Quantum vacuum energy in general relativity*

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The vacuum energy of a quantized field in a non-Minkowskian spacetime is discussed. The approach taken emphasizes the analogy between this vacuum energy and the energy of the vacuum state of the quantized electromagnetic field in the presence of a pair of parallel conducting plates, the Casimir energy. The energy of the vacuum state of a quantized scalar field in a one-dimensional box of length L (the spacetime manifold $S^1\times R$) is shown to be $-\pi\hbar c/6L$. A massless, conformal scalar field in the Einstein universe ($S^3\times R$) also possesses a nonzero vacuum energy. The vacuum energy density in this case is $\rho = \hbar c(480\pi^2 a_0^4)^{-1}$, where a_0 is the radius of the universe. The pressure P is $\frac{1}{3}\rho$, so the energy-momentum tensor associated with these zero-point fluctuations is of the same form as that for classical radiation. It is shown that a closed Robertson-Walker universe has the same vacuum energy density and pressure as a static universe of instantaneously equal radius. The electromagnetic, neutrino, and minimally coupled scalar fields in the Einstein universe cannot be treated successfully by these techniques. Finally, the vacuum energy of a scalar field in the presence of a linearized plane gravitational wave is discussed. It is shown that for a certain choice of vacuum state, which is an eigenstate of the Hamiltonian so no pair production occurs, the vacuum energy and pressure vanish. This result holds for both the conformal and nonconformal energy-momentum tensors.

I. INTRODUCTION

One of the basic questions concerning the relationship of quantum field theory and gravitation is that of the zero-point or vacuum energy. Does the zero-point energy of a quantized field act as the source of a gravitational field? In other areas of physics, the zero-point energy is usually dispensed with on the grounds that only differences in energy are measurable and the scale of energy may be made to start wherever we wish. In gravitational physics this luxury does not exist; the actual value of the energy-momentum tensor of matter determines the geometry of spacetime.

A number of authors have addressed themselves to the question of determining the energy-momentum tensor for a quantized field in a curved background spacetime. One of the earliest investigations was carried out by Utiyama and DeWitt,[1] who gave a covariant prescription for removing the divergent terms in the energy-momentum tensor by means of an expansion in powers of the gravitational constant. Further work was done by Halpern.[2] This technique promises to be a powerful tool, but has not yet been developed to the point that it can be readily applied to specific problems. One of the cases of particular interest is that of the cosmological models. Here investigations have been made by Zel'dovich and Starobinsky,[3] Parker and Fulling,[4] and by Fulling, Parker, and Hu.[5] They have shown that by making appropriate subtractions, a renormalized energy-momentum tensor may be found which seems to be unique. The question of uniqueness does not appear to be fully settled, however. It is hoped that there exists only one generally covariant subtraction procedure so that all methods for performing the subtractions will necessarily lead to the same result. At the present time only a limited understanding of the physical aspects of the vacuum energy problem exists, so the theoretical basis of all of the above techniques is still rather obscure. A review of work in this subject has recently been given by DeWitt.[6]

One may make some general remarks concerning the origin of the vacuum energy. It is only those modes whose wavelengths are of the order of or longer than l, the local radius of curvature, which are affected by the presence of spacetime curvature. One may find a local inertial frame which transforms the gravitational field away over a region whose linear dimensions are of the order of l, so much shorter wavelengths are not essentially affected by the presence of the gravitational field. Those modes of wavelength greater than l have a zero-point energy density[7] of the order of l^{-4}, so we might expect that the energy density induced into the vacuum state of a quantized field by the presence of an external gravitational field will necessarily also be of the order of l^{-4}. Hawking[8] has expressed this by noting that an unambiguous definition of particle number in a local inertial frame may only be given for modes of wavelength much less than l. The uncertainty in particle number of modes of longer wavelength is associated with an energy density of the order of l^{-4}.

An energy density of this magnitude will necessarily cause a significant correction to the original gravitational field when l approaches the Planck length ($\lambda_P = 1$, or 1.62×10^{-33} cm in conventional units). It is not clear at this time whether this energy density is necessarily positive, or whether it always satisfies the various other assumptions on the energy-momentum tensor which are used in the proof of singularity theorems. In all of the works cited above and in the present paper, the gravitational field is unquantized and is assumed to be precisely measurable. This is presumably only a semiclassical approximation to a more exact theory in which all fields in nature are quantized. It is probably at least as good an approximation to the full theory as that obtained by quantizing only the gravitational field and neglecting the effects of other fields.

The creation of a nonzero vacuum energy by an external gravitational field is analogous to the Casimir effect. Casimir[9] demonstrated that the vacuum fluctuations of the electromagnetic field give rise to an attractive force between a pair of parallel conducting plates. Quantize the electromagnetic field subject to the appropriate boundary conditions at the plates and calculate the vacuum energy with a wavelength cutoff. One finds that as the separation between the plates changes, the vacuum energy per unit area changes by a finite, cutoff-independent amount. Thus, in spite of the formal divergence of the vacuum energy, a change in the configuration of the system causes a finite shift in the energy of the vacuum state. If the vacuum energy of the system for infinite separation is set equal to zero, then the energy per unit area of the plates for any finite separation R is $-\pi^2(720R^3)^{-1}$. One may show that this energy is uniformly distributed between the plates.[10] Thus there is a constant negative energy density $\propto R^{-4}$, corresponding to the fact that it is primarily modes whose wavelength is of the order of R which contribute to the vacuum energy.

It is reasonable that the vacuum energy of a free quantized field in Minkowski space be zero. This corresponds to the infinitely separated plates in the case of the Casimir effect. Poincaré invariance rules out any nonzero vacuum expectation value of the energy-momentum tensor except possibly a constant multiple of $\eta_{\mu\nu}$, the Minkowski metric. Takahashi and Shimodaira[11] have shown that in fact the vacuum expectation value must be zero in order that the generators of the Poincaré group satisfy the correct commutation relations. A further argument to this effect is that Minkowski space will not be a solution of Einstein's equations with zero cosmological constant unless the total energy-momentum tensor for matter vanishes. (On the other hand, if the cosmological constant is nonzero, one is forced to assign a nonzero energy-momentum tensor to Minkowski space.) We thus will adopt the point of view that the physical vacuum energy of a free quantized field in Minkowski space must be zero. This applies only to the full manifold R^4, however. Other manifolds may have a nonzero vacuum energy even though the curvature tensor vanishes in part or all of the spacetime. See Sec. II.

In this paper a particular approach to the problem of the vacuum energy will be discussed. The Casimir effect suggests that we look for examples where the presence of a gravitational field is a perturbation which shifts the vacuum energy by a finite amount from its Minkowski space value. If we can then deform a given manifold into Minkowski space, we can ask whether or not the deformation changes the vacuum energy density by a finite amount; if so, this difference is the physical vacuum energy density of the original manifold. More precisely, define a cutoff energy-momentum tensor whose vacuum expectation value is always finite. Vary some parameter which determines the strength of the gravitational field. If the vacuum expectation value of the cutoff energy-momentum tensor changes by a cutoff-independent quantity, this is the change in the physical energy-momentum tensor associated with the zero-point fluctuations of the field in question.

This procedure has a number of limitations. The least serious of these is that the actual calculations must be done in a noncovariant fashion. This does not prevent the final result from being covariant. As long as the quantity to be subtracted can be identified unambiguously on physical grounds, the need for manifest covariance is not as great as in those techniques which have no other basis for finding the correct subtraction. Another question which arises is whether the final result depends on the nature of the cutoff function which is used. This seems unlikely to be the case, provided that a cutoff-independent result is in fact obtained, but a proof of this is lacking. The most serious limitation of the method is that it usually does not work. It is only in special cases that a cutoff-independent change in the vacuum energy can be identified. Two such examples are a conformally invariant scalar field in a closed Robertson-Walker metric and a scalar field in the presence of a linearized gravitational wave. The first example has been discussed recently by Fulling, Parker, and Hu,[5] who show that the renormalizations required to remove the divergences of the energy-momentum tensor are especially simple in this case. In most cases, such as that of the minimally coupled scalar field in the Robertson-Walker metric, more elaborate renormalization is required. These authors do not actually calculate the finite remainder after subtractions have been made.

The second example has been discussed by

Gibbons[12] who treats an exact plane wave which is an exact solution of Einstein's equations. He argues that the renormalized energy-momentum tensor must be identically zero, which agrees with the first-order result obtained in Sec. IV. The analogous case of a plane electromagnetic wave interacting with the Dirac field was investigated by Schwinger,[13] who found that there are no nonlinear vacuum effects due to a single plane wave.

In Sec. II the example of a scalar field in a one-dimensional box with periodic boundary conditions is studied. It is shown that the energy of the vacuum state is nonzero if the box is finite. In Sec. III the conformal scalar field in the Einstein universe and a general closed Robertson-Walker metric is studied. Again it is shown that the vacuum state has associated with it a nonzero energy density and pressure. In Sec. IV, a scalar field in the background of a linearized gravitational wave is examined. In this case the vacuum energy density and pressure vanish.

II. A SIMPLE EXAMPLE

The simplest example of a quantum field theory in which the energy of the vacuum state is nonzero is that of a massless scalar field in a one-dimensional box subject to periodic boundary conditions. If the length of the box is L, then the eigenfrequencies are

$$\omega_n = \frac{2\pi n}{L}, \quad n = 0, 1, 2, \ldots . \tag{1}$$

Each frequency is two-fold degenerate. The zero-point energy, with a frequency cutoff, is

$$\overline{E}_0 = 2 \sum_n \tfrac{1}{2}\omega_n e^{-\alpha\omega_n} . \tag{2}$$

It is easy to show that

$$\begin{aligned}\overline{E}_0 &= \frac{2\pi}{L} e^{-2\pi\alpha/L}(1 - e^{-2\pi\alpha/L})^{-2} \\ &= \frac{L}{2\pi\alpha^2} - \frac{\pi}{6L} + (\text{terms in positive powers of } \alpha).\end{aligned} \tag{3}$$

If one divides $\overline{E}_0$ by L and lets $L \to \infty$, it is apparent that the cutoff energy density in an infinite box is $(2\pi\alpha^2)^{-1}$. The Casimir prescription tells us to subtract this energy density from that for a finite box; if the result is a finite, cutoff-independent quantity it may be identified as being the physical zero-point energy density in the finite box. Let

$$\tilde{E}_0 = \frac{L}{2\pi\alpha^2} . \tag{4}$$

Then the physical zero-point energy in the finite box is

$$E_0 = \lim_{\alpha \to 0} (\overline{E}_0 - \tilde{E}_0) = -\frac{\pi}{6L} , \tag{5}$$

which is to be identified as the energy of the vacuum state of the quantized scalar field.

This is essentially a one-dimensional version of the original Casimir effect. However, it is interesting to note that we are dealing with a two-dimensional spacetime with the topology $S^1 \times R$. It is the topological structure of this spacetime (the fact that it is closed) which is responsible for the nonzero energy of the vacuum state.

III. THE CONFORMAL SCALAR FIELD IN A CLOSED UNIVERSE

A. The Einstein universe

We will now investigate the conformally invariant scalar field in the Einstein universe, which has the spatial geometry of a 3-sphere. The metric may be expressed as

$$ds^2 = -dt^2 + a_0^2 d\sigma^2 , \tag{6}$$

where

$$d\sigma^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2) . \tag{7}$$

Here χ and θ run from 0 to π and ϕ runs from 0 to 2π. The radius of the universe, a_0, is constant. The conformally invariant Klein-Gordon equation for a massless scalar field is

$$\Box\psi - \tfrac{1}{6}R\psi = 0 , \tag{8}$$

where $\Box\psi = \psi^{,\rho}{}_{;\rho}$ and $R = 6a_0^{-2}$ is the scalar curvature[14] corresponding to Eq. (6).

Let $\psi = X(\chi)P_l^m(\cos\theta)e^{im\phi}e^{-i\omega t}$, where P_l^m is the associated Legendre function. Then Eq. (8) becomes

$$\frac{d}{d\chi}\left(\sin^2\chi \frac{dX}{d\chi}\right) + (\omega^2 - \tfrac{1}{6}R)\sin^2\chi\, X - l(l+1)X = 0 . \tag{9}$$

The solutions of this equation are

$$X \propto \sin^l\chi\, C_{n-l}^{l+1}(\cos\chi), \tag{10}$$

where the $C_{n-l}^{l+1}(\cos\chi)$ are Gegenbauer functions and $n = 0, 1, 2, \ldots$. The eigenfrequencies are

$$\omega_n = \left(\frac{n(n+2)}{a_0^2} + \tfrac{1}{6}R\right)^{1/2} = \frac{n+1}{a_0} . \tag{11}$$

For fixed n, l takes on the values $0, 1, \ldots, n$. Thus, the degeneracy of each eigenfrequency is

$$\sum_{l=0}^{n} (2l+1) = (n+1)^2 . \tag{12}$$

The solutions of the minimally coupled wave equation, $\Box\psi=0$, in the Einstein universe were first given by Schrödinger.[15,16] The solutions of the conformal equation are studied in Ref. 5.

The energy-momentum tensor for a conformally invariant scalar field in a Riemannian spacetime has been discussed by Parker.[17] It is, in our notation,

$$T_\alpha{}^\beta = -\psi_{,\alpha}\psi^{,\beta} + \tfrac{1}{2}\delta_\alpha{}^\beta g^{\rho\sigma}\psi_{,\rho}\psi_{,\sigma} + \tfrac{1}{6}(\psi^2)^{,\beta}{}_{;\alpha} - \tfrac{1}{6}\delta_\alpha{}^\beta(\psi^2)^{,\rho}{}_{;\rho} - \tfrac{1}{6}G_\alpha{}^\beta\psi^2\,, \tag{13}$$

where $G_\alpha{}^\beta = R_\alpha{}^\beta - \frac{1}{2}\delta_\alpha{}^\beta R$ is the Einstein tensor. In particular, for the metric Eq. (6), $G^0{}_0 = -3a_0{}^{-2}$.

The Hamiltonian is

$$H = \int T_0{}^0\sqrt{-g}\,d^3x\,. \tag{14}$$

It may be shown in our case that

$$H = \tfrac{1}{2}\int (\psi_{,0}{}^2 - \psi\psi_{,00})\sqrt{-g}\,d^3x\,. \tag{15}$$

If ψ is a Hermitian field operator, it may be expressed as

$$\psi = \sum_\lambda (a_\lambda F_\lambda + a_\lambda F_\lambda^*)\,, \tag{16}$$

where $\lambda=\omega,\ l,$ and m. Here $F_\lambda = f_\lambda e^{-i\omega t}$ is a solution of Eq. (8). The spatial functions f_λ may be normalized so that

$$\int f_{\lambda_1} f^*_{\lambda_2}\sqrt{-g}\,d^3x = (2\omega)^{-1}\delta_{\lambda_1\lambda_2}\,. \tag{17}$$

The Hamiltonian now takes the familiar form

$$H = \tfrac{1}{2}\sum_\lambda \omega(a_\lambda a_\lambda^\dagger + a_\lambda^\dagger a_\lambda)\,. \tag{18}$$

The formal quantization may be carried out in the usual manner.[5] The operators a_λ and $a_\lambda^\dagger$ satisfy the usual commutation relations for creation and annihilation operators and may be used to define a Fock space. The vacuum state is defined by $a_\lambda|0\rangle=0$ for all λ. In this particular spacetime, the existence of a global timelike Killing vector leads naturally to a unique choice for the vacuum state.

We may now proceed as in Sec. II. The cutoff zero-point energy is

$$\begin{aligned}\bar{E}_0 &= \sum_\lambda \tfrac{1}{2}\omega e^{-\alpha\omega}\\ &= \frac{1}{2a_0}\sum_{n=0}^{\infty}(n+1)^3 e^{-(n+1)\alpha/a_0}\\ &= \frac{1}{2a_0}(e^{3\alpha/a_0} + 4e^{2\alpha/a_0} + e^{\alpha/a_0})(e^{\alpha/a_0}-1)^{-4}\,.\end{aligned} \tag{19}$$

If this quantity is expanded in powers of α, one finds that

$$\bar{E}_0 = 3\frac{a_0{}^3}{\alpha^4} + \frac{1}{240a_0} + (\text{terms in positive powers of } \alpha/a_0)\,. \tag{20}$$

The cutoff zero-point energy in an equal volume of Minkowski space is just

$$\tilde{E}_0 = 3\frac{a_0{}^3}{\alpha^4}\,, \tag{21}$$

so that the physical vacuum energy in the closed Einstein universe is

$$E_0 = \lim_{\alpha\to 0}(\bar{E}_0 - \tilde{E}_0) = \frac{1}{240a_0}\,. \tag{22}$$

The spatial volume of the universe is $2\pi^2 a_0{}^3$, so the energy density is

$$\rho = \frac{1}{480\pi^2 a_0{}^4}\,. \tag{23}$$

The corresponding pressure may be obtained from the requirement that the conformally invariant energy-momentum tensor be traceless. Since our renormalization procedure consists in subtracting the Minkowski-space energy-momentum tensor from the Einstein-universe energy-momentum tensor and both are traceless, the renormalized energy-momentum tensor is also traceless. Hence the pressure is

$$P = \tfrac{1}{3}\rho\,.$$

B. The closed expanding universe

We now turn to the case of a closed Robertson-Walker metric and show that the time dependence of the metric has no effect upon the vacuum energy and pressure. Since this metric may be obtained from that of Eq. (6) by a conformal transformation, we first study the behavior of the Hamiltonian under such a transformation. Let the metric $\tilde{g}_{\mu\nu}$ be conformally related to $g_{\mu\nu}$:

$$\tilde{g}_{\mu\nu} = \Omega^{-2}g_{\mu\nu}\,. \tag{24}$$

Under such a transformation, Eq. (8) is invariant provided that the field becomes $\tilde{\psi}=\Omega\psi$. The energy-momentum tensor becomes[17]

$$\tilde{T}^\mu{}_\nu = \Omega^4 T^\mu{}_\nu\,. \tag{25}$$

The Hamiltonian is an invariant:

$$\tilde{H} = \int \tilde{T}^0{}_0(-\tilde{g})^{1/2}d^3x = H \tag{26}$$

since $(-\tilde{g})^{1/2} = \Omega^{-4}\sqrt{-g}$. The new field operator may be expanded as

$$\tilde{\psi} = \sum_\lambda (a_\lambda\tilde{F}_\lambda + a_\lambda^\dagger\tilde{F}_\lambda^*)\,, \tag{27}$$

where $\tilde{F}_\lambda = \Omega F_\lambda$ are solutions of the transformed Klein-Gordon equation. Since the creation and annihilation operators are unchanged by the transformation, states of definite particle number in the metric $g_{\mu\nu}$ will also be states of definite particle number in $\tilde{g}_{\mu\nu}$.

The second-quantized Hamiltonian, $\tilde{H}$, thus has the same expression, Eq. (18), as H. The vacuum state is an eigenstate of $\tilde{H}$, so there will be no production of (conformal) scalar particles in a Robertson-Walker metric, as was first pointed out by Parker.[18]

If the time coordinate is rescaled, so that $f(t)\,d\eta = dt$ defines a new time coordinate η, the Hamiltonian is multiplied by a factor of f. This is the case because $T^0{}_0$ is invariant and $\sqrt{-g}$ is multiplied by f under such a rescaling. The Hamiltonian for a scalar field in the closed Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2 d\sigma^2\,, \tag{28}$$

may be obtained from that in the Einstein universe as follows. Rescale the time coordinate in Eq. (6) by $a_0 d\eta = dt$. Perform a conformal transformation for which $\Omega = a_0/a$. Finally, rescale the time coordinate again by $d\eta = a^{-1}(t)\,dt$. The net effect on the Hamiltonian is to multiply it by a factor of a_0/a. The vacuum state of the theory is unchanged by this transformation, so the vacuum expectation value of the Hamiltonian is multiplied by the same factor. The essential point is that the time dependence of a does not affect the form of the vacuum energy, which is now given by Eq. (22) with a_0 replaced by a. Thus, the physical vacuum energy in a closed Robertson-Walker metric must be the same as that in an Einstein universe of instantaneously equal radius.

One may infer that the vacuum energy density vanishes in the flat and open Robertson-Walker metrics. Fulling, Parker, and Hu[5] have shown that in both cases the vacuum expectation value of $T^0{}_0$ may be expressed as the integral $(4\pi^2)^{-1}\int \omega^3 d\omega$, which is also the expression appropriate to flat space. Thus, their difference is zero and the physical vacuum energy and pressure for a massless conformal scalar field vanish in these cases.

We have seen that the energy-momentum tensor associated with the vacuum fluctuations of the massless conformal scalar field in a closed Robertson-Walker metric is identical to that corresponding to classical radiation; specifically, $P = \frac{1}{3}\rho$ and $\rho > 0$. In this case, there is no avoidance of the singularity theorems. A universe in which the vacuum fluctuations are the sole contribution to $T_{\mu\nu}$ will behave just as does the radiation-filled Friedmann universe. The maximum radius of such a universe is given by

$$a_{\max}{}^2 = \frac{8\pi}{3}\,\rho a^4 = \frac{1}{180\pi}\,. \tag{29}$$

Thus, a closed universe filled only with vacuum energy expands to a maximum radius of $(180\pi)^{-1/2}\lambda_P \simeq 10^{-34}$ cm before recontracting.

One might hope that the electromagnetic and neutrino fields, which also satisfy conformally invariant equations, would also lead to a finite vacuum energy and pressure in the same way that the conformal scalar field does. Unfortunately, this is not the case. In the metric Eq. (6), the solution of Maxwell's equations leads to the eigenfrequencies[19]

$$\omega_n = \frac{n}{a_0}\,, \quad n = 2, 3, \ldots\,. \tag{30}$$

Each frequency is $2(n^2-1)$-fold degenerate. The solution of the neutrino wave equation leads to the eigenfrequencies[15]

$$\omega_n = \frac{2n+1}{2a_0}\,, \quad n = 1, 2, \ldots \tag{31}$$

with a degeneracy of $2n(n+1)$. In either case, if one repeats the calculation above, one finds that the analog of Eq. (20) contains not only an α^{-4} term but also an α^{-2} term. (In the case of fermions, each mode contributes an amount $-\frac{1}{2}\omega$ to the vacuum energy.) This means that not all of the divergences of the energy-momentum tensor may be removed by subtracting the flat-space energy-momentum tensor.

We should also note that the minimally coupled scalar field also cannot be treated successfully by these techniques. In this case, the wave equation is Eq. (8) except without the $-\frac{1}{6}R$ term. The resulting eigenfrequencies are

$$\omega_n = \frac{1}{a_0}\,[n(n+2)]^{1/2}\,, \quad n = 0, 1, 2, \ldots \tag{32}$$

with a degeneracy of $(n+1)^2$. This case has recently been investigated by Streeruwitz[20] who considers the possibility of renormalizing the energy-momentum tensor using a method proposed by Zel'dovich.[21]

In light of the fact that the Casimir method fails for the minimally coupled scalar field, it is interesting that it still succeeds for a massive scalar field which satisfies the equation $\Box\psi - \frac{1}{6}R\psi - \mu^2\psi = 0$. Although this equation is not conformally invariant if $\mu \neq 0$, it still leads to a finite vacuum energy. The details of this result will be discussed elsewhere.

IV. A LINEARIZED GRAVITATIONAL WAVE

In this section the interaction of a quantized scalar field with a plane gravitational wave will be considered. The wave will be assumed to be weak and will be treated in the linearized approximation. Such a wave can be described by the metric $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ where

$$h_{11}=-h_{22}=A\cos K(z-t) \tag{33}$$

and all other components of $h_{\mu\nu}$ vanish. This is a polarized wave propagating in the $+z$ direction. Our results will be independent of this choice of polarization. The other independent polarization state is represented by the metric $h_{12}=h_{21}=A\cos K(z-t)$, which may be related to that of Eq. (33) by a coordinate transformation.

The Klein-Gordon equation for a massive scalar field in this metric becomes

$$\Box\psi-\mu^2\psi=\nabla^2\psi-\frac{\partial^2\psi}{\partial t^2}-\mu^2\psi -A\cos K(z-t)\left(\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}\right) =0. \tag{34}$$

The scalar curvature R vanishes in this spacetime, so there is no ambiguity in the choice of Eq. (34). Let $\psi=e^{i(k_x x+k_y y-\omega t)}f(z)$. Then

$$\frac{d^2f}{dz^2}+[k_z{}^2+A(k_x{}^2-k_y{}^2)\cos K(z-t)]f(z)=0 \tag{35}$$

if $k_z{}^2=\omega^2-k_x{}^2-k_y{}^2-\mu^2$. A solution of this equation is

$$f(z)=c_{\vec{k}}\exp\left\{i\left[k_z z-\frac{A(k_x{}^2-k_y{}^2)}{2K(\omega-k_z)}\sin K(z-t)\right]\right\}. \tag{36}$$

If $\mu=0$, and $k_x=k_y=0$, the second term in the argument of the exponential is not present in Eq. (36).

We wish to define a set of solutions of Eq. (34) which satisfy periodic boundary conditions at the boundaries of a cube of edge L. This leads to a discrete set of eigenvalues for k_x, k_y, and k_z. The eigenvalues for k_x and k_y have the usual form, $2\pi/L$ times an integer. Those for k_z are more complicated; they are functions of k_x, k_y, and t. This is not an essential difficulty, however. For sufficiently large L, k_z is real and as $L\to\infty$, $k_z\to 2\pi n/L$ where n is an integer. Thus, when the infinite-volume limit is taken, the eigenvalues of $\vec{k}$ are the usual ones in flat space and $\sum_{\vec{k}}$ becomes $(2\pi)^{-3}V\int d^3k$. Here $V=L^3$.

Let $\{F_{\vec{k}}\}$ be a complete set of solutions of Eq. (34). If we choose $c_{\vec{k}}=(2\omega V)^{-1/2}$, then the $F_{\vec{k}}$ satisfy the orthonormality relations

$$\int F_{\vec{k}_1}F^*_{\vec{k}_2}d^3x=\frac{1}{2\omega}\delta_{\vec{k}_1,\vec{k}_2} =\int F_{\vec{k}_1}F_{-\vec{k}_2}d^3x\ , \tag{37}$$

where the integration is taken over the volume of the cube. In addition, the fact that $\int(d/dz)(f_1f_2)dz=0$ for any two solutions of Eq. (36) leads to the relations

$$\int F_{\vec{k}_1}F_{\vec{k}_2}\cos K(z-t)\,d^3x=0 \tag{38}$$

and

$$\int F_{\vec{k}_1}F^*_{\vec{k}_2}\cos K(z-t)\,d^3x=\delta_{\vec{k}_1,\vec{k}_2}(2\omega V)^{-1}\times\int\cos K(z-t)\,d^3x \to 0 \text{ as } V\to\infty \tag{39}$$

The Lagrangian density for the scalar field is

$$\mathcal{L}=-(g^{\alpha\beta}\psi_{,\alpha}\psi^\dagger_{,\beta}+\mu^2\psi\psi^\dagger), \tag{40}$$

and the associated energy-momentum tensor is

$$\tau_{\rho\nu}=\frac{\delta\mathcal{L}}{\delta\psi^{,\rho}}\psi_{,\nu}+\frac{\delta\mathcal{L}}{\delta\psi^{\dagger,\rho}}\psi^\dagger_{,\nu}-g_{\rho\nu}\mathcal{L} =-\psi^\dagger_{,\rho}\psi_{,\nu}-\psi_{,\rho}\psi^\dagger_{,\nu}+g_{\rho\nu}(g^{\alpha\beta}\psi_{,\alpha}\psi^\dagger_{,\beta}+\mu^2\psi\psi^\dagger). \tag{41}$$

This is the nonconformal tensor. Although the minimally coupled and conformal Klein-Gordon equations are identical here, the energy-momentum tensors are not. The conformal tensor is given by

$$T_{\rho\nu}=\tau_{\rho\nu}+\tfrac{1}{3}[(\psi\psi^\dagger)_{;\rho\nu}-g_{\rho\nu}\Box(\psi\psi^\dagger)-G_{\rho\nu}\psi\psi^\dagger]. \tag{42}$$

Since $G_{\rho\nu}=0$ in this spacetime, $T_{\rho\nu}$ may be expressed as

$$T_{\rho\nu}=\tau_{\rho\nu}+\tfrac{1}{3}[(\psi_{,\rho}\psi^\dagger_{,\nu}+\psi_{,\nu}\psi^\dagger_{,\rho})-2g_{\rho\nu}(\psi^{,\alpha}\psi^\dagger_{,\alpha}+\mu^2\psi\psi^\dagger) +(\psi_{;\rho\nu}\psi^\dagger+\psi\psi^\dagger_{;\rho\nu})]. \tag{43}$$

The canonical momentum is defined to be

$$\pi=\frac{\delta\mathcal{L}}{\delta\psi_{,0}}=\psi^\dagger_{,0}. \tag{44}$$

The field operator ψ and its momentum are required to satisfy

$$[\psi(x),\pi(y)]=[\psi^\dagger(x),\pi^\dagger(y)]=i\delta(x,y) \tag{45}$$

on a t-constant hypersurface, where $\delta(x,y)$ is the usual 3-dimensional Dirac δ function.

If ψ is expanded in terms of the basis function $F_{\vec{k}}$,

$$\psi=\sum_{\vec{k}}(a_{\vec{k}}F_{\vec{k}}+b^\dagger_{\vec{k}}F^*_{\vec{k}}), \tag{46}$$

then the creation and annihilation operators satisfy the usual commutation relations

$$[a_{\vec{k}_1},a^\dagger_{\vec{k}_2}]=[b_{\vec{k}_1},b^\dagger_{\vec{k}_2}]=\delta_{\vec{k}_1,\vec{k}_2}. \tag{47}$$

The vacuum state of the system is defined by $a_{\vec{k}}|0\rangle = b_{\vec{k}}|0\rangle = 0$ for all $\vec{k}$.

The Hamiltonian operator is $H = \int \tau_0{}^0 d^3x$. If one inserts Eq. (46) in this expression using the relations Eqs. (37), (38), and (39) and then lets $L \to \infty$, the result is

$$H = V(2\pi)^{-3} \int d^3k \, \frac{\omega}{2} \, (a_{\vec{k}} a_{\vec{k}}^{\dagger} + a_{\vec{k}}^{\dagger} a_{\vec{k}} + b_{\vec{k}} b_{\vec{k}}^{\dagger} + b_{\vec{k}}^{\dagger} b_{\vec{k}}) . \tag{48}$$

Thus the Hamiltonian is diagonal and the vacuum is an eigenstate. There is no particle production by a single plane gravitational wave.

It is important to note that a crucial assumption in this work is this choice of vacuum state. It seems to be a natural choice to make in this quantization procedure and reduces to the usual Minkowski vacuum as $A \to 0$. It is equivalent to the choice used by Gibbons[12] who defines an in and an out vacuum on either side of an exact plane wave. Since there is no particle creation, the two states are identical and may be identified as being *the* vacuum state for all time. Similarly, if the plane gravitational wave considered here were to be switched on in the distant past and off in the distant future, the vacuum state defined here would also be the in and out vacua.

The vacuum expectation value of the energy-momentum tensor is

$$\begin{aligned}\langle 0|\tau_{\rho\nu}|0\rangle &= -\sum_{\vec{k}} [F^*_{\vec{k},\rho} F_{\vec{k},\nu} + F_{\vec{k},\rho} F^*_{\vec{k},\nu} \\ &\qquad - g_{\rho\nu}(g^{\alpha\beta} F_{\vec{k},\alpha} F^*_{\vec{k},\beta} + \mu^2 |F_{\vec{k}}|^2)] \\ &= -\sum_{\vec{k}} (F^*_{\vec{k},\rho} F_{\vec{k},\nu} + F_{\vec{k},\rho} F^*_{\vec{k},\nu}) ,\end{aligned} \tag{49}$$

which is infinite as usual. By use of Eq. (36), one may show that $g^{\alpha\beta}(F_{\vec{k},\alpha} F^*_{\vec{k},\beta} + \mu^2 |F_{\vec{k}}|^2)$ vanishes.

We now introduce a frequency cutoff and define the cutoff vacuum energy-momentum tensor by

$$\langle 0|\tau_{\rho\nu}|0\rangle_c = -\sum_{\vec{k}} (F_{\vec{k},\rho} F^*_{\vec{k},\nu} + F^*_{\vec{k},\rho} F_{\vec{k},\nu}) e^{-\alpha\omega} . \tag{50}$$

Let $L \to \infty$. Then

$$\begin{aligned}\langle 0|\tau_{00}|0\rangle_c &= -\frac{1}{(2\pi)^3} \int \frac{1}{\omega} \left[\omega - \frac{A(k_x{}^2 - k_y{}^2)}{2(\omega - k_z)} \cos K(z-t) \right]^2 \\ &\qquad \times e^{-\alpha\omega} d^3k \\ &= -\frac{1}{(2\pi)^3} \int \omega e^{-\alpha\omega} d^3k ,\end{aligned} \tag{51}$$

where the term proportional to A^2 has been dropped and the term proportional to A vanishes (as may be seen by interchange of k_x and k_y). Similarly, one finds that

$$\begin{aligned}\langle 0|\tau_{11}|0\rangle_c &= \langle 0|\tau_{22}|0\rangle_c \\ &= \langle 0|\tau_{33}|0\rangle_c \\ &= -\frac{1}{(2\pi)^3} \int \frac{k_x{}^2}{\omega} e^{-\alpha\omega} d^3x ,\end{aligned} \tag{52}$$

and all other components of $\langle 0|\tau_{\mu\nu}|0\rangle_c$ vanish to first order in A. These expressions for the cutoff vacuum energy and pressure are, however, identical to those in Minkowski space (since they are independent of A). Thus the physical vacuum expectation value of the energy-momentum tensor, Eq. (41), is zero to first order in A. One may also show that the conformal tensor, Eq. (43), leads to the same result.

This is perhaps not a startling result. One might have guessed at the beginning that the vacuum energy should be independent of the sign of A, which may be changed by shifting the phase of the gravitational wave by π, and hence that there should be no linear term in the final result. It is nonetheless gratifying that this result follows from a successful Casimir calculation. This calculation demonstrates directly that the physical vacuum energy and pressure must be zero to first order in A but leaves the question of higher-order contributions unanswered. That in fact there are no such higher-order contributions is indicated by the work of Gibbons, which deals with an exact solution of the Einstein equations.

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