi^2S^3)^{-1}\sum_{l=0}^{\infty}\frac{(l+1)^2l(l+2)S^{-2}}{[l(l+2)S^{-2}+m^2]^{1/2}} \;, \qquad (3.5)\end{aligned}$$

$$\langle T^i{}_k\rangle_0 = 0, \quad i\neq k \;. \qquad (3.6)$$

Substitution of $k=l+1$ in (3.4) gives

$$\epsilon(S,m) = (2\pi^2S)^{-1}m^3\sum_{k=0}^{\infty}(k/\nu)^2[(k/\nu)^2+\mu^2]^{1/2} \;, \qquad (3.7)$$

where

$$\nu = mS \qquad (3.8)$$

is the ratio of radius of universe: Compton wavelength of field, and

$$\mu^2 = 1-\nu^{-2} \;. \qquad (3.9)$$

Assuming that the Compton wavelength is smaller than S or $\nu=mS>1$, the energy density (3.7) can be rewritten by means of the Euler-Maclaurin formula introducing a cutoff $k_{\max}=N$ as

$$\begin{aligned}\epsilon(S,m,N) = (2\pi^2S)^{-1}\Bigg(& m^3\nu\left[\frac{1}{4}\left(\frac{N}{\nu}\right)^4+\frac{\mu^2}{2}\left(\frac{N}{\nu}\right)^2+\frac{\mu^4}{16}-\frac{\mu^4}{8}\ln 2N+\frac{\mu^4}{16}\ln\nu^2\mu^2\right] \\ &+m^3\left\{\frac{1}{2}\left(\frac{N}{\nu}\right)^3+\frac{\mu^2}{2}\left(\frac{N}{\nu}\right)+\frac{B_2}{2}\nu^{-1}\left[\left(\frac{3N}{\nu}\right)^2+\frac{\mu^2}{2}\right]+\frac{B_4}{4}\nu^{-2}\right\}+O\left(\frac{\nu}{N}\right)\Bigg); \qquad (3.10)\end{aligned}$$

here B_n are the Bernoulli numbers. The vacuum energy density (3.10) is regularized by integrating over a mass spectrum $\rho(m^2)$ under the conditions

$$\int\rho(m^2)\mu^n\,dm^2 = 0, \quad n=0,2,4 \qquad (3.11)$$

or, equivalently,

$$\int\rho(m^2)m^n\,dm^2 = 0, \quad n=0,2,4 \;. \qquad (3.12)$$

By conditions (3.11) all divergences are eliminated from

$$\epsilon(S) = \lim_{N\to\infty}\int\rho(m^2)\epsilon(S,m,N)dm^2 \;, \qquad (3.13)$$

which reduces (3.10) to

$$\begin{aligned}\epsilon(S) &= (32\pi^2)^{-1}\int m^4\mu^4\ln(\nu^2\mu^2)\rho(m^2)dm^2 \\ &= (32\pi^2)^{-1}\int dm^2\rho(m^2)m^4\left(1-\frac{1}{m^2S^2}\right)^2 \\ &\quad\times\ln\left[(mS)^2\left(1-\frac{1}{m^2S^2}\right)\right]. \qquad (3.14)\end{aligned}$$

This expression for the vacuum energy density is exact, if we assume the validity of the regularization conditions (3.12). The effects of curvature are all exactly contained via $\mu^2=1-m^{-2}S^{-2}$, depending only on the ratio Compton wavelength: radius of universe. As easily seen for Minkowski space, the same procedure leads us to

$$\epsilon(S) = (32\pi^2)^{-1}\int dm^2\rho(m^2)m^4\ln m^2 \;. \qquad (3.15)$$

Now $\epsilon(S)$ given by (3.14) can be expanded in powers of $(mS)^{-2}$, leading to

$$\epsilon(S)=(32\pi^2)^{-1}\Big(\int\rho(m^2)m^4\ln(m^2S^2)dm^2 - 2S^{-2}\int\rho(m^2)m^2\ln(m^2S^2)dm^2 + S^{-4}\int\rho(m^2)\ln(m^2S^2)dm^2+O(m^{-2}S^{-6})\Big)$$
$$\simeq(32\pi^2)^{-1}(L_4-2S^{-2}L_2+L_0S^{-4}), \quad (3.16)$$

where in the last step we introduced

$$L_n=\int\rho(m^2)m^n\ln(S^2m^2)dm^2, \quad (3.17)$$

which is independent of S as can be seen from conditions (3.12) for $n=0, 2, 4$.

Variation of the most general gravitational Lagrangian containing terms of second order in the curvature,

$$\mathcal{L}=\sqrt{-g}\,(A_0+A_1R+A_2R^2+A_3R_{\mu\nu}R^{\mu\nu})+\mathcal{L}_{\text{matter}}, \quad (3.18)$$

gives us the following equations of motion for the gravitational field:

$$C_0g_{\mu\nu}+C_1G_{\mu\nu}+C_2\tilde G_{\mu\nu}\propto T_{\mu\nu}, \quad (3.19)$$

where $G_{\mu\nu}$ is the Einstein tensor and $\tilde G_{\mu\nu}$ is obtained by variation of $A_2R^2+A_3R_{\mu\nu}R^{\mu\nu}$ in (3.18).

In the case of a static Robertson-Walker universe the tensors $G_{\mu\nu}$ and $\tilde G_{\mu\nu}$ are

$$G^0{}_0=-3S^{-2}, \quad \tilde G^0{}_0=-18S^{-4}, \quad \tilde G^i{}_k=G^i{}_k=0,$$
$$G^i{}_i=-S^{-2}, \quad \tilde G^i{}_i=6S^{-4} \quad (i\neq k). \quad (3.20)$$

The vacuum energy density (3.16) can therefore be written as

$$\epsilon(S)=\langle T^0{}_0\rangle_0\simeq(32\pi^2)^{-1}(L_4g^0{}_0+\tfrac{2}{3}L_2G^0{}_0-\tfrac{1}{18}L_0\tilde G^0{}_0). \quad (3.21)$$

The same regularization procedure leads us from the pressure equation (3.5) to

$$-p(S)=\langle T^i{}_i\rangle_0\simeq(32\pi^2)^{-1}(L_4g^i{}_i+\tfrac{2}{3}L_2G^i{}_i-\tfrac{1}{18}L_0\tilde G^i{}_i). \quad (3.22)$$

Thus equations (3.21) and (3.22) can be summarized in covariant form as

$$\langle T^\mu{}_\nu\rangle_0\simeq(32\pi^2)^{-1}(L_4g^\mu{}_\nu+\tfrac{2}{3}L_2G^\mu{}_\nu-\tfrac{1}{18}L_0\tilde G^\mu{}_\nu). \quad (3.23)$$

From the general expression (3.14) and (3.16) we learn that also terms of higher order in $(mS)^{-2}$ will be present. These terms are to be expected also from the results of DeWitt[8] but are too complicated to be given explicitly in tensorial form by calculation of the coincidence limits of covariant derivatives of the world function, as done by DeWitt up to the "quadratic curvature" terms.

In the following section we will generalize our method to nonstatic Robertson-Walker cosmologies. This generalization is important because the discussion of collapse behavior includes all aspects of a dynamical metric in its extreme development.

IV. ROBERTSON-WALKER COSMOLOGY

For our present considerations we have first of all to solve the time-dependent equation (2.6) for the scalar field. This is done starting from the ansatz

$$\psi_l(t)=[2S^3W_l(t)]^{-1/2}\exp\left(-i\int^t W_l(t')dt'\right). \quad (4.1)$$

For time-dependent $\omega_l(t)$ the function $W_l(t)$ can be approximated by a refined WKB procedure.[11] The energy function $W_l(t)$ is given explicitly by Parker and Fulling[9] and takes the form

$$W_l(t)=\omega_l(1+\epsilon_2^{(l)})(1+\epsilon_4^{(l)}), \quad (4.2)$$

where $\epsilon_2^{(l)}$ and $\epsilon_4^{(l)}$ contain time derivatives of the radius $S(t)$ up to second and fourth order, respectively. From $W_l(t)$ one finds straightforwardly using the energy-momentum vacuum expectation value (2.7)

$$\langle T^0{}_0\rangle_0\simeq(2\pi^2S^3)^{-1}\sum_l(l+1)^2\Big\{\omega_l+\frac{1}{2}\Big(\frac{\dot S}{S}\Big)^2\omega_l^{-5}\Big(l(l+2)S^{-2}+\frac{3m^2}{2}\Big)^2 + \frac{1}{8}\Big(3\frac{\dot S^4}{S^4}-2\frac{\dot S\dddot S}{S^2}-2\frac{\dot S^2\ddot S}{S^3}+\frac{\ddot S^2}{S^2}\Big)\omega_l^{-11}[(l(l+2)S^{-2})^4+O(l^3(l+2)^3m^2S^{-6})]\Big\}. \quad (4.3)$$

This approximation for the vacuum energy density is correct up to fourth order in the time derivatives of $S(t)$. Now the same regularization as in the static case is performed: We start using the Euler-Maclaurin formula to transform the sum in (4.3) into an integral and regularize the resulting expression choosing a regularizing mass spectrum which again obeys conditions (3.12). One obtains

$$\langle T^0{}_0\rangle_0 \simeq (2\pi^2)^{-1}\int dm^2\rho(m^2)\left\{\frac{m^4}{16}\mu^4\ln\mu^2\nu^2-\left(\frac{\dot S}{S}\right)^2\left(\frac{m^2}{8}\mu^2+\frac{1-\mu^2}{4}\right)\ln\mu^2\nu^2 - \left[3\left(\frac{\dot S}{S}\right)^4-2\frac{\ddot S\dot S^2}{S^3}-2\frac{\dot S\dddot S}{S^2}+\frac{\ddot S^2}{S^2}\right]\tfrac{1}{16}\ln\mu^2\nu^2\right\} \tag{4.4}$$

where ν and μ^2 are given by Eqs. (3.8) and (3.9). Expansion in orders of S^{-2} yields finally

$$\langle T^0{}_0\rangle_0 \simeq \frac{1}{32\pi^2}\left\{L_4-\frac{2L_2}{3}\left[\frac{\dot S^2}{S^2}+\frac{1}{S^2}\right]-\frac{L_0}{18}\left[18\left(-\frac{1}{S^4}+\frac{2\dot S^2}{S^4}+\frac{3\dot S^4}{S^4}-\frac{2\dot S^2\ddot S}{S^3}-\frac{2\dot S\dddot S}{S^2}+\frac{\ddot S^2}{S^2}\right)\right]\right\}. \tag{4.5}$$

We also need the corresponding expression for the vacuum pressure, which by Eq. (2.8) now becomes

$$\langle T^i{}_i\rangle_0 \simeq \frac{1}{32\pi^2}\left\{L_4-\frac{2L_2}{3}\left[-\frac{2\ddot S}{S^2}-\frac{\dot S^2}{S^2}-\frac{1}{S^2}\right] -\frac{L_0}{18}\left[6\left(\frac{1}{S^4}-\frac{2\dot S^2}{S^4}+\frac{4\ddot S}{S^3}-\frac{3\dot S^4}{S^4}+\frac{12\dot S^2\ddot S}{S^3}-\frac{3\ddot S^2}{S^2}-\frac{4\dot S\dddot S}{S^2}-\frac{2\ddddot S}{S}\right)\right]\right\}. \tag{4.6}$$

To join Eqs. (4.5) and (4.6) in one covariant form we establish the agreement of the square-bracketed expressions in Eqs. (4.5) and (4.6) with the time-time and space-space components, respectively, of the Einstein tensor $G^\mu{}_\nu$ and the tensor $\tilde G^\mu{}_\nu$, which is obtained by variation of the "curvature-quadratic" Lagrangian in (3.18). One obtains

$$\langle T^\mu{}_\nu\rangle \simeq (32\pi^2)^{-1}(L_4 g^\mu{}_\nu+\tfrac{2}{3}L_2G^\mu{}_\nu-\tfrac{1}{18}\tilde G^\mu{}_\nu)\ . \tag{4.7}$$

Therefore, up to fourth order in ∂_t and S^{-1} the vacuum expectation value of the quantized scalar field in a Robertson-Walker universe again contains three terms:

(a) a cosmological term proportional to $g_{\mu\nu}$,
(b) an Einstein term proportional to $G_{\mu\nu}$, and
(c) "quadratic curvature" terms proportional to $\tilde G_{\mu\nu}$.

In general, also terms of higher order in ∂_t and S^{-1} will be present, but they are too complicated to show their tensor character explicitly. (In this connection see also Ref. 8.)

V. ORDERS OF MAGNITUDE

From (4.5) and (4.6) and the form of L_n we can easily see that the contributions (a), (b), and (c) are of equal order of magnitude if the universe attains the size of the dominating Compton wavelength of the regularizing mass spectrum or if the universe's time development shows a characteristic time of the order Compton wavelength/velocity of light. Such situations are to be expected for highly collapsed states of the universe.

If, on the other hand, gravitation is a pure "metric elasticity" effect of the vacuum,[3] we have to set $L_0=L_4=0$, implying further conditions on the mass spectrum, and the dominating mass $\overline{m}$ of the spectrum becomes $\overline{m}^2\sim\kappa^{-1}$ (κ is the gravitational constant, $\hbar=c=1$), i.e., $\overline{m}=10^{-5}$ g, which is the Planck mass. Without supposing $L_4=0$ by further cancellations, we would obtain a cosmological density of $\rho_0\sim10^{94}$ g/cm^3.

Starting from the astrophysically founded upper limit of the cosmological density, 10^{-29} g/cm^3 we estimate $\overline{m}=10^{-35}$ g, a quantity showing no connection to known masses of elementary particle physics.

Finally, if we neglect the cosmological term, $L_4\equiv0$, and put in the gravitational constant by hand, the "quadratic" terms $\tilde G_{\mu\nu}$ will take the same order of magnitude as the Einstein term $\kappa^{-1}G_{\mu\nu}$ in a domain where $\kappa^{-1}S^{-2}\sim S^{-4}$, $\kappa\sim S^{-2}$, or S is of the order of Planck length, namely 10^{-33} cm. If the universe collapses to the Planck length, the quadratic corrections to the Einstein equation will become equally important as the classical Einstein terms themselves.

VI. SUMMARY

We have carried out a calculation of the vacuum expectation value of the energy-momentum tensor of a massive scalar field coupled minimally to a closed nonstatic Robertson-Walker metric. The occurring divergences were regularized by averaging over a mass spectrum which fulfills certain regularization conditions. These conditions are the same as needed for regularization of the vacuum energy-momentum density in a Minkowski metric. Up to higher orders in ∂_t and S^{-1} three types of contributions to the energy and pressure of the vacuum are obtained: First a cosmological term present also in the case of flat space, secondly a term of Einstein type renormalizing the gravitational constant, and finally terms resulting from a gravitational Lagrangian quadratic in the curvature quantities. The orders of magnitude of all these terms were estimated, with the following results: The "quadratic term" will dominate the classical gravitational Lagrangian if the universe collapses to the Planck length; to obtain gravitation as a pure "metric elasticity" effect we need particle vacua corresponding to particles with Planck's mass in our universe; the cosmological term has to be dropped in this case (or it dominates all); all three corrections induced by the vacuum energy-momentum tensor on a classical Robertson-Walker universe are of the same order of magnitude if the universe attains the size of the dominating Compton wavelength or if its time development shows a characteristic time of order Compton wavelength/velocity of light.

Our calculations do not provide evidence that gravitational collapse of the universe is averted at some physically realistic distance or that gravitation is described in a natural way by the elastic properties of the vacuum of some particles. But on the other hand there is given some evidence that quantum corrections of the type shown must be taken into account in general relativity and are important at least for highly collapsed states of the early universe.

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