Even-wave harmonic oscillator theory of baryonic states: A new classification

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(Received 13 March 1974; revised manuscript received 2 July 1974)

An even-wave harmonic oscillator (h.o.) model for the quark-quark interaction proposed recently for the baryon spectrum is described with a detailed mathematical formulation. The mechanism, which formally admits of a relativistic extension of the Feynman-Kislinger-Ravndal type, leaves unchanged the usual h.o. predictions for $\underline{56}$ states (symmetric) for all L values even and odd, but totally keeps out the $\underline{20}$ states (antisymmetric). It changes the structure of the $\underline{70}$ states considerably, while retaining the principal feature of linear rise of (mass)^2 with J through the interplay of two reduced slopes of magnitudes $\frac{1}{2}\alpha$ and $\frac{1}{2}\sqrt{3}\alpha$, compared to α for the $\underline{56}$ spectrum. The new features of the $\underline{70}$ states are (i) a dual spectrum leading to considerable mass splitting compared to the usual h.o. model without SU(6)-breaking effects, (ii) prediction of a unique $(\underline{70}, 0^+)$ supermultiplet *lower* than the $(\underline{70}, 1^-)$, and (iii) the prediction of low radial excitations because of the reduced slopes. The immediate experimental successes are (i) an understanding of $P_{11}(1470)$ together with possible Δ, Σ, Λ counterparts, (ii) two distinct mass groupings manifest in $(\underline{70}, 1^-)$ states, and (iii) plausible explanation of $P'_{11}(1750)$ as a radial excitation of $P_{11}(1470)$. The mass splittings of Δ, Σ, Λ from their N counterparts, compared for $\underline{56}$ and $\underline{70}$ states, conform extremely well to the ratio of the *average* slope $\delta = \frac{1}{4}\alpha(1+\sqrt{3}) \approx 0.68\alpha$ for $\underline{70}$ states to that (α) for the $\underline{56}$, thus facilitating the prediction of Δ, Σ, Λ positions from those of N states for different quantum numbers. Extra predictions of states are discussed in terms of an extended classification scheme given by an ordered set of four quantum numbers $(n_x l_x n_y l_y)$ defined in the text.

I. INTRODUCTION

The success of the harmonic-oscillator (h.o.) quark model[1-3] stems primarily from its capacity to predict a linear rise of squared masses with J. All versions of this model, relativistic or otherwise, incorporate this feature in an essential manner. Therefore, any attempt to modify the h.o. model in order to respond to the weaker areas of agreement with experiment is likely to prove futile unless the "straight line" feature is maintained, and this fact severely limits the possible alternatives.

Certain experimental features provide important clues to the direction of modification. For example, one of the natural predictions of the h.o. model[1] is the existence of $\underline{20}$ states of $L^P = 1^+$, corresponding to the spectroscopic notation $P(1p)^2$, at the same principal quantum number which describes the first Regge recurrence of the $\underline{56}$ supermultiplet. So far there has been no evidence, direct or indirect, for any of the $\underline{20}$ states, after almost a decade of the discovery of the symmetrical quark model.[1] Another vexing anomaly is the existence of the $P_{11}(1470)$ at an unusually low mass compared to what its most favorable status in the h.o. model (viz., the first radial excitation of the $\underline{56}$ nucleon) would entitle it to. No less significant is the fact that its mass is lower than even the lower-mass negative-parity nucleons $D_{13}(1520)$ and $S_{11}(1550)$. To understand such huge mass shifts would require strong symmetry-breaking effects which can be incorporated in appropriate mass operators with several free parameters to be determined from the data. Several attempts which were made in this direction in the mid-sixties[4-6] have recently been stepped up more systematically in the light of much richer data that are now available.[7,8] Still another feature is the existence of two distinct mass groups in the $(\underline{70}, 1^-)$ spectrum for each of the particle types N, Λ, Σ. The spin-orbit splittings in these groups are relatively small, but there is as yet no theory to understand these group separations, except through a parametric mass operator.

This paper represents an attempt to formulate an alternative mechanism with which we hope to face some of the above questions, without giving up the "straight line" behavior. The attempt stems from the consideration that since an SU(6)-symmetric h.o. model has a much higher symmetry content than mere SU(6) symmetry, one does not necessarily have to break the latter in order to produce some structure in the mass spectrum by breaking only the former. [In the usual description of hadron spectra, the h.o. model and

SU(6) symmetry have got so much intermingled that it is useful to remind the reader of their distinct roles.] The mass splittings, if any, produced by an SU(6)-invariant mechanism would have to be strictly *orbital* [*not* spin or SU(3)] in character.

The mechanism we propose in this paper is that the pairwise quark-quark (QQ) interaction be still h.o. in character, but (for some reasons which essentially beg the answer) the odd partial waves be heavily depressed with respect to the even ones. Ideally, therefore, we choose a Serber-type[9] h.o. potential in which the proportions of direct and exchange forces are exactly 50:50, leading to an h.o. interaction only in the *even* partial waves. Such a mechanism, which is an offshoot of (and generalization from) an earlier suggestion of only s-wave forces,[10] has now a built-in mechanism for yielding straight-line trajectories, by the inclusion of an infinite number of partial waves (a feature which the s-wave model did not possess). This mechanism, like the earlier s-wave model, clearly keeps out the $\underline{20}$ states, thus responding to one of the observed features (at least so far) of the baryon spectrum. As for the $\underline{56}$ and $\underline{70}$ states, the mechanism leaves the former unchanged, but yields a drastically altered spectrum, dual in character, for the latter.

The physical features of the dual $\underline{70}$ spectrum, together with its experimental successes, have already been described in a recent note by the author.[11] This paper is mainly devoted to the mathematical formulation of the even-wave h.o. theory and a new classification scheme, especially for $\underline{70}$ states, necessitated by the introduction of a more elaborate set of quantum numbers than is customary in the usual h.o. theory. Unfortunately, a certain degree of approximation appears unavoidable for an explicit derivation of the mass formula, as well as the associated wave functions. We shall seek, in this formulation, to achieve a physical meaning of this approximation (and hence its plausibility) in terms of a certain quantum number ν which, roughly speaking, is a measure of the degree of departure of a given state of $[2,1]$ symmetry from a totally symmetric state. The formulation also requires the introduction of an alternative set of variables (λ_i, y_i), as well as the associated quantum numbers, in preference to the usual internal coordinates (ξ_i, η_i) and corresponding quantum numbers, for a three-body system. (The necessary connections will be defined in the text.) This alternative set of quantum numbers (n_x, l_x, n_y, l_y), rather than the more usual set $(n_\xi, l_\xi, n_\eta, l_\eta)$, will form the basis of this new approach both for the mass spectrum and for the structure of the wave functions.

In Sec. II, we derive two coupled spatial equations for the $\underline{70}$ wave functions (ψ', ψ''), after eliminating the SU(6) variables. Section III is mainly concerned with the eigenvalue problem for $\underline{70}$ states with $L=0$, using certain variables (γ, ρ, λ) introduced by Simonov,[12] to obtain a pair of uncoupled equations in suitable linear combinations (ϕ', ϕ'') of the functions (ψ', ψ''), together with a discussion of their significance. Section IV gives a more general formulation of the problem using the Bargmann-Moshinsky[13] method. This leads to an explicit derivation of the dual mass formula and associated wave functions in terms of the variables χ_i, y_i, and λ, for any general state. The nature of the approximation used in the process is clarified (and hence made plausible). An alternative derivation in the Appendix, using a spherical basis, also leads to a closely similar structure for the dual mass spectrum, thus lending additional support to the latter. Section V is devoted to a discussion of an alternative classification scheme brought about by the introduction of the new quantum numbers (n_x, l_x, n_y, l_y) and the difficulties of their visual indentification in terms of the observed mass spectra, without a simultaneous examination of more sensitive physical parameters such as decay matrix elements. (The latter, which moreover requires a detailed formulation of techniques within this framework, is, however, left out of the scope of this paper.) Section VI discusses the essential features of experimental support for the new mass formula for $\underline{70}$ states, and also summarizes our conclusions.

II. DYNAMICAL EQUATIONS FOR QQQ STATES

The techniques of treatment of the different degrees of freedom in the symmetrical QQQ model[1] are extensively available in the literature[3, 14-16] to which we refer the interested reader for details. In the notation of Ref. 16, and SU(6)-invariant Q-Q potential of the h.o. type in *even* partial waves is given by

$$V^{(+)}(ij)=[P^+_\sigma(ij)P^+_u(ij)+P^-_\sigma(ij)P^-_u(ij)] \times P^+_x(ij)(\tfrac{1}{2}m\omega_0{}^2 r_{ij}{}^2+V_0)\,, \tag{2.1}$$

where the even-odd projection operators $P^\pm(ij)$ in the different variables are expressible in terms of the corresponding permutation operators (ij) as

$$P^\pm_{\sigma,u,x}=\tfrac{1}{2}[1\pm(ij)_{\sigma,u,x}]\,, \tag{2.2}$$

$$(ij)_\sigma=\tfrac{1}{2}[1+\vec{\sigma}_i,\vec{\sigma}_j]\,, \tag{2.3}$$

$$(ij)_u=\tfrac{1}{2}\left(\tfrac{2}{3}+\sum_{\alpha=1}^{8}\lambda_{i\alpha}\lambda_{j\alpha}\right). \tag{2.4}$$

$(ij)_x$ is the space-exchange operator and $\lambda_{i\alpha}$ are the Gell-Mann matrices. V_0 is a constant addition to the h.o. potential, which would now affect $\underline{56}$ and $\underline{70}$ states differently, in contrast to the full h.o. model where such an inclusion would produce a constant shift in all the levels. The $\underline{20}$ states cannot be supported by the operator P_x^+ in (2.1); rather one requires the *complementary part* $V^{(-)}$ which together with (2.1) would lead to the full, SU(6)-invariant h.o. potential with Bose statistics:

$$V^{(-)}(ij) = [P_\sigma^+(ij)P_u^-(ij) + P_\sigma^-(ij)P_u^+(ij)] \times P_x^-(ij)(\tfrac{1}{2}m\omega_0{}^2 r_{ij}{}^2 + V_0) . \tag{2.5}$$

Since the potentials are SU(6) invariant, the equations for the spatial functions, ψ^S for $\underline{56}$ and (ψ', ψ'') for $\underline{70}$, are most quickly recovered by considering their $\underline{10}_q$ and $\underline{8}_q$ multiplets whose complete wave functions are given by[16]

$$|\underline{10}_q\rangle = \psi^S \chi^S \phi^S , \tag{2.6}$$

$$|\underline{8}_q\rangle = 2^{-1/2}(\psi'\phi' + \psi''\phi'')\chi^S , \tag{2.7}$$

where χ^S and ϕ are the spin and SU(3) wave functions of appropriate permutation symmetries described in Ref. 16. Thus, after some manipulations with normalization, the use of (2.6) for $\underline{56}$ states leads to the following eigenvalue equation:

$$H\alpha^{-1}\psi^S \equiv [-\tfrac{1}{2}(\nabla_\xi{}^2 + \nabla_\eta{}^2) + \tfrac{1}{2}(\xi^2 + \eta^2)]\psi^S = (M^2 + \beta)\alpha^{-1}\psi^S , \tag{2.8}$$

where

$$\alpha = \sqrt{3}\, m\omega_0 , \quad \beta = -3mV_0 , \tag{2.9}$$

and all lengths are expressed in units of $\alpha^{-1/2}$. $\vec{\xi}$ and $\vec{\eta}$ are normalized internal coordinates of the QQQ system defined by

$$(2/\alpha)^{1/2}\vec{\xi} = \vec{r}_3 - \vec{r}_2 , \qquad (6/\alpha)^{1/2}\vec{\eta} = -2\vec{r}_1 + \vec{r}_2 + \vec{r}_3 . \tag{2.10}$$

In the above normalization, the quantity M^2 $(= mE)$ which substitutes for the energy (E) may be regarded directly as the *square of the relativistic mass*, in the sense of Ref. 2 or FKR. Further, the effect (β) of the constant part (V_0) of the body potential has been displayed along with the M^2 term in Eq. (2.8).

Equation (2.8) is obviously identical to the full h.o. prediction, since the part (2.5) of the QQ potential does not contribute to the $\underline{56}$ states. The $\underline{56}$ spectrum solution is therefore described by the formula

$$M_{56}{}^2 \equiv mE = \alpha(2n_\xi + l_\xi + 2n_\eta + l_\eta + 3) - \beta \tag{2.11}$$

in terms of two sets of h.o. quantum[2,3] numbers (n_ξ, l_ξ) and (n_η, l_η).

For the $\underline{70}$ states, the use of the wave function (2.7) yields in the same notation and normalization the coupled differential equations

$$K_0\psi' \equiv [-\nabla_\xi{}^2 - \nabla_\eta{}^2 - (2M^2+\beta)\alpha^{-1} + \tfrac{1}{2}(\xi^2+\eta^2)]\psi' = \tfrac{1}{4}[2\vec{\xi}\cdot\vec{\eta}\psi'' + (\xi^2 - \eta^2)\psi'] , \tag{2.12}$$

$$K_0\psi'' \equiv [-\nabla_\xi{}^2 - \nabla_\eta{}^2 - (2M^2+\beta)\alpha^{-1} + \tfrac{1}{2}(\xi^2+\eta^2)]\psi'' = \tfrac{1}{4}[2\vec{\xi}\cdot\vec{\eta}\psi' - (\xi^2 - \eta^2)\psi''] , \tag{2.13}$$

whose structure is vastly different from that of the full h.o. model. The latter, which can be recovered only with the addition of (2.5) to (2.1), would lead exactly to Eq. (2.8) *separately* for ψ' and ψ'', as also for the $\underline{20}$ wave function ψ^a which comes entirely from (2.5). In the absence of the portion (2.5), the $\underline{70}$ dynamics is governed by the new equations (2.12)–(2.13), while $\underline{20}$ is totally kept out.

III. THE EIGENVALUE PROBLEM FOR $\underline{70}$ STATES

The solution of the coupled Eqs. (2.12)–(2.13) for ψ', ψ'' cannot, unfortunately, be found exactly, but it is possible to recast them in a form suggestive enough for suitable approximations with clear physical interpretation. The method requires some familiarity with the language of three-particle coordinates of the type developed by Simonov *et al.*,[12,17] to which we refer the interested reader for details. To keep the discussion reasonably self-contained, we shall summarize the roles of some important variables which enter the calculations in an essential manner.

Of the six coordinates ξ_i and η_i, the three scalar coordinates may be chosen as γ, λ, ρ:

$$\xi^2 - \eta^2 = \gamma\cos\lambda , \quad 2\xi_i\eta_i = \gamma\sin\lambda , \tag{3.1}$$

$$\rho = \xi^2 + \eta^2 . \tag{3.2}$$

(These variables are somewhat different from those used by Simonov[12] whose ρ^2 is our ρ and whose A is our $\gamma\rho^{-1}$.) The significance of the variable λ, defined by

$$\tan\lambda = 2\xi_i\eta_i(\xi^2 - \eta^2)^{-1} , \tag{3.3}$$

stems from the fact that it is the main carrier of the $[2,1]$ symmetry among scalar wave functions, since the other two variables ρ and γ are totally symmetric.

The dependence of the three-body wave function on λ which preserves the necessary $[2,1]$ symmetry is given by $e^{\pm i\nu\lambda}$ (or their sine-cosine forms), where ν is the eigenvalue of the operator $\hat{\nu}$ defined by

$$2\hat{\nu} \equiv -i\left(\xi_i \frac{\partial}{\partial \eta_i} - \eta_i \frac{\partial}{\partial \xi_i}\right) = -2i \frac{\partial}{\partial \lambda} . \tag{3.4}$$

The operator $\hat{\nu}$ commutes with the free Hamiltonian, or even with the one involving the *full* h.o., and in these cases, its eigenvalues are integral, as has been shown by Simonov.[12] There will, however, be found some additional problems with our (2×2 matrix) Hamiltonian for ψ' and ψ'' since it has an additional λ dependence.

The other three coordinates (angles) which specify the orientation of the QQQ system are more relevant for the consideration of higher L (>0) states, but remain implicit for purely scalar wave functions ($L=0$) which we consider first in this section. They appear explicitly for higher L states which can be treated in either of two alternative ways: a tensorial formalism of the Bargmann-Moshinsky type[12] (as a generalization of the Simonov differential techniques[17] for the free Hamiltonian) or a direct angular momentum analysis in terms of the D^L_{mk} functions involving the three Euler-angle variables of the QQQ system. We shall describe both these methods (the latter in the Appendix) for the solution of the 70 spectrum. The introduction of the tensorial method is facilitated by a prior consideration of the $L=1$ case which we take up in this section immediately following the $L=0$ case.

(a) Reduction for $L=0$ states. The crucial step in the reduction of Eqs. (2.12)–(2.13) lies in finding suitable linear combinations of (ψ', ψ'') which will decouple them. Now Eqs. (3.1) indicate that the right-hand sides as well as the nondifferential portions of the left-hand sides of (2.12)–(2.13) will separate out with the combinations

$$\begin{aligned} \phi' &= \psi' \cos\tfrac{1}{2}\lambda + \psi'' \sin\tfrac{1}{2}\lambda , \\ \phi'' &= \psi' \sin\tfrac{1}{2}\lambda - \psi'' \cos\tfrac{1}{2}\lambda , \end{aligned} \tag{3.5}$$

indicating, as it were, that $\cos\frac{1}{2}\lambda$ and $\sin\frac{1}{2}\lambda$ are analogous to the $[2,1]_{S,a}$ functions ψ'', ψ', respectively, but such an interpretation clearly requires going beyond integral eigenvalues of the quantum number ν.[12] The combinations (3.5) which can be achieved by multiplying the two equations (2.12) and (2.13) by the indicated half-harmonics must also pass the test of the differential operators whose λ dependence is expressed (for $L=0$) by

$$\nabla_\xi{}^2 + \nabla_\eta{}^2 = \frac{4\rho}{\gamma^2} \frac{\partial^2}{\partial \lambda^2} + (\text{non-}\lambda \text{ terms}) . \tag{3.6}$$

Now the condition that the result of commuting the half-harmonics through the differential operator $\partial_\lambda{}^2$ should be to reproduce the ϕ' and ϕ'' functions is expressed by the requirement that the quantities

$$\begin{aligned} &\cos\tfrac{1}{2}\lambda\, \partial_\lambda{}^2\psi' + \sin\tfrac{1}{2}\lambda\, \partial_\lambda{}^2\psi'' = \partial_\lambda{}^2\phi' - \tfrac{1}{4}\phi' + \partial_\lambda \phi'' , \\ &\sin\tfrac{1}{2}\lambda\, \partial_\lambda{}^2\psi' - \cos\tfrac{1}{2}\lambda\, \partial_\lambda{}^2\psi'' = \partial_\lambda{}^2\phi'' - \tfrac{1}{4}\phi'' - \partial_\lambda \phi' \end{aligned} \tag{3.7}$$

be proportional to ϕ' and ϕ'', respectively, preferably with the same constant of proportionality. This leads to the following λ dependence of ϕ' and ϕ'':

$$\phi' \propto \sin(\nu+\tfrac{1}{2})\lambda , \quad \phi'' \propto \cos(\nu+\tfrac{1}{2})\lambda , \tag{3.8}$$

with the right-hand sides of (3.7) replaced effectively by $-(\nu+1)^2\phi'$ and $-(\nu+1)^2\phi''$, respectively. Thus, the quantity $-(\nu+1)^2$ may be regarded as the eigenvalue of the operator $\partial_\lambda{}^2$ *as modified by* the extra terms on the right-hand side of (3.7). As to the nature of the quantum number ν, we take the view that since the actual wave functions are ψ' and ψ'', rather than ϕ' and ϕ'', the former should have an integral harmonic dependence on λ. In that case, the quantity ν should be integral (≥ 0) as in the usual description.[12] (If, on the other hand, we were to insist on ϕ' and ϕ'' having integral harmonic dependence on λ, the values of ν would have to be half-integral).

The (ψ', ψ'') versus (ϕ', ϕ'') controversy is, of course, a new feature of our description of 70 states, arising out of the *even-wave* h.o. potential.

The decoupled equations with the λ dependence indicated by (3.8) are now

$$K^+\phi' = 0 , \quad K^-\phi'' = 0 , \tag{3.9}$$

where $\gamma^2 = \rho^2\omega$ and the operators $K^\pm$ are given by

$$K^\pm = \mp\tfrac{1}{4}\gamma + \tfrac{1}{2}\rho - (2M^2+\beta)\alpha^{-1} - \nabla_\xi{}^2 - \nabla_\eta{}^2 \tag{3.10}$$

and for $L=0$

$$\nabla_\xi{}^2 + \nabla_\eta{}^2 \Rightarrow 4\rho \frac{\partial^2}{\partial \rho^2} + 12 \frac{\partial}{\partial \rho} + \frac{16}{\rho}\left[(1-2\omega)\frac{\partial}{\partial \omega} + \omega(1-\omega)\frac{\partial^2}{\partial \omega^2} - \frac{4}{\rho\omega}(\nu+1)^2\right] . \tag{3.11}$$

But for the $\frac{1}{4}\gamma$ terms in K^+, Eqs. (3.9) would have been separable in ρ and γ, the latter being involved only in the Jacobi polynomials $P_n^{(\nu+1,0)}(1-2\omega)$, according to the structure of (3.11).[12, 18] As they stand, these equations are not exactly soluble, yet they show a most interesting feature: the existence of two distinct branches of the spectrum, corresponding to the two modes ϕ' and ϕ''. These branches would be coincident for the full h.o. model, so as to conform to the standard (degen-

erate) solution given by Eq. (2.11). This partial lifting of degeneracy over the h.o. model, which is still achieved within SU(6) symmetry, is a result of h.o. symmetry breaking due to the suppression of the odd partial waves in the Q-$\bar{Q}$ potential. However, since this result is likely to raise misgivings about an apparent doubling of the number of states, the following immediate clarification is relevant. The complete wave function for a $\underline{70}$ state, e.g., (2.7), involves *both* ψ' and ψ'', and *hence* both ϕ' and ϕ'' as well. Reexpressing Eq. (3.5) as

$$\psi' = \phi' \cos\tfrac{1}{2}\lambda + \phi'' \sin\tfrac{1}{2}\lambda\,, \qquad \psi'' = \phi' \sin\tfrac{1}{2}\lambda - \phi'' \cos\tfrac{1}{2}\lambda\,, \tag{3.12}$$

it follows that ψ' involves the mass spectra predicted by ϕ' and ϕ'' with the probabilities $\cos^2\frac{1}{2}\lambda$ and $\sin^2\frac{1}{2}\lambda$, respectively, while ψ'' involves the same spectra with the *opposite* probabilities. The complete effect of such mixing of masses on physical observables requiring the spatial wave function will depend on the full λ structure of the latter, including the role of λ mixing given by Eq. (3.12). A fuller discussion of these problems will be given in Sec. V in connection with the detailed classification of state in terms of a new set of quantum numbers derived in Sec. IV.

(b) Cartesian treatment of $L=1$ states. To construct $L=1$ states of the correct $[2,1]$ symmetry, in the Cartesian basis introduce the vector functions ψ_i', ψ_i'' and the corresponding vector combinations φ_i', φ_i'' as

$$\phi_i' = \cos\tfrac{1}{2}\lambda\,\psi_i' + \sin\tfrac{1}{2}\lambda\,\psi_i'', \qquad \phi_i'' = \sin\tfrac{1}{2}\lambda\,\psi_i' - \cos\tfrac{1}{2}\lambda\,\psi_i''\,. \tag{3.13}$$

Since ψ_i' and ψ_i'' satisfy Eqs. (2.12)–(2.13), multiplication of the latter, *from the left*, by the indicated half-harmonics in (3.12) and (3.13) leads without approximation to the equations

$$\cos\tfrac{1}{2}\lambda K^+\psi_i' + \sin\tfrac{1}{2}\lambda K^+\psi_i'' = 0\,, \tag{3.14}$$

$$\sin\tfrac{1}{2}\lambda K^-\psi_i' - \cos\tfrac{1}{2}\lambda K^-\psi_i'' = 0\,, \tag{3.15}$$

where the operators $K^\pm$ are given by Eq. (3.10) but without the validity of Eq. (3.11) any more. In particular, the half-harmonics can *no longer* be commuted through $K^\pm$ to permit an exact inference of the equations

$$K^+\phi_i' = K^-\phi_i'' = 0\,. \tag{3.16}$$

The latter are now only *approximately* true, unlike their scalar counterparts, Eq. (3.9). This approximation lies in the neglect of certain portions of $K^\pm$ which depend on the Euler angle variables and hence would now come into play because of the nonzero angular momenta of ψ_i', ψ_i''. The same difficulty appears for functions of higher L values.

In the next section we outline a more systematic method (following Bargmann and Moshinsky) by which it is not only possible to understand somewhat precisely the meaning of the approximation involved in commuting $\sin\frac{1}{2}\lambda$, etc., through $K^\pm$ in Eqs. (3.14)–(3.15), but also to extend the corresponding approximation to any L value exhibited by a tensor of arbitrary rank.

IV. THE BARGMANN-MOSHINSKY METHOD

We present here an adaptation of the Bargmann-Moshinsky method[12] for our even-wave h.o. formalism, which turns out to be rather convenient for the construction of wave functions. Though the method is somewhat approximate, it is possible to give a rather precise meaning to the approximation through the neglect of couplings to certain types of states.

Define the operators

$$\tfrac{1}{2}\xi_i \pm \partial_{\xi_i} = a_{\xi i},\ a_{\xi i}^\dagger\,, \tag{4.1}$$

$$\tfrac{1}{2}\eta_i \pm \partial_{\eta_i} = a_{\eta i},\ a_{\eta i}^\dagger\,, \tag{4.2}$$

which obey the normalized commutation relations

$$[a_{\xi i}, a_{\xi j}^\dagger] = [a_{\eta i}, a_{\eta j}^\dagger] = \delta_{ij}\,, \tag{4.3}$$

while all other pairs commute. While these forms differ from those of the full h.o. theory,[12] their normalized nature warrants the deduction of integral eigenvalues for the operators

$$N_{\xi i} = a_{\xi i}^\dagger a_{\xi i}\,, \quad N_{\eta i} = a_{\eta i}^\dagger a_{\eta i}\,. \tag{4.4}$$

These operators diagonalize a good part of our (2×2) $(\text{mass})^2$ operator K defined through Eqs. (2.12) and (2.13) in the (ψ', ψ'') basis as

$$K_\psi \begin{pmatrix}\psi'\\ \psi''\end{pmatrix} = \begin{pmatrix} K_+ & -\frac{1}{2}\xi\cdot\eta \\ -\frac{1}{2}\xi\cdot\eta & K_-\end{pmatrix}\begin{pmatrix}\psi'\\ \psi''\end{pmatrix}\,, \tag{4.5}$$

where

$$K_\pm = -\nabla_\xi{}^2 - \nabla_\eta{}^2 + \tfrac{1}{2}\rho \mp (\xi^2 - \eta^2) - (2M^2+\beta)\alpha^{-1}\,, \tag{4.6}$$

and in the (ϕ', ϕ'') basis as

$$K_\phi \begin{pmatrix}\phi'\\ \phi''\end{pmatrix} = \begin{pmatrix} K^+ & 0 \\ 0 & K^-\end{pmatrix}\begin{pmatrix}\phi'\\ \phi''\end{pmatrix}\,, \tag{4.7}$$

where $K^\pm$ are given as in Eq. (3.10).

The last form, while rigorously true of *scalar* functions (ϕ', ϕ'') with the λ dependence, Eq. (3.8), is only approximately true of higher angular momenta states, as mentioned after Eqs. (3.14) and (3.15). In order to see the effect of this ap-

proximation more closely, and also to give a more transparent look to the resulting structure for $K^{\pm}$, we define, analogously to (3.5), the new variables:

$$x_i = \xi_i \cos\tfrac{1}{2}\lambda + \eta_i \sin\tfrac{1}{2}\lambda,$$
$$y_i = \xi_i \sin\tfrac{1}{2}\lambda - \eta_i \cos\tfrac{1}{2}\lambda. \tag{4.8}$$

Using Eqs. (3.1)–(3.3), we find

$$\rho = x^2 + y^2,$$
$$\gamma = x^2 - y^2, \tag{4.9}$$
$$x_i\, y_i = 0.$$

The last equation shows that the vectors x_i and y_i are *not* independent. Nevertheless, we can define the operators a_{xi} and a_{yi}, analogously to (4.7), as

$$a_{xi} = \cos\tfrac{1}{2}\lambda\, a_{\xi i} + \sin\tfrac{1}{2}\lambda\, a_{\eta i}, \tag{4.10}$$

$$a_{yi} = \sin\tfrac{1}{2}\lambda\, a_{\xi i} - \cos\tfrac{1}{2}\lambda\, a_{\eta i}, \tag{4.11}$$

together with their Hermitian conjugated relations, and deduce the commutation relations among these from (3.3) and (4.1)–(4.3). Some sample relations are

$$[a_{xi}, a_{xj}^{\dagger}] = \delta_{ij} + 2\gamma^{-1} y_j\, a_{yi} + 2a_{yj}^{\dagger}\gamma^{-1} y_i + 4\gamma^{-2} x_i x_j, \tag{4.12}$$

$$[a_{xi}, a_{xj}] = 2\gamma^{-1}(y_i\, a_{yj} - y_j\, a_{yi}), \tag{4.13}$$

$$[a_{yi}, a_{yj}^{\dagger}] = \delta_{ij} - 2\gamma^{-1} x_j a_{xi} - 2a_{xj}^{\dagger}\gamma^{-1} x_i + 4\gamma^{-2} y_i y_j. \tag{4.14}$$

These equations show that a_{xi}, a_{yi} have almost the significance of the operators $a_{\xi i}$, $a_{\eta i}$, except for certain correction terms. The latter arise because of the (ξ, η) dependence of the λ variable which is also responsible for the lack of independence ($x_i\, y_i = 0$) of the x_i and y_i variables. However, these correction terms have a simple significance: They involve *two* units of the quantity ν defined by Eq. (3.4), thus implying coupling of a given state to those differing from it by $\Delta\nu = 2$. [A marginal exception to this rule arising from Eqs. (4.12) and (4.14) when summed over i and j is discussed at the end of this section.]

Unfortunately, we have not been able to solve in a *closed* form (through suitable transformations or otherwise), the algebra implied by Eqs. (4.12)–(4.14), and a host (~ 20) of others. We have therefore chosen the next best course for a solution of this modified h.o. problem, viz., to *ignore the coupling to states differing from a chosen one by* $\Delta\nu = 2$. This immediately leads to the simplified commutation relations

$$[a_{xi}, a_{xj}^{\dagger}] = \delta_{ij} = [a_{yi}, a_{yj}^{\dagger}], \tag{4.15}$$

$$[a_{xi}, x_j] = -[a_{xi}^{\dagger}, x_j] = \delta_{ij}, \tag{4.16}$$

$$[a_{yi}, y_j] = -[a_{yi}^{\dagger}, y_j] = \delta_{ij}, \tag{4.17}$$

with all other pairs commuting.

Indeed, these relations amount to regarding x_i and y_i as two sets of *effectively independent* variables, ignoring the constraining relation $x_i\, y_i = 0$. The same approximation also justifies commuting the half-harmonics $\cos\frac{1}{2}\lambda$ and $\sin\frac{1}{2}\lambda$ through to the right of the operators K^{+} in Eqs. (3.14)–(3.15), and hence also the reduction of the 2×2 K_{ψ} matrix, Eq. (4.5) to the K_{ϕ} matrix, Eq. (4.7) for any angular momentum state (not merely $L = 0$). The elements of the K_{ϕ} matrix, Eq. (4.7), now take the following forms in terms of the x and y variables (summation implied):

$$K^{+} + (2M^2 + \beta)\alpha^{-1} = a_{xi}^{\dagger} a_{xi} + a_{yi}^{\dagger} a_{yi} + 3 + \tfrac{1}{2}y^2, \tag{4.18}$$

$$K^{-} + (2M^2 + \beta)\alpha^{-1} = a_{xi}^{\dagger} a_{xi} + a_{yi}^{\dagger} a_{yi} + 3 + \tfrac{1}{2}x^2, \tag{4.19}$$

where use has been made of Eq. (4.9). These operators are still not diagonal in Hilbert space, and we need two "Bogoliubov" transformations, noting the relations

$$x_i = a_{xi} + a_{xi}^{\dagger}, \quad y_i = a_{yi} + a_{yi}^{\dagger}. \tag{4.20}$$

These work out as

$$(1 - \epsilon^2)^{1/2} b_{xi} = a_{xi} - \epsilon a_{xi}^{\dagger},$$
$$(1 - \epsilon^2)^{1/2} b_{yi} = a_{yi} - \epsilon a_{yi}^{\dagger}, \tag{4.21}$$

where

$$\epsilon = -2 + \sqrt{3} \tag{4.22}$$

and

$$[b_{xi}, b_{xj}^{\dagger}] = [b_{yi}, b_{yj}^{\dagger}] = \delta_{ij}. \tag{4.23}$$

These lead to the diagonalized operators (summation implied)

$$K^{+} + (2M^2 + \beta)\alpha^{-1} = (a_{xi}^{\dagger} a_{xi} + \tfrac{3}{2}) + \sqrt{3}\,(b_{yi}^{\dagger} b_{yi} + \tfrac{3}{2}), \tag{4.24}$$

$$K^{-} + (2M^2 + \beta)\alpha^{-1} = \sqrt{3}\,(b_{xi}^{\dagger} b_{xi} + \tfrac{3}{2}) + (a_{yi}^{\dagger} a_{yi} + \tfrac{3}{2}). \tag{4.25}$$

The latter in turn give rise to a dual spectrum (corresponding to the null eigenvalues of $K^{\pm}$)

$$(2M^2 + \beta)\alpha^{-1} = \sum_{i=1}^{3} (n_{xi} + \sqrt{3}\, m_{yi}) + \tfrac{3}{2}(1 + \sqrt{3}), \tag{4.26}$$

$$(2M^2 + \beta)\alpha^{-1} = \sum_{i=1}^{3} (\sqrt{3}\, m_{xi} + n_{yi}) + \tfrac{3}{2}(1 + \sqrt{3}), \tag{4.27}$$

where n_{xi}, m_{xi}, etc., are integers (≥ 0).

The associated wave functions which can be con-

structed through the action of suitable creation operators on the ground states, ϕ_0' and ϕ_0'' defined by

$$a_{xi}\phi_0' = b_{yi}\phi_0'' = 0\,, \tag{4.28}$$

$$b_{xi}\phi_0'' = a_{yi}\phi_0' = 0\,, \tag{4.29}$$

are as follows:

$$\Phi' = (a_{xi}^\dagger a_{xj}^\dagger \cdots)(b_{yp}^\dagger b_{yq}^\dagger \cdots)\phi_0'\,, \tag{4.30}$$

$$\Phi'' = (b_{xi}^\dagger b_{xj}^\dagger \cdots)(a_{yp}^\dagger a_{yq}^\dagger \cdots)\phi_0''\,, \tag{4.31}$$

where the *sum* of the exponents in each bracket adds up to the appropriate group of excitation quantum numbers (n, m) specified in Eqs. (4.26)–(4.27).

To get a more concrete picture of the above structures in x and y coordinates, we note that in the "decoupling approximation," the commutation relations (4.16) and (4.17) imply the *effective* representations

$$a_{xi},\ a_{xi}^\dagger = \tfrac{1}{2}x_i \pm \frac{\partial}{\partial x_i}\,,$$
$$a_{yi},\ a_{yi}^\dagger = \tfrac{1}{2}y_i \pm \frac{\partial}{\partial y_i}\,, \tag{4.32}$$

as if x and y were completely independent variables. In the differential operator representation, moreover,

$$K^+ = -\nabla_x{}^2 - \nabla_y{}^2 + \tfrac{1}{4}x^2 + \tfrac{3}{4}y^2 - (2M^2+\beta)\alpha^{-1}\,,$$
$$K^- = -\nabla_x{}^2 - \nabla_y{}^2 + \tfrac{3}{4}x^2 + \tfrac{1}{4}y^2 - (2M^2+\beta)\alpha^{-1}\,. \tag{4.33}$$

Equations (4.31)–(4.33) lead to the following structures of the ground-state wave function

$$\phi_0' = N' \exp(-\tfrac{1}{4}x^2 - \tfrac{1}{4}\sqrt{3}\,y^2)\,, \tag{4.34}$$

$$\phi_0'' = N'' \exp(-\tfrac{1}{4}\sqrt{3}\,x^2 - \tfrac{1}{4}y^2)\,, \tag{4.35}$$

while the excited states can be expressed alternatively in terms of standard Hermite functions in x_i and y_i. As to the spectra (4.26) and (4.27), these can alternatively be represented by radial (n) and orbital (l) quantum numbers, so that the squared masses are finally given by

$$M_{70}'^2 = \alpha(n_x + \tfrac{1}{2}l_x + \tfrac{3}{4}) + \alpha\sqrt{3}\,(n_y + \tfrac{1}{2}l_y + \tfrac{3}{4}) - \tfrac{1}{2}\beta\,, \tag{4.36}$$

$$M_{70}''^2 = \alpha\sqrt{3}\,(n_x + \tfrac{1}{2}l_x + \tfrac{3}{4}) + \alpha(n_y + \tfrac{1}{2}l_y + \tfrac{3}{4}) - \tfrac{1}{2}\beta \tag{4.37}$$

in terms of the two independent sets (n_x, l_x) and (n_y, l_y) according to the identifications

$$\sum_{i=1}^{3} n_{xi} = 2n_x + l_x\,, \text{ etc.} \tag{4.38}$$

We add a word of caution about ignoring the constraining relation $x_i\, y_i = 0$ which would apparently produce one too many independent variables λ. In the "decoupling" approximation, however, λ is largely passive so that the above procedure does not constitute any real danger as long as this variable does not appear explicitly in the wave function, e.g., (4.30)–(4.35), since the a and b operators can be expressed entirely in terms of the x and y variables. (There are some marginal difficulties in calculations involving transition matrix elements but these can be surmounted.)

To obtain some further insight into the general form of the spectra (4.36) and (4.37), we present in the Appendix an alternative (spherical) approach to the problem in which the independent variables chosen are the three internal coordinates (x, y, λ) and the three Euler angles (θ, x, ϕ) specifying the orientation in space of the "effective rigid body" defined by the vectors x_i and y_i constrained by $x_i\, y_i = 0$. In essential details the spectral structure is identical to (4.36) and (4.37), except for an alternative classification for the quantum states involved (consequent on rigorously incorporating the constraint $x_i\, y_i = 0$).

A brief comparison between the two approaches is given in the next section where arguments are advanced in favor of the Cartesian treatment. We now close this section with a remark on the effect of errors in Eqs. (4.12) and (4.14) due to noninclusion of $\Delta\nu = 0$ effects from the neglected terms. This error, which can be most easily estimated through an inspection of Eqs. (4.18)–(4.19), gives rise to a correction to the K^+ operators which is simply half the sum of the neglected terms in (4.12) and (4.14), contracted with $i = j$, and reduces to

$$\gamma^{-1}y_i(a_{yi}^\dagger + a_{yi}) - \gamma^{-1}x_i(a_{xi}^\dagger + a_{xi}) = -1\,. \tag{4.39}$$

The correction is thus merely a constant, and does not change the relative shape of the spectra.

For completeness we also record the corresponding spectrum for $\underline{56}$ states in the (approximate) x, y representation:

$$M_{56}{}^2 = \alpha(2n_x + l_x + 2n_y + l_y + 3) - \beta\,, \tag{4.40}$$

with a similar correction (-2), analogously to (4.39).

V. INTERPRETATION OF THE SPECTRA

We have thus obtained two alternative descriptions of the spectra, each with a certain degree of approximation, yet their general features bear strong resemblance to each other. We hope this lends some credibility to the techniques used, but more importantly points to a certain degree of stability of the spectrum structure against our approximate treatments. This stability arises from the dominant role of the *potential energy terms* which remain unaffected by the approximations

used and are directly responsible for the slopes. In particular, the two independent slopes $\frac{1}{2}\alpha$ and $\frac{1}{2}\sqrt{3}\,\alpha$ are a direct manifestation of the potential energy in the two different modes (ϕ', ϕ'') of solution and are in no way affected by our approximations. The latter, which affect certain kinetic energy terms of less sensitive ranking (except perhaps for very short distances), give rise to much lower order effects. For example, the decoupling approximation in the treatment of Sec. IV, which resulted in x_i and y_i being regarded as effectively two sets of independent vectors, gives somewhat higher "zero-point energies" (3, $\frac{3}{4}$, etc.) than their lack of independence would warrant. On the other hand, the apparently smaller zero-point energies in the alternative treatment given in Appendix A are partly a result of "correct" counting of variables and partly of the absorption of the former in the symbol $a_{Lk\nu}$ incorporating the angular momentum effect in the energy spectrum.

Both methods require *two* radial quantum numbers (n_x, n_y) and two angular momenta for a complete description of the spectrum. The latter have been expressed in two sets of alternatives: (l_x, l_y) in the "Cartesian" treatment of Sec. IV, and (L, k) in the "spherical" treatment given in Appendix A. In the 56 states, the dependence of the spectrum is on the total L only, as in the full h.o. model, but the 70 spectra now see a more detailed dependence on both l_x *and* l_y (or equivalently on L *and* k). The same remarks are true of the radial quantum numbers n_x and n_y. The quantum number ν, which has the significance of a difference between the (n_ξ, l_ξ), (n_η, l_η) sets,[13]

$$2\nu = 2n_\xi + l_\xi - 2n_\eta - l_\eta\,, \tag{5.1}$$

or in the decoupling approximation, of a corresponding difference between the "x" and "y" sets, does not appear in the full h.o. calculations or, for that matter, the 56 spectrum in the even-wave h.o. treatment, but appears nontrivially in the 70 spectrum. In the Cartesian description, the two combinations $2n_x + l_x$ and $2n_y + l_y$ appear with different coefficients, thus bringing out the ν dependence of the spectrum, while in the spherical treatment, this quantum number appears through the coefficient $a_{Lk\nu}$, which in a sense plays the role of a "sum of L and 2ν."

The extra structure exhibited in the 70 spectrum as a result of lowering the full h.o. symmetry by keeping only the even partial waves requires a more detailed classification of states than has been the general practice in the full h.o. model (because of the mass degeneracy associated with it). This can, in principle, be done in terms of either of the two sets

$$(n_x, n_y, l_x, l_y) \text{ and } (n_x, n_y, L, k)\,. \tag{5.2}$$

Since both have entailed some approximations in their respective derivations, and since both exhibit strong similarity of structures, it is not useful to continue giving a comparative discussion of both, instead of making a conscious choice at this stage between them for the purpose of further exposition. In making the "Cartesian" choice we have been guided by the following considerations.

The Cartesian form is explicitly *more symmetrical* in the quantum numbers associated with the x and y variables than its spherical counterpart. All the quantum numbers appear with explicit linearity in the Cartesian form, in contrast to the spherical form. The main drawback of the Cartesian treatment, viz., the assumption of effective independence of x_i and y_i, ignoring the constraint $x_i\, y_i = 0$, is no doubt avoided in the spherical version, but adequate care has been taken to avoid duplication of variables by omitting any further reference to the λ variable after incorporating its effect through the connecting relations (3.5), (3.13), etc. Finally, a great advantage of the Cartesian treatment is the immediate facility it provides for a straightforward relativistic extension,[3] while the spherical treatment offers no such possibility in any simple way. Moreover, since a relativistic extension increases ξ and η vectors (hence also x and y) by one component each, the effect of the constraining relation $x_i y_i - x_\mu y_\mu = 0$ on the independence of the variables x_μ and y_μ should be somewhat diluted with respect to the 3-dimensional version, thus reducing the relative error involved in its neglect.

Having thus chosen the Cartesian version, Sec. IV, for further presentation, we devote the rest of this section to a discussion of a few important features of the 70 spectrum as well as the new classifications. The two slopes, $\frac{1}{2}\alpha$ and $\frac{1}{2}\sqrt{3}\,\alpha$ which are associated with corresponding strengths of the "x" and "y" parts of the potential in the two modes ϕ' and ϕ'', appear in complementary forms in the two spectra. In contrast, the 56 spectrum involves these strengths symmetrically and has a unique slope α.

The "half-spacing" of the quantum numbers for 70 masses compared to their 56 counterparts is directly traceable to the very nature of the potential used. Thus, by cutting out the odd partial waves from the full h.o. potential, one is not losing any effective strength for the 56 states since the former are not operative in 56 at all. On the other hand, the 20 states which depend entirely on the odd partial waves are completely eliminated in the process. The 70 states, which receive half their

strength from the even waves and the remainder from the odd ones, are only "half affected," so to say. It is this half-strength of the h.o. that is manifest in the appearance of corresponding slopes in the 70 masses, compared to 56.

From the observational point of view, the most interesting feature is perhaps the doubling of the mass spectra for 70 states in general, the only exceptions being those which correspond to $n_x = n_y$ *and* $l_x = l_y$, and hence have degenerate masses for both the ϕ' and ϕ'' modes. In particular, the *lowest* state, $n_x = n_y = l_x = l_y = 0$, has a unique mass and corresponds to a $(\underline{70}, 0^+)$ supermultiplet. The prediction of such a low-lying 70, even below the $(\underline{70}, 1^-)$ supermultiplet, is a unique feature of the even-wave h.o. model, and has no analog in the usual h.o. model which predicts a $(\underline{70}, 0^+)$ at a much higher mass value.[3]

The classification of levels in this new description must be made in terms of the set of numbers (n_x, l_x, n_y, l_y), prefixed by 56 or 70 as the case may be. Since the (degenerate) mass prediction for 56 states is identical to the usual h.o. the correspondence to the usual notation $(\underline{56}, L^P)$ is easily established. However, for 70 states in general there is enough structure in the mass formula to necessitate a more detailed description (in terms of the set of four quantum numbers) to avoid possible confusion. Thus, the two first excited states, corresponding to $L^P = 1^-$, are now denoted by (0100) and (0001), in the order $(n_x l_x,\ n_y, l_y)$. Each of these states has two distinct masses associated with it.

The next excited states have several competing possibilities, which in the above notation are given by the sequence

$$(0200), (0002), (0101);\ (1000), (0010). \qquad (5.3)$$

Of the three orbital excitations $(L^P = 2^+)$ each of the first two, (0200) and (0002), predicts a pair of distinct masses (coincident among themselves) while the third (0101) corresponds to a unique mass lying between (0200) and (0002). Since the radial quantum number (n) appears with twice the weight of the corresponding orbital quantum number (l), it is clear that the states (1000) and (0010) have exactly the same masses as (0200) and (0002), respectively, while there is no radial counterpart to (0101). The corresponding $(\underline{56}, 2^+)$ states have exactly similar classifications but all their masses are degenerate (as in the usual h.o. theory) at a level which is distinct from the three 70 levels described above. For still higher 70 levels there is a further proliferation of states, yet with an appreciable structure in the mass patterns compared to the predictions of the full h.o.

To identify these quantum numbers with the observed mass doublets such as (S_{11}, D_{13}) versus $(S'_{11}, D'_{13}, D_{15})$, *before* breaking SU(6) symmetry, it should be possible to define the spatial wave functions (ψ', ψ'') of the correct $[2, 1]$ symmetry for *each* of the lower (l) mass and higher (h) mass states. For the case for $L^P = 1^-$ $(n_x = n_y = 0)$ these (vector) wave functions ψ_{li}, ψ_{ui} turn out to be as follows:

$$\psi'_{li},\ \psi'_{ui} = \cos\tfrac{1}{2}\lambda\,(\phi'_{xi},\ \phi''_{xi}) + \sin\tfrac{1}{2}\lambda\,(\phi''_{yi},\ -\phi'_{yi}),$$

$$\psi''_{li},\ \psi''_{ui} = \sin\tfrac{1}{2}\lambda\,(\phi'_{xi},\ \phi''_{xi}) - \cos\tfrac{1}{2}\lambda\,(\phi''_{yi},\ -\phi'_{yi}).$$

Here the φ functions ϕ'_{xi}, ϕ''_{xi} correspond to the $l_x = 1$ excitations of amounts $\alpha/2$ and $\alpha\sqrt{3}/2$, respectively, while ϕ'_{yi}, ϕ''_{yi} give $l_y = 1$ excitations of amounts $\alpha\sqrt{3}/2$ and $\alpha/2$, respectively. This explains the occurrence of the indicated $\phi_{x,y}$ components in the ψ_l, ψ_u functions and ensures that the (mass)^2 operator has the excitations $\alpha/2$ and $\alpha\sqrt{3}/2$ for the ψ_l and ψ_u states, respectively. Similar expressions hold for the wave functions for higher excitations. The effect of these structures is more interesting for decay matrix elements which will form the subject matter for a separate communication, while the present study is concerned only with the mass spectra.

VI. DISCUSSION OF THE SPECTRUM

The 70 mass spectrum, Eqs. (4.36) and (4.37) has a constant $-\frac{1}{2}\beta$ arising out of the constant term V_0 in the QQ interaction whose corresponding effect on the 56 spectrum is the addition of a term $-\beta$. Thus, while this constant has no role to play within a given supermultiplet, it is important for a comparison of the levels of 56 and 70 states. This situation contrasts with that of the full h.o. potential, which would provide a uniform shift of levels for all states 56 and 70 alike. Considering only nucleonlike states in 56 and 70, $\beta \approx 4$ GeV^2 was found to describe the relative spacings of several states,[11] but at the cost of depressing the entire spectrum (56 and 70) by about 2 GeV^2. To compensate for this effect one requires an overall additive constant C of magnitude ~ 2 GeV^2 (≈ 1.88 GeV^2 to normalize to the nucleon mass) for both the 56 and 70 states. A possible interpretation of this term comes from the FKR[3] treatment of the relativistic extension of the Hamiltonian H through the following successive steps:

$$mH = \tfrac{1}{2}(m+H)^2 - \tfrac{1}{2}m^2 - \tfrac{1}{2}H^2$$

$$\approx \tfrac{1}{2}(m+H)^2 - \tfrac{1}{2}m^2. \qquad (6.1)$$

FKR identified $\frac{1}{2}(m+H)^2$ as the "square of the relativistic mass," absorbing the additive constant $\frac{1}{2}m^2$ in the definition of their overall constant C.

Since the role of this overall constant is uniform for 56 and 70 states (unlike that of the constant V_0 in $Q\bar{Q}$ potential), one obtains a formal justification for its addition in our treatment which clearly admits of a straightforward FKR-type relativistic extension without any change in the structure of the spectrum, except the inclusion of timelike excitations.[3,4]

The ground state of 70, i.e., $n_x = n_y = l_x = l_y = 0$, has now the unique mass

$$M_0'^2 = M_0''^2 = \tfrac{3}{4}\alpha + \tfrac{3}{4}\sqrt{3}\,\alpha - \tfrac{1}{2}\beta + C\,, \qquad (6.2)$$

while the masses of the excited states rise in units of $\frac{1}{2}\alpha$ and $\frac{1}{2}\sqrt{3}\,\alpha$ for each unit of l excitation in the appropriate term of the mass formula, and of α and $\sqrt{3}\,\alpha$ for each of the corresponding units of n excitation. In a recent note we have discussed the possibilities of accommodation of several N states up to the third resonance region, within the above scheme.[11] These possibilities are based on three main features:

(1) The dual spectrum which, even without SU(6) symmetry breaking, predicts two distinct groups of masses, with obvious relevance to the $L^P = 1^-$ states;

(2) low radial excitation compared with the full h.o. model, a feature which naturally accommodates states like $P'_{11}(1750)$;

(3) existence of $P_{11}(1470)$ below the $L^P = 1^-$ states, as a rather natural candidate for $(\underline{70}, 0^+)$.

The partners of the N component of $(\underline{70}, 0^+)$ are no doubt a problem for the even-wave h.o. model just as much as they are for the regular h.o. model. In the full h.o. model these are in 56, and hence require the identification of fewer particles. On the other hand, these masses, corresponding at least to the first radial excitations of 56, require strong symmetry-breaking effects to bring down their values. It may be argued that only the $P_{11}(1470)$ needs to be brought down in mass, while the others (Δ, Λ, Σ, etc.) yet to be properly established may still afford to remain hidden in a somewhat higher-lying region of mass, thus keeping the low-mass region relatively peaceful. On the other hand, it is difficult to see how a symmetry-breaking mechanism which can be so effective in bringing down the N mass would be so ineffective for bringing down the other partners correspondingly without the mass operator having some complicated properties. Indeed, there seems to be some mild evidence for relatively low-lying Δ, Λ, and Σ states, according to a recent compilation and analysis[8] from the Particle Data Group data and other sources,[19-23] viz., $P_{33}(1690)$, $P_{01}(1570)$, and $P_{21}(1620)$. Since the P_{33} spin is not established[24] and $J^P = \frac{1}{2}^+$ is not ruled out, one may expect that these are just the mass levels of particles needed for the low-lying $(\underline{70}, 0^+)$ supermultiplet, without an excessive reliance on strong, symmetry-breaking effects. (We give below some estimates of its mass in terms of our 70 mass formula). As to the other missing masses, while the 1_d state would be difficult to distinguish from its 8_d counterpart, an outstanding difficulty with $(\underline{70}, 0^+)$ would still remain with the missing $\underline{8}_q$ state with $J = \frac{3}{2}$, again at a comparatively low mass (1500–1600 MeV). It will be difficult to detect such an N state easily, in view of a photoproduction selection rule,[25] so one has primarily to rely on a more intensive phase-shift analysis[21] for a P^q_{13} state in the 1500–1600 MeV region.

A. 70 versus 56 mass differences

For a more quantitative comparison of the differences of our predictions from the usual h.o. model we need to go beyond the N states which have already been so compared.[11] In this respect we are considerably inhibited by the lack of adequate reliable data of each variety (Δ, Λ, Σ). We therefore do the next best thing, i.e., make use of certain empirical mass difference regularities recognized by FKR,[3] to normalize the non-N-type masses to the level of the N states before comparing with our mass formula. We further recognize that we are entitled to borrow the FKR regularities only for 56 states with which our predictions agree, but instead of a literal takeover of the FKR regularities for the 70 states, which may well be in disharmony with our own results for such states, we would rather predict the latter in terms of our model, with only the 56 cases as input.

Since the two slopes for 70 states are $\frac{1}{2}\alpha$ and $\frac{1}{2}\alpha\sqrt{3}$, our basic unit of (mass)^2 intervals for 70 may be taken as the mean of these two, i.e.,

$$\delta \equiv \tfrac{1}{4}\alpha + \tfrac{1}{4}\alpha\sqrt{3} \approx 0.68\alpha\,, \qquad (6.3)$$

to be compared with α for 56 states. This estimate we shall use as a "conversion factor" from 56 to 70 before making the FKR regularities for 56 on symmetry breaking in (mass)^2 applicable to the 70 states. We do not, of course, use the factor for predicting the SU(6) *unbroken* N masses which are still given by the complete dual mass formula, but believe that the averaged interval is adequate for conversion of the (fine-structure) mass-difference effects such as $\Delta^2 - N^2$ and $\Sigma^2 - N^2$ from 56 to 70 states.

Before giving a comparison of our predictions for non-nucleonic 70 masses in terms of FKR-like estimates let us first check the working of the conversion factor $\delta/\alpha = 0.68$ for at least two types of

mass differences, $\Delta^2 - N^2$ and $\Sigma^2 - N^2$, in $\underline{56}$ and $\underline{70}$ states. In GeV^2 units, $\Delta^2 - N^2$ has the values 0.65 and 0.42 for $\underline{56}$ and $\underline{70}$ states, respectively, taking for the latter the average of $D_{33}{}^2 - D_{13}{}^2$ and $S_{31}{}^2 - S_{11}{}^2$. This gives a ratio $\delta/\alpha \approx 0.66$ in rather good agreement with our theoretical estimate 0.68 of δ/α given by Eq. (6.3), i.e., $\Delta^2 - N^2 \approx 0.44\alpha$. In $\Sigma^2 - N^2$, we take the three states $\Sigma(1190)$, $\Sigma(1670)$, and $\Sigma(1765)$ to be compared with their respective J counterparts $N(938)$, $N(1520)$, and $N(1670)$. They yield the respective numbers 0.54, 0.43, and 0.33, the average of the last two figures, viz., 0.38α, again agreeing rather well with the predicted value of 0.37α, according to Eq. (6.3).

For Λ states we take the point of view that a direct comparison is difficult in view of the alternating signs for $\Sigma^2 - \Lambda^2$ between $\underline{56}$ and $\underline{70}$, of magnitude 0.16 GeV^2, as noticed by FKR. Since we cannot explain the last feature, we prefer to incorporate this correction by hand to reduce the Λ-mass problem to an effective Σ-mass problem, before making the predictions on Λ masses. (This point of view is somewhat different from FKR's who consider the average mass of Σ^2 and Λ^2.)

B. Comparison with the data

Having thus tested the working of our conversion factor δ/α for mass differences within $\underline{56}$ and $\underline{70}$ states, we can make specific predictions for Δ, Σ, Λ masses in terms of the N masses, after taking into account the following precautions.

(a) The experimental slope for N is $\alpha \approx 1$ GeV^2, as used in Ref. 11, which is also reasonable for Λ states, but the corresponding slopes for Δ, Σ are more near the average value 1.05 GeV^2 as given by FKR; this introduces a small correction.

(b) As noted by FKR, the 1_d Λ states should be negatively corrected for strangeness before comparing with an 8_d Λ state. In other words, a 1_d state can be directly compared with an 8_d N state and should therefore have nearly the same mass, except for an (upward) correction due to the Σ-Λ mass difference effect. This should make plausible the close equality of the $D_{13}(1520)$ and $D_{03}(1520)$ masses, but would still not explain the low mass of $S_{01}(1405)$.

(c) Finally, since our theory is SU(6) invariant, the predictions must be understood in the sense of spin-averaged mass squared, as in Ref. 11.

Table I gives the predicted Δ, Σ, Λ masses for the low-lying states together with the observed ones, taking into account the various corrections noted above. We have taken (in GeV^2)

$$\Delta^2 - N^2 = 0.43\,, \quad \Sigma^2 - N^2 = 0.37 \tag{6.4}$$

and used slopes $\alpha \approx 1.05$ GeV^2 for (Δ, Σ) and 1.0 for Λ states. Also listed are the *predicted* (spin-averaged) masses for the N states, and *these* values (rather than masses of the observed nucleon states) with specified (lj) values have been used for predicting the positions of Δ, Σ, Λ, according to (6.3) and (6.4), in terms of their most plausible (lj) assignments as appropriate companions for the corresponding N states. For the Λ states we give two numbers, one each for the 1_d and 8_d, respectively (only the latter requires a strangeness correction according to the above arguments). Under the "observed" columns we indicate what we believe could be reasonable candidates, according to the current listings.[19]

It is not possible to make any comments on quantitative comparison with observations beyond indicating the generally hopeful pattern of the predictions against the data, since in the first place the theory does not as yet include SU(6)-violating effects. Indeed, since the theory merely claims to provide a much reduced role for SU(6)-violating

TABLE I. Low-lying mass predictions ($N, \Delta, \Sigma, \Lambda$) in GeV; spin assignments (in $2I\,2J$ notation), together with observed particles. For the Λ states, the upper and lower figures correspond to the 1_d and 8_d multiplet, respectively. For N states, see also Ref. 11.

N (GeV)		Δ (GeV)		Σ (GeV)		Λ (GeV)	
Pred.	Obs.	Pred.	Obs.	Pred.	Obs.	Pred.	Obs.
$P(1.41)$	$P_{11}(1.46)$	$P_{31}(1.60)$	$P_{33}(1.69)$	$P_{21}(1.58)$	$P_{21}(1.62)$	$P_{01}\binom{1.48}{1.59}$	$P_{01}(1.57)$
$D(1.56)$	$D_{13}(1.52)$	$D_{33}(1.73)$	$D_{33}(1.69)$	$D_{23}(1.71)$	$D_{23}(1.67)$	$D_{03}\binom{1.60}{1.72}$	$D_{03}\binom{1.52}{1.69}$
$D(1.68)$	$D_{15}(1.67)$	...	...	$D_{25}(1.81)$	$D_{25}(1.77)$	$D_{05}(1.83)$	$D_{05}(1.83)$
$P'(1.72)$	$P'_{11}(1.75)$	$P'_{31}(1.89)$	$P_{31}(1.91)$	$P'_{21}(1.88)$	$P_{21}(1.92)$	$P'_{01}(1.76)$ $P'_{03}(1.87)$	$P_{01}(1.75)$ $P_{03}(1.87)$

effects over the usual h.o. treatment (which seems to require the latter in a stronger and more essential manner[8]), any quantitative comparison of the prediction with the data would be premature. Therefore, some of the cases of "good" agreement in Table I must be regarded as partially accidental, rather than causal, at this stage. As a general trend we notice that the predictions tend to underestimate the observed masses for even parity states, but the opposite seems to hold for the negative parity states. Further, the observed Λ states seem to lie between the predicted masses of the corresponding 1_d and 8_d states.

The pattern of agreement for the low-lying states seems, on the whole, reasonable enough to warrant the conclusion of a better zero-order description than with the full h.o. model without SU(6) symmetry-breaking terms. The following additional comments for higher-lying states are also largely qualitative and concern some of the better established states only.

In the third resonance region, there is clearly a much larger proliferation of states. Thus, the prediction of a Δ state of $P'_{31}(1.91)$ would seem to put it rather too close to a corresponding observed $P_{31}(1.91)$ state which traditionally has been regarded as a $(\underline{56}, 2^+)$. The difference lies in the quark-spin assignment which in our prediction of a $(\underline{70}, 0^+)$ radial excitation is a doublet. Since a $(\underline{70}, 2^+)$ Δ state, on the other hand, can at best show up as a P_{33} (not P_{31}), the competition is between $(\underline{56}, 2^+)$ and $(\underline{70}, 0^+)$, presumably with little mixing effects (since $\Delta L = 2$). Indeed, a fairly unambiguous way of distinguishing between these two possibilities would be through a direct photo-production experiment—only a $(\underline{70}, 0^+)$ Δ_{31} would be easily photoproduced, while the $(\underline{56}, 2^+)$ Δ_{31} would *not* (since such a low J with high L state of Δ cannot be easily produced by the magnetic interaction[26]).

Because of the mass splitting patterns, the need for mixing among the "third resonance" states, such as $(\underline{56}, 2^+)$ and $(\underline{70}, 2^+)$ or $(\underline{56}, 0^+)_2$ and $(\underline{70}, 0^+)_2$, is much less urgent in our model than in the usual h.o. theory. Thus, even before the introduction of mixing effects our assignment of a $(\underline{70}, 2^+)$ to the $P_{13}(1860)$ seems to agree with current thinking on this state.[3,19] In Table II we have listed separately some of our predictions for the higher N states in the extended notation $(n_x l_x n_y l_y)$. The $G_{17}(2190)$ seems difficult to accommodate in the $\underline{70}$, but finds fairly easy accommodation as a $(\underline{56}, 3^-)$ which predicts the mass as in a full h.o. model, viz., Eq. (4.40). Most of the other predictions in Table II are speculative at this stage, and are intended merely to indicate the rapid proliferation of states with the fairly narrow mass intervals as a characteristic of this model.

C. Summary and conclusion

To summarize, we have tried to present a considerably modified version of the usual h.o. model, one characterized only by even waves, as a means of keeping out $\underline{20}$ states. The $\underline{56}$ structure is left unchanged, but the dynamics of $\underline{70}$ states changes considerably over the usual h.o. model while maintaining a linear rise of M^2 with J. Its prediction of a low-lying $(\underline{70}, 0^+)$ presents an attractive candidate for $P_{11}(1470)$ and similar low-lying Δ, Σ, Λ resonances in the p wave for which some evidence seems to exist. The dual mass spectrum, even within SU(6) invariance, provides some understanding of certain distinct mass groupings among $(\underline{70}, 1^-)$ states. The considerably small spacing predicted for radial excitations is a great help for states like $P'_{11}(1750)$ and $P_{13}(1860)$. The average (mass)^2 spacing for $\underline{70}$ states which is predicted as about $\frac{2}{3}$ for $\underline{70}$ states compared with $\underline{56}$ seems to account extremely well for $\Delta^2 - N^2$ and $\Sigma^2 - N^2$ mass differences in these two supermultiplets, a feature which has been used in this paper to build up our predictions for Δ, Σ, Λ states, in terms of N states.

On the not-so-bright side of the picture we must reckon with the fact that the calculation of $\underline{70}$ states is not yet exact, though the plausibility of the approximation and the stability of the structure of the mass spectrum has been shown by two independent methods. Again, the (as yet) absence of a low-lying $\underline{8}_q$ multiplet of $(\underline{70}, 0^+)$ exhibiting $J = \frac{3}{2}$ states could be another serious difficulty. Finally, the large proliferation of higher $\underline{70}$ states with comparatively smaller spacing would call for a far more fine-grained analysis of the data than available at present.

The classification is also more involved, re-

TABLE II. Predictions on higher lying $\underline{70}$ N states in the $(n_x l_x n_y l_y)$ classification, together with a list of observed N states with possible relevance. For discussions, see Sec. VI.

N state	Pred. mass (GeV)	Observed states (GeV)
{(1000), (0200)} {(0010), (0002)}	1.72, 1.92	$P_{11}(1.75), P_{13}(1.86)$
(0101)	1.82	
(1101), (0111)	2.08, 2.25	$F_{17}(1.99), F_{15}(2.18)$
{(0300), (0003)} {(0201), (0102)}	{1.86, 2.14} {1.95, 2.05}	$D_{13}(2.04), S_{11}(2.10)$ $D_{15}(2.10), G_{17}(2.19)$
{(1201), (1102)} {(0211), etc.}	{2.19, 2.28} {2.36, etc.}	

quiring as many as a set of four quantum numbers $(n_x l_x n_y l_y)$ for the identification of a state. This, unfortunately, requires looking beyond the mere mass spectrum, and more closely into the decay characteristics. The latter in turn requires a detailed formulation in terms of wave functions (and perhaps also some mixing effects). While some preliminary results have been obtained in this direction,[27] a more detailed study is still in progress.

To conclude, we draw attention to two theoretically attractive features of the new proposal. Like the full h.o. model the present one also admits of a straightforward relativistic extension on the lines of FKR[3] and it appears that such extensions can now be defended by more formal field theory as well.[4] Secondly, since the even-wave h.o. model predicts all the $\underline{56}$ and $\underline{70}$ supermultiplets (with *both* even and odd L), it is fully compatible with exchange degeneracy requirements as well.[28] Finally, despite these attractive features, however, a direct and unambiguous experimental determination of a $\underline{20}$ $(1p)^2$ state would be sufficient to demolish this model.

ACKNOWLEDGMENTS

This paper was written during the author's stay at International Centre for Theoretical Physics (ICTP) (Trieste). He would like to thank Professor Abdus Salam and P. Budini as well as the International Atomic Energy Agency and the UNESCO for hospitality at ICTP, Trieste. He is also grateful to Professor W. Sandhas for the hospitality of the Physikalische Institut der Universität Bonn, and to Dr. R. F. Meyer for discussions.

APPENDIX A

We outline here an alternative derivation of the spectrum in a spherical basis, using as independent variables the three internal coordinates x, y, λ and the Euler angles θ, ϕ, χ according to the following scheme[29]:

$$\begin{aligned} x_1 &= x\sin\theta\cos\phi\,, \\ x_2 &= x\sin\theta\sin\phi\,, \\ x_3 &= x\cos\theta\,; \end{aligned} \tag{A1}$$

$$\begin{aligned} y_1 &= -y\cos\chi\sin\phi - y\cos\theta\cos\phi\sin\chi\,, \\ y_2 &= y\cos\chi\cos\phi - y\cos\theta\sin\phi\sin\chi\,, \\ y_3 &= y\sin\theta\sin\chi\,, \end{aligned} \tag{A2}$$

ensuring that $x_i y_i = 0$.

The conversion from the six independent variables ξ_i, η_i to the above set is completed through the inverse relations to Eq. (4.8) involving λ:

$$\begin{aligned} \xi_i &= x_i\cos\tfrac{1}{2}\lambda + y_i\sin\tfrac{1}{2}\lambda\,, \\ \eta_i &= x_i\sin\tfrac{1}{2}\lambda - y_i\cos\tfrac{1}{2}\lambda\,. \end{aligned} \tag{A3}$$

These relations lead to the six-dimensional metric

$$\begin{aligned} \sum_i (d\xi_i^2 + d\eta_i^2) = {} & dx^2 + dy^2 + \tfrac{1}{4}\rho\, d\lambda^2 + d\theta^2(x^2 + y^2\sin^2\chi) \\ & + d\phi^2(x^2\sin^2\theta + y^2\cos^2\chi + y^2\sin^2\chi\cos^2\theta) \\ & + y^2 d\chi^2 + 2d\lambda\, d\phi\, xy\sin\theta\cos\chi \\ & - 2d\theta\, d\lambda\, xy\sin\chi \\ & - 2d\theta\, d\phi\, y^2\sin\theta\sin\chi\cos\chi \\ & + 2d\phi\, d\chi\, y^2\cos\theta\,. \end{aligned} \tag{A4}$$

But for the cross terms which involve the cyclic variable χ, there would have been a neat separation between the external and internal coordinates. This is the well-known difficulty associated with the corresponding kinetic energy operator for a three-body system as recognized long ago by Derrick.[30] For a simple closed form solution we propose only one approximation, viz., to *average out over the dependence on the cyclic variable* χ wherever it appears in (A3). This yields the reduced structure $(\rho = x^2 + y^2)$:

$$\begin{aligned} ds^2 = {} & dx^2 + dy^2 + \tfrac{1}{4}\rho d\lambda^2 \\ & + (x^2 + \tfrac{1}{2}y^2)(d\theta^2 + \sin^2\theta\, d\phi^2) \\ & + y^2(d\chi + \cos\theta\, d\phi)^2\,. \end{aligned} \tag{A5}$$

From this expression one can *deduce* a corresponding structure for the six-dimensional "kinetic-energy operator" in the standard mathematical manner[18]:

$$\nabla_\xi^2 + \nabla_\eta^2 \Rightarrow \frac{1}{\rho}\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}\right) + \frac{1}{\rho}\frac{\partial}{\partial y}\left(\rho\frac{\partial}{\partial y}\right) + \frac{4}{\rho}\frac{\partial^2}{\partial\lambda^2} - \frac{J_{\rho\rho}^2}{A}\,, \tag{A6}$$

where

$$\begin{aligned} -J_{0\rho}^2 = {} & \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \left(\cot^2\theta + \frac{A}{C}\right)\frac{\partial^2}{\partial\chi^2} \\ & - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial^2}{\partial\phi\,\partial\chi}\,, \end{aligned} \tag{A7}$$

and

$$A = x^2 + \tfrac{1}{2}y^2\,, \quad C = y^2 \tag{A8}$$

are the two "principal moments of inertia."

As a result of the decoupling of the internal $(xy\lambda)$ and external $(\theta\phi\chi)$ coordinates, the six-dimensional Schrödinger equation is now solvable, since the potentials depend only on the internal coordinates. Moreover, the diagonal (ϕ', ϕ'') representation is now valid for any angular momentum, as long as its λ dependence, which now factors out, is taken exactly as in Eq. (3.8). Solution of

Eq. (A7) leads to the eigenvalue determination

$$A^{-1}J^2 = [L(L+1) - k^2](x^2 + \tfrac{1}{2}y^2)^{-1} + k^2 y^{-2}, \qquad \text{(A9)}$$

corresponding to the eigenfunctions $D^L_{km}(\theta\phi\chi)$, where L is the resultant angular momentum, and k, the projection of L on the body-fixed axis, is associated with the angular coordinate χ.

The reduction of the main equations $K^+\phi' = K^-\phi'' = 0$ which now involve the operator (A6) is facilitated through the replacement, as in the steps immediately following Eq. (3.6), of the "modified operator" $\partial_\lambda{}^2$ by $-(\nu+1)^2$ corresponding to the λ dependence of ϕ', ϕ'' given by Eq. (3.8). Finally, the change of variables

$$X = x^2 = \tfrac{1}{2}(\rho+\gamma); \quad Y = y^2 = \tfrac{1}{2}(\rho-\gamma) \qquad \text{(A10)}$$

leads to the eigenvalue equations

$$H^{\pm}_{Lk\nu} = (2M^2 + \beta)\alpha^{-1} \qquad \text{(A11)}$$

for the ϕ' and ϕ'' modes, respectively, where

$$H^{\pm}_{Lk\nu} = -4X\frac{\partial^2}{\partial X^2} - 4Y\frac{\partial^2}{\partial Y^2} - 4\left(\frac{\partial}{\partial X} + \frac{\partial}{\partial Y}\right) + \frac{4(\nu+1)^2}{X+Y} + \frac{L(L+1)-k^2}{X+\frac{1}{2}Y} + \frac{k^2}{Y} + \tfrac{1}{2}(X+Y) \mp \tfrac{1}{4}(X-Y). \qquad \text{(A12)}$$

These equations are almost separable, but not completely so. However, from Eq. (A10), and the fact that γ is always positive (and less than ρ), it is reasonable to assume that over a wide region (except possibly for a thin strip where γ is small), the inequality $X \gg Y$ holds adequately. With this last assumption, Eq. (A12) becomes separable in X and Y and one obtains, in a straightforward way, the dual mass spectrum:

$$M'^2_{Lk\nu} = \alpha(n_X + \tfrac{1}{2}a_{Lk\nu} + \tfrac{1}{2}) + \sqrt{3}\,\alpha(n_Y + \tfrac{1}{2}k + \tfrac{1}{2}) - \tfrac{1}{2}\beta, \qquad \text{(A13)}$$

$$M''^2_{Lk\nu} = \sqrt{3}\,\alpha(n_X + \tfrac{1}{2}a_{Lk\nu} + \tfrac{1}{2}) + \alpha(n_Y + \tfrac{1}{2}k + \tfrac{1}{2}) - \tfrac{1}{2}\beta, \qquad \text{(A14)}$$

where

$$a_{Lk\nu}{}^2 = L(L+1) + 4(\nu+1)^2 - k^2. \qquad \text{(A15)}$$

Comparison of this structure with the Cartesian form given by Eqs. (4.36) and (4.37) of Sec. IV, is given in Sec. V of the text.

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