PHYSICAL REVIEW D VOLUME 11, NUMBER 11 1 JUNE 1975

# SU(4) and SU(8) mass formulas and weak interactions*

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(Received 11 March 1975)

Mass formulas for SU(4) and SU(8) groups have been obtained and are especially simple for degenerate representations. Also, tensor representations of baryon multiplets are discussed together with some applications to weak decay processes.

## I. INFINITESIMAL GENERATORS OF U($n$) GROUP

Recent spectacular discoveries[1] of $\psi$ and $\psi'$ renewed a great interest in the study of the SU(4) symmetry proposed by many authors[2] some years ago, since the SU(4) group is important in the construction of a consistent unified gauge theory.[3,4]

In this note we will study mass formulas of the SU(4) as well as SU(8) groups. We shall also discuss some applications of the SU(4) for weak decay processes. As we shall see shortly, the SU(4) mass formula is much more complicated than the ordinary SU(3) formula, and we shall consider mostly special classes of the SU(4) irreducible representations which are of physical relevance. With this in view, we shall, in this section, recapitulate some basic properties of the irreducible representation theory of the $n$-dimensional unitary group U($n$). We shall consider here the U($n$) group rather than the SU($n$) group, since it is far easier to deal with for our purpose. Besides, the transition to its SU($n$) subgroup is trivial. Some possible and physically different consequences between the two groups will be discussed in Sec. IV.

Let $A^\mu_\nu\,(\mu,\nu=1,2,\ldots,n)$ be infinitesimal generators of the U($n$) group satisfying the Lie commutation relation

$$[A^\mu_\nu, A^\alpha_\beta]=\delta^\mu_\beta A^\alpha_\nu-\delta^\alpha_\nu A^\mu_\beta\,. \tag{1.1}$$

If we are interested in the SU($n$) subgroup, then we have to consider traceless generators $B^\mu_\nu$ given by

$$B^\mu_\nu=A^\mu_\nu-\frac{1}{n}\delta^\mu_\nu\sum_{\lambda=1}^{n}A^\lambda_\lambda\,, \tag{1.2}$$

which satisfy the same commutation relation as $A^\mu_\nu$.

Hereafter, we shall restrict ourselves to a given irreducible representation of the U($n$) specified by $n$ integers $f_1, f_2,\ldots,f_n$ satisfying

$$f_1\geq f_2\geq f_3\geq\cdots\geq f_n\,. \tag{1.3}$$

The dimension $N$ of the representation is calculable from the Weyl formula[5]

$$N=\frac{\prod^n_{\mu<\nu}(l_\mu-l_\nu)}{1!2!\cdots(n-1)!}\,, \tag{1.4}$$

where we have set

$$l_\lambda=f_\lambda+n-\lambda\,,\quad \lambda=1,2,\ldots,n\,. \tag{1.5}$$

If $n^2$ operators $T^\mu_\nu$ $(\mu,\nu=1,2,\ldots,n)$ in this space satisfy the commutation relation

$$[A^\mu_\nu, T^\alpha_\beta]=\delta^\mu_\beta T^\alpha_\nu-\delta^\alpha_\nu T^\mu_\beta\,, \tag{1.6}$$

then we call $T^\mu_\nu$ a vector operator. Especially, $A^\mu_\nu$ itself is a vector operator. For any two vector operators $S^\mu_\nu$ and $T^\mu_\nu$, we can define a product vector operator $R^\mu_\nu$ by

$$R^\mu_\nu=\sum_{\lambda=1}^{n}S^\lambda_\nu T^\mu_\lambda\,. \tag{1.7}$$

For simplicity, we shall hereafter write this relation simply as

$$R=ST\,, \tag{1.7'}$$

dropping all Greek indices. The product thus defined obeys the associative law $(ST)U=S(TU)$ for three-vector operators. Moreover, the unit vector operator $I$ is given by

$$I^\mu_\nu=\delta^\mu_\nu E\,, \tag{1.8}$$

where $E$ is an identity operator in our irreducible space. Then we can define the $j$th power $A^j$ recursively by

$$A^{j+1}=A^jA\,,\quad A^0=I\,,\quad j=0,1,2,\ldots\,. \tag{1.9}$$

Next, for any vector operator $T^\mu_\nu$, we can associate a scalar $\langle T\rangle$ by

$$\sum_{\lambda=1}^{n}T^\lambda_\lambda=\langle T\rangle E\,, \tag{1.10}$$

Since $\sum^n_{\lambda=1}T^\lambda_\lambda$ commutes with all $A^\mu_\nu$ so that it must be a constant multiple of a unit operator $E$ owing to Schur's lemma. Especially, if we set

$$M^{(n)}_j=\langle A^j\rangle\,,\quad j=0,1,2,\ldots \tag{1.11}$$

then these are eigenvalues of generalized Casimir operators of the U($n$) group. Its explicit values have been computed by Louck and Biedenharn[6] to be

$$M_j^{(n)} = \sum_{\lambda=1}^{n} (l_\lambda)^j \prod_{\nu=1}^{n}{}' \frac{1 + l_\nu - l_\lambda}{l_\nu - l_\lambda}, \tag{1.12}$$

where the product on $\nu$ omits the singular point $\nu = \lambda$.

Now as we have proved elsewhere (see Ref. 7, hereafter referred to as I), the U($n$) generators $A_\nu^\mu$ satisfy a simple $n$th-order polynomial equation

$$A(l_1)A(l_2)\cdots A(l_{n-1})A(l_n) = 0, \tag{1.13}$$

where $A(l_j)$ is defined by

$$A(l_j) = A - l_j I, \tag{1.14}$$

and the product is the vector product in the sense of (1.7′). However, we can find a stronger identity if we have $f_\mu = f_\nu$ for some $\mu \neq \nu$. Let us suppose that we have $f_{j+1} = f_j$ for some integer $j$. Then we call $A(l_j)$ the redundant factor, and $f_j$ the corresponding redundant signature. In such a case, we can omit[7] all redundant factors in Eq. (1.13). As an illustration, let us consider a special case $n = 8$ with

$$f_1 = f_2 = f_3 > f_4 = f_5 > f_6 > f_7 = f_8. \tag{1.15}$$

Then, all factors $A(l_1)$, $A(l_2)$, $A(l_4)$, and $A(l_7)$ are redundant, and we have a stronger identity,

$$A(l_3)A(l_5)A(l_6)A(l_8) = 0. \tag{1.16}$$

Of course, the validity of (1.16) automatically implies that of (1.13) with $n = 8$. As we have shown in I, Eq. (1.16) is the minimum polynomial equation satisfied by $A_\nu^\mu$ for this case.

Next, following the terminology of I, let us name a given irreducible representation to be degenerate, if we have

$$f_1 = f_2 = \cdots = f_j > f_{j+1} = f_{j+2} = \cdots = f_n \tag{1.17}$$

for an integer $j$. Then, the above redundant factor rule implies the validity of

$$A(l_j)A(l_n) = 0, \tag{1.18}$$

or equivalently of

$$\sum_{\lambda=1}^{n} A_\nu^\lambda A_\lambda^\mu = (l_j + l_n)A_\nu^\mu - l_j l_n \delta_\nu^\mu E. \tag{1.18′}$$

The most interesting special case of the degenerate representation is that of the completely symmetric representation with $f_2 = f_3 = \cdots = f_n = 0$, i.e., $f_j = 0$ for $j \neq 1$, where we can prove[7,8] an extra stronger relation

$$(A_\nu^\mu + \delta_\nu^\mu)A_\beta^\alpha = (A_\beta^\mu + \delta_\beta^\mu)A_\nu^\alpha. \tag{1.19}$$

We remark that such a simple relation does not necessarily apply for other degenerate representations. If we relax the condition to a weaker one $f_3 = f_4 = \cdots = f_n = 0$, i.e., $f_j = 0$ for $j \neq 1, 2$, we find a slightly more complicated formula,[9]

$$(A_\nu^\mu + \delta_\nu^\mu)K_\beta^\alpha + (A_\beta^\alpha + \delta_\beta^\alpha)K_\nu^\mu = (A_\beta^\mu + \delta_\beta^\mu)K_\nu^\alpha + (A_\nu^\alpha + \delta_\nu^\alpha)K_\beta^\mu, \tag{1.20a}$$

where $K_\nu^\mu$ is given by

$$K_\nu^\mu = (AA)_\nu^\mu - \tfrac{1}{2}(f_1 + f_2 + 2n - 4)A_\nu^\mu. \tag{1.20b}$$

We note that (1.20) is consistent with (1.18′) and (1.19) for a special case, $f_2 = 0$. Also, it becomes a trivial identity for the case $f_1 = f_2$ in view of (1.18′).

Last, we can prove[7] that any vector operator $T_\nu^\mu$ can be expressed as

$$T_\nu^\mu = \sum_{j=0}^{n-1} a_j (A^j)_\nu^\mu \tag{1.21}$$

in the irreducible representation space under consideration where $a_j$ ($j = 0, 1, 2, \ldots, n-1$) are some unspecified constants. Because of (1.18) this formula can be further simplified to become

$$T_\nu^\mu = a\delta_\nu^\mu E + bA_\nu^\mu \tag{1.22}$$

for any degenerate representation.

## II. SU(4) MASS FORMULA

As in the SU(3) theory, we assume that the SU(4)-breaking interaction $H_1$ has the tensor structure

$$H_1 = T_3^3 + yT_4^4, \tag{2.1}$$

where $y$ is a real constant.[13] Then, as we noted in (1.21), $T_\nu^\mu$ can be expressed as

$$T_\nu^\mu = a\delta_\nu^\mu E + bA_\nu^\mu + c(AA)_\nu^\mu + d(AAA)_\nu^\mu \tag{2.2}$$

in any given irreducible representation, so that the most general SU(4) mass formula contains five unknown parameters. Now, physical states are labeled by quantum numbers specified by the canonical chain decomposition

$$U(4) \supset U(3) \supset U(2) \supset U(1). \tag{2.3}$$

Explicit expressions for matrix elements of $(AA)_4^4$ and $(AAA)_4^4$ in this canonical decomposition can be easily obtained.[10] However, the difficulty is to evaluate matrix elements of operators $(AA)_3^3$ and $(AAA)_3^3$ in the closed form. Of course, any matrix element of generator $A_\nu^\mu$ and hence of $(AA)_3^3$ and $(AAA)_3^3$ can be calculated in principle from the general formula given by Baird and Biedenharn.[11] However, this procedure is in reality very complicated to be of much practical use, as has recently been noted by Bose,[12] who also derived Eq. (2.2). Hence, we shall restrict ourselves to special classes of representations which are of physical interest. First, let us consider the case of degenerate representations.

*(a) Degenerate representations.* As we noted in

the previous section, this implies that we have either $f_1 > f_2 = f_3 = f_4$, or $f_1 = f_2 > f_3 = f_4$, or $f_1 = f_2 = f_3 > f_4$. Especially, this class contains 20-dimensional representations (3, 0, 0, 0) and a 4-dimensional one (1, 1, 1, 0) for baryons as well as another 20-dimensional representation (1, 1, −1, −1) for possible exotic bosons. For degenerate representations, we can express $(AA)^\mu_\nu$ and $(AAA)^\mu_\nu$ in terms of $\delta^\mu_\nu E$ and $A^\mu_\nu$ because of Eq. (1.18′). Therefore, (2.2) can be simplified to become Eq. (1.22), i.e.,

$$T^\mu_\nu = a\delta^\mu_\nu E + bA^\mu_\nu \,. \tag{2.4}$$

As a result, we have the mass formula

$$M = M_0 + b(N_3 + yN_4)\,, \tag{2.5a}$$

where we have set

$$N_j = A^j_j\,, \quad j = 1, 2, 3, 4\,. \tag{2.5b}$$

Note that in the quark model, $N_j$ is simply the number operator of the $j$th quark. Then, our mass formula (2.5) implies that it gives exactly the same answer as is given by a naive additive quark model, where the mass difference in the multiplet results solely from masses of constituent quarks. In that case, the parameter $y$ is related to the bare quark mass $m_j$ for the $j$th quark $q_j$ by

$$y = \frac{m_4 - m_1}{m_3 - m_1}\,. \tag{2.6}$$

As usual, we identify the completely symmetric representation (3, 0, 0, 0) as the lowest $J^P = \frac{3}{2}^+$ baryon multiplet. Then, the mass formula (2.5) gives the familiar equal-spacing mass rule. However, since their explicit mass relations have been given elsewhere,[13] we shall not reproduce them here. The same equal-spacing relation also applies for the completely antisymmetric case (1, 1, 1, 0), of which $Y_0^*$(1405 MeV) is presumably a member. With respect to bosons, so far no exotic state has been found. However, Iwasaki[14] attempts to classify $\psi'$(3.700 MeV) in an exotic representation which may correspond to (1, 1, −1, −1). In that case, we can again apply our formula (2.5) with $b = 0$ since the mass of the bosons must be invariant under $N_j \to -N_j$ because of the charge conjugation invariance.

For the completely symmetric representation (3, 0, 0, 0), we could considerably simplify the second-order mass formula for the second-order perturbation interaction of the form

$$H_2 = T^{33}_{33} + \alpha T^{44}_{44} + \beta T^{34}_{34}\,. \tag{2.7}$$

For this case, we can utilize (1.18) as well as (1.19) to reduce complicated tensors formed out of generators $A^\mu_\nu$. Then the final result is simply to add terms proportional to $A^3_3 A^3_3$, $A^4_4 A^4_4$, and $A^3_3 A^4_4$ to our mass formula (2.5a). We need not consider a term $A^4_3 A^3_4$ since (1.19) leads to

$$A^4_3 A^3_4 = A^3_3 A^4_4 + A^3_3\,.$$

The situation is very much analogous[15] to the electromagnetic mass difference of the decouplet baryon state in the SU(3) symmetry.

*(b) Representation with one degeneracy.* This is the case where only one pair among $f_1$, $f_2$, $f_3$, and $f_4$ coincides, i.e., we have either $f_1 = f_2 > f_3 > f_4$, or $f_1 > f_2 = f_3 > f_4$, or $f_1 > f_2 > f_3 = f_4$. Especially, this class contains physically interesting cases of a 20′-dimensional baryon representation (2, 1, 0, 0) with $f_3 = f_4 = 0$ and a 15-dimensional boson representation (1, 0, 0, −1) with $f_2 = f_3 = 0$. Then, our redundant factor rule implies that we can now express $(AAA)^\mu_\nu$ in a linear combination of $\delta^\mu_\nu E$, $A^\mu_\nu$, and $(AA)^\mu_\nu$. Therefore, the mass formula is rewritten as

$$M = M_0 + b(A^3_3 + yA^4_4) + c[(AA)^3_3 + y(AA)^4_4]\,. \tag{2.8}$$

In terms of the baryon number $B$, hypercharge $Y$, and the charm quantum number $C$, we can identify[16]

$$N_3 = A^3_3 = B - C - Y\,, \quad N_4 = A^4_4 = C\,, \qquad \sum_{\lambda=1}^{4} A^\lambda_\lambda = 3B\,. \tag{2.9}$$

Then, $(AA)^4_4$ is easily computed as in the SU(3) case[17,10] to be

$$2(AA)^4_4 = M_2^{(4)} - M_1^{(4)} - M_2^{(3)} + (N_4)^2 + 4N_4\,, \tag{2.10}$$

where the $j$th Casimir operator $M_j^{(n)}$ of the U($n$) group with $n = 3, 4$ can be evaluated by means of Eq. (1.12). Note that $M_2^{(3)}$ is the second-order Casimir operator of the U(3) group contained in the canonical chain (2.3).

Similarly, $(AA)^3_3$ can be expressed as

$$2(AA)^3_3 = M_2^{(3)} - M_1^{(3)} - M_2^{(2)} + (N_3)^2 + 3N_3 + 2A^4_3 A^3_4\,, \tag{2.11}$$

or equivalently[17] as

$$(AA)^3_3 = \tfrac{1}{4}Y^2 - I(I+1) - (2B - C + \tfrac{3}{2})Y - (1+B)C + \tfrac{1}{2}(C^2 - B^2) + \tfrac{1}{2}M_2^{(3)} + A^4_3 A^3_4\,. \tag{2.11′}$$

However, the difficult part is the evaluation of the matrix element $A^4_3 A^3_4$ contained in (2.11). As we remarked earlier, we could compute its matrix element from the formula of Baird and Biedenharn[11] or from that of Sen.[18] But the general expression proves to be too complicated to be of practical use, the main reason being that two irreducible representations of the U(3) subgroup now mix together by this operator. Hence, it is far easier in practice to handle specific cases separately.

*(i) Representation (2, 1, 0, 0).* This is a 20-dimensional representation which contains the following U(3) components:

$$(2,1,0)\oplus(2,0,0)\oplus(1,1,0)\oplus(1,0,0)=\underline{8}\oplus\underline{6}\oplus\underline{\overline{3}}\oplus\underline{3}\,. \tag{2.12}$$

Note that except for the 8-dimensional case, all other U(3) components are degenerate. Then, the final mass formula is calculated to be

$$\begin{aligned}
&\underline{8}:\ M(8)=m_0+\alpha Y+\beta[I(I+1)-\tfrac{1}{4}Y^2],\\
&\underline{6}:\ M(6)=m_0-(y-1)\alpha+\tfrac{1}{2}(3y+1)\beta+\alpha Y,\\
&\underline{3}:\ M(3)=m_0-2(y-1)\alpha-\beta+(\alpha-\tfrac{3}{2}\beta)Y,\\
&\underline{\overline{3}}:\ M(\overline{3})=m_0-(y-1)\alpha-\tfrac{1}{2}(y-1)\beta+(\alpha-\beta)Y,\\
&\underline{6}-\underline{\overline{3}}:\ M(\overline{3}-6)=-\frac{\sqrt{3}}{2}\beta\sum_{j=1}^{2}(|S_j\rangle\langle A_j|+|A_j\rangle\langle S_j|),
\end{aligned} \tag{2.13}$$

where $m_0$, $\alpha$, and $\beta$ are some unknown parameters, and $M(6-\overline{3})$ is the mass mixing operator between two states $|S_j\rangle$ and $|A_j\rangle$ $(j=1,2)$ with $I=\frac{1}{2}$, $Y=1$, and $C=1$ belonging to the $\underline{6}$ and $\underline{\overline{3}}$ representations, respectively. The index $j=1,2$, refers to two isotopic spin states $I_z=\pm\frac{1}{2}$. Also $S$ and $A$ designate symmetric and antisymmetric tensor states, respectively, as we will show in Sec. IV.

If we diagonalize the mixing problem, we can obtain physical masses in terms of three unknown parameters, $m_0$, $\alpha$, and $\beta$. When we eliminate these, we find various mass relations which have been given in Refs. 4 and 13. We should also mention that the mixing between $S$ and $A$ is very small[13] and that we can practically ignore its effect.

*(ii) Representation (1, 0, 0, −1).* This contains the following U(3) components:

$$(1,0,-1)\oplus(1,0,0)\oplus(0,0,-1)\oplus(0,0,0)=\underline{8}\oplus\underline{3}\oplus\underline{\overline{3}}\oplus\underline{1}\,. \tag{2.14}$$

Now, the operator $A_3^4A_4^3$ mixes $\underline{8}$ and $\underline{1}$ multiplets. After some calculations, we find the mass formulas

$$\begin{aligned}
&\underline{8}:\ M(8)=m_0+\beta[I(I+1)-\tfrac{1}{4}Y^2],\\
&\underline{1}:\ M(1)=m_0+\tfrac{1}{4}\beta(7-9y),\\
&\underline{1}-\underline{8}:\ M(1-8)=\frac{1}{\sqrt{2}}\beta(|\omega_0\rangle\langle\omega_8|+|\omega_8\rangle\langle\omega_0|),\\
&\underline{3},\underline{\overline{3}}:\ M(3)=M(\overline{3})\\
&\qquad=m_0+\tfrac{1}{2}\beta(1-3y)+3\beta[I(I+1)-\tfrac{1}{4}Y^2],
\end{aligned} \tag{2.15}$$

where we assumed the charge-conjugation invariance of the mass. Diagonalizing the mixing in the sector containing $\underline{1}$ and $\underline{8}$, and eliminating unknown parameters $m_0$ and $\beta$, we find one mass relation,

$$[\phi-\tfrac{1}{3}(4K^*-\rho)][\omega-\tfrac{1}{3}(4K^*-\rho)]=-\tfrac{2}{9}(K^*-\rho)^2\,, \tag{2.16}$$

which has been originally found by Bjorken and Glashow[2] and by Gerstein and Whippman.[19] Note that this formula differs from the more familiar Schwinger mass formula[19] by a factor of 4 on the right-hand side. For the $1^-$ nonet, this formula is badly satisfied. Moreover, (2.15) predicts very low masses for charmed mesons with masses lower than that of $\rho$. Hence, the pure 15-plet assignment for $1^-$ mesons is ruled out. Indeed, a more popular assignment[20,21] is to assume the $1^-$ nonet as well as $\psi$(3100 MeV) form a part of a $(\underline{15}\oplus\underline{1})$-dimensional multiplet of the U(4) group. In this case, we have to consider an additional mixing between the 15-plet and the singlet. Since this problem has been discussed by many authors,[20,21] I will not repeat it.

With respect to $0^-$ mesons, the mass formula (2.16) is well satisfied for nonets $\pi$, $K$, $\overline{K}$, $\eta$, and $\eta'$(960 MeV). However, (2.15) again predicts then very low masses ($\approx$ 750 MeV) for charmed mesons. Besides, this assignment gives the value $y=2.43$, which is at variance with the value $y=20.7$ determined[13,21] from $\underline{15}\oplus\underline{1}$ plets of $1^-$ mesons. For these reasons, we have to assign $\underline{15}\oplus\underline{1}$ structure rather than a pure 15-plet also for $0^-$ bosons.

### III. SU(8) MASS FORMULA

As a generalization of the SU(6) quark model,[22] we may consider the SU(8) group.[23]

Let us designate the capital Latin indices $A$ and $B$ to represent pairs such as

$$A=(\mu,j)\,,\quad B=(\nu,k)\,, \tag{3.1}$$

where the Greek indices $\mu,\nu=1,2,3,4$ refer to the U(4) group, while $j,k=1,2$ refer to the U(2) spin subgroup of the U(8). In order to avoid possible confusion, we shall now use a notation $X_B^A$ $(A,B=1,2,\ldots,8)$ to represent infinitesimal generators of the U(8) group.

As usual, we assume that the U(8)-breaking interaction is given by

$$H_1=T^{\{3,j\}}_{\{3,j\}}+yT^{\{4,j\}}_{\{4,j\}}+S^{\{\mu,j\},\{\nu,k\}}_{\{\mu,k\},\{\nu,j\}}\,, \tag{3.2}$$

where the tensor $S_{CD}^{AB}$ $(A,B,C,D=1,2,\ldots,8)$ represents spin-spin interaction between two quarks, and repeated indices over $j$ and $k$ and over $\mu$ and $\nu$ imply automatic summations on values 1, 2, and on values 1, 2, 3, 4, respectively.

We can write down the most general expressions for $H_1$ in terms of the U(8) generators $X^A_B$. However, the result is too complicated with little gain so that we shall concentrate on the most interesting case of the completely symmetric baryon representation with $f_1=3$, $f_j=0$ ($j=2,3,\ldots,8$), corresponding to a 120-dimensional degenerate representation. As has been noted by many authors,[23] this has a nice decomposition,

$$\underline{120}=(\underline{20}',2)\oplus(\underline{20},4)\,,$$

with respect to the chain $U(8)\supset U(4)\otimes SU(2)$. For the completely symmetric representation, we can express $(XX)^E_F$ in terms of $\delta^E_F$ and $X^E_F$ because of (1.18′). Also, (1.19) is now rewritten as

$$X^B_C X^F_G - X^B_G X^F_C = \delta^B_G X^F_C - \delta^B_C X^F_G\,. \qquad (3.3)$$

Defining U(4) and U(2) generators by[9]

$$A^\mu_\nu=\sum_{j=1}^{2} X\{^{\mu,j}_{\nu,j}\}\,,\quad \mu,\nu=1,2,3,4$$
$$B^j_k=\sum_{\mu=1}^{4} X\{^{\mu,j}_{\mu,k}\}\,,\quad j,k=1,2 \qquad (3.4)$$

the general mass formula based upon (3.2) is rewritten as

$$M=m_0+\alpha(A^3_3+yA^4_4)+\beta J(J+1)\,, \qquad (3.5)$$

if we use (3.3) and (1.18) together with (3.4). Here, $J$ is the spin of the system given by

$$J(J+1)=\tfrac{1}{2}\sum_{j,k=1}^{2} B^j_k B^k_j-\tfrac{1}{4}B^2\,,$$
$$B=\sum_{j=1}^{2}B^j_j=\sum_{\mu=1}^{4}A^\mu_\mu\,. \qquad (3.6)$$

Our formula (3.5) predicts exactly the same mass relations as the ordinary simple quark model. Especially, it leads to $m(\Lambda)=m(\Sigma)$, which is typical of the quark model. If we want to remove this degeneracy, then we have only to introduce an additional interaction of the form

$$H_2'=S\{^{3,j}_{3,k}\}{}^{,\{\nu,k\}}_{,\{\nu,j\}}+T\{^{4,j}_{4,j}\}{}^{,\{4,k\}}_{,\{4,k\}}\,. \qquad (3.7)$$

Then, again utilizing the identity (3.3), the effect of $H_2'$ is essentially equivalent to adding new terms

$$\alpha'(AA)^3_3+\beta'(N_4)^2+\gamma' N_3+\delta' N_4+\epsilon' E \qquad (3.8)$$

to the right-hand side of (3.5). The evaluation of the matrix element of $(AA)^3_3$ has been already performed in the previous section. The final formula is very analogous to (but slightly different from) mass formulas proposed by some authors.[23]

With respect to bosons, we may identify them to belong to a representation with a signature $f_1=1$, $f_2=f_3=\cdots=f_7=0$, and $f_8=-1$. Then, five signatures $f_2$, $f_3$, $f_4$, $f_5$, and $f_6$ are redundant so that our redundant factor rule now demands the validity of

$$(XXX)^A_B=a\delta^A_B E+bX^A_B+c(XX)^A_B\,. \qquad (3.9)$$

However, the evaluation of these matrix elements is rather involved. Besides, we have an additional complication of the following nature. The multiplet decomposes into[23]

$$\underline{63}=(\underline{1},3)\oplus(\underline{15},3)\oplus(\underline{15},1)$$

under $U(8)\to U(4)\otimes SU(2)$. This implies that we have 15-plets of $1^-$ and $0^-$ mesons as well as a $1^-$ singlet. However, a pure 15-plet for the $0^-$ mesons leads to various undesirable results, as we have already noted in the previous section. Therefore, we have to consider the $\underline{63}\oplus\underline{1}$ multiplet instead of a pure $\underline{63}$-plet assignment. This gives an additional mixing problem, just as in the U(4) case. The treatment of such a case can be better handled by the tensor method which will be explained in the next section. However, we will not discuss the boson case further in this note.

## IV. TENSOR REPRESENTATIONS

In many calculations involving the U(4) group, the tensor notation is quite often very convenient. First, let us consider the representation (3, 0, 0, 0), corresponding to the completely symmetric tensor $S_{\mu\nu\lambda}$,

$$S_{\mu\nu\lambda}=S_{\nu\mu\lambda}=S_{\lambda\nu\mu}=S_{\mu\lambda\nu}\,. \qquad (4.1)$$

Then, the properly orthonormalized state is simply given by

$$|n_1,n_2,n_3,n_4\rangle=\left[\frac{3!}{n_1!n_2!n_3!n_4!}\right]^{1/2}S_{\mu\nu\lambda}\,, \qquad (4.2)$$

where $n_j$ is the number of the integer $j$ (= 1, 2, 3, 4) contained in the indices $\mu,\nu,\lambda$. For example, we have $n_1=2$, $n_2=1$, $n_3=n_4=0$ for $S_{112}$.

A slightly more complicated case is the representation (2, 1, 0, 0). This is specified by a tensor $\psi_{\mu\nu\lambda}$ satisfying conditions

$$\psi_{\mu\nu\lambda}+\psi_{\nu\lambda\mu}+\psi_{\lambda\mu\nu}=0\,, \qquad (4.3)$$

$$\psi_{\mu\nu\lambda}=\psi_{\nu\mu\lambda}\,. \qquad (4.4)$$

Properly orthonormalized states under the chain decomposition (2.3) are given below:

*(a)* $\underline{8}$: $(C=0)$.

(i) $I=1$, $Y=0$:

$$\Sigma^+=\psi_{113}\,,\quad \Sigma^0=\sqrt{2}\,\psi_{123}\,,\quad \Sigma^-=\psi_{223}\,;$$

(ii) $I=0$, $Y=0$: $\Lambda=(\tfrac{2}{3})^{1/2}(\psi_{321}-\psi_{312})$; (4.5a)

(iii) $I=\tfrac{1}{2}$, $Y=1$: $p=\psi_{112}$, $n=\psi_{221}$;

(iv) $I=\tfrac{1}{2}$, $Y=-1$: $\Xi^0=\psi_{331}$, $\Xi^-=\psi_{332}$.

*(b)* $\underline{6}$*: (C = 1).*

(i) $I=1$, $Y=0$:

$$B^{++}=\psi_{114}\,,\quad B^{+}=\sqrt{2}\,\psi_{124}\,,\quad B^{0}=\psi_{224}\,;$$

(ii) $I=\frac{1}{2}$, $Y=-1$: (4.5b)

$$B^{+}\equiv S_1=\sqrt{2}\,\psi_{134}\,,\quad B^{0}\equiv S_2=\sqrt{2}\,\psi_{234}\,;$$

(iii) $I=0$, $Y=-2$: $B^0=\psi_{334}$.

*(c)* $\underline{\overline{3}}$*: (C = 1).*

(i) $I=\frac{1}{2}$, $Y=-1$:

$$B^{+}\equiv A_1=(\tfrac{2}{3})^{1/2}(\psi_{413}-\psi_{431})\,, \tag{4.5c}$$

$$B^{0}\equiv A_2=(\tfrac{2}{3})^{1/2}(\psi_{423}-\psi_{432})\,;$$

(ii) $I=0$, $Y=0$: $B^{+}\equiv A^{+}=(\frac{2}{3})^{1/2}(\psi_{421}-\psi_{412})$.

*(d)* $\underline{3}$*: (C = 2).*

(i) $I=\frac{1}{2}$, $Y=-1$: $B^{++}=\psi_{441}$, $B^{+}=\psi_{442}$; (4.5d)

(ii) $I=0$, $Y=-2$: $B^{+}=\psi_{443}$.

For $\underline{6}$ and $\overline{3}$, $S_j$ and $A_j$ ($j=1,2$) are the same objects we encountered in Sec. II. Actually, we have to diagonalize the mass matrix as a linear combination of $S_j$ and $A_j$ because of the mixing. However, since the mixing is very small,[13] we can neglect the complication in practice.

We have normalized our states as

$$\sum_{j=1}^{20} N_j^* N_j=\tfrac{2}{3}\sum_{\lambda,\mu,\nu=1}^{4}\psi^*_{\mu\nu\lambda}\psi_{\mu\nu\lambda}\,, \tag{4.6}$$

where $N_j$ ($j=1,2,\ldots,20$) are 20 states listed in (4.5). This can be easily shown if we utilize Eqs. (4.3) and (4.4), which especially lead to a relation such as

$$\psi_{jjk}=-2\psi_{kjj}=-2\psi_{jkj}\,,\quad j\neq k\,.$$

We can easily reproduce the mass formula of the second section by the tensor method[17] if we write the mass operator as

$$M=\tfrac{2}{3}m_0'\sum_{\lambda,\mu,\nu=1}^{4}\psi^*_{\mu\nu\lambda}\psi_{\mu\nu\lambda} +\alpha'\sum_{\mu,\nu=1}^{4}(\psi^*_{3\mu\nu}\psi_{3\mu\nu}+y\psi^*_{4\mu\nu}\psi_{4\mu\nu}) +\beta'\sum_{\mu,\nu=1}^{4}(\psi^*_{3\mu\nu}\psi_{3\nu\mu}+y\psi^*_{4\mu\nu}\psi_{4\nu\mu})\,, \tag{4.7}$$

with some unknown parameters $m_0'$, $\alpha'$, and $\beta'$.

We remark that the $(2,1,0,0)$ representation can be also represented by another tensor $\phi_{\mu\nu\lambda}$ satisfying

$$\phi_{\mu\nu\lambda}+\phi_{\nu\lambda\mu}+\phi_{\lambda\mu\nu}=0\,, \tag{4.8}$$

$$\phi_{\mu\nu\lambda}=-\phi_{\nu\mu\lambda} \tag{4.9}$$

instead of $\psi_{\mu\nu\lambda}$ satisfying (4.3) and (4.4). However, they are related to each other by

$$\phi_{\mu\nu\lambda}=\frac{1}{\sqrt{3}}(\psi_{\mu\lambda\nu}-\psi_{\nu\lambda\mu})\,,\qquad \psi_{\mu\nu\lambda}=\frac{1}{\sqrt{3}}(\phi_{\mu\lambda\nu}+\phi_{\nu\lambda\mu})\,. \tag{4.10}$$

In terms of $\phi_{\mu\nu\lambda}$, we can rewrite (4.5). For example, we find

$$\Lambda=\sqrt{2}\,\phi_{213}\,,\quad \Sigma_0=(\tfrac{2}{3})^{1/2}(\phi_{132}+\phi_{231})\,.$$

Next, as is well known,[5] all U(4) representations with signature $(f_1+e,f_2+e,f_3+e,f_4+e)$ represent the same single SU(4) representation for all integral values of $e$. Especially choosing $e=-1$, we see that $(2,1,0,0)$ is equivalent to $(1,0,-1,-1)$ as far as the SU(4) subgroup is concerned. This implies that we could have used a traceless tensor $T_\lambda^{\mu\nu}$ satisfying

$$T_\lambda^{\mu\nu}=-T_\lambda^{\nu\mu}\,,\quad \sum_{\lambda=1}^{4}T_\lambda^{\lambda\nu}=0 \tag{4.11}$$

instead of $\psi_{\mu\nu\lambda}$ or $\phi_{\mu\nu\lambda}$ to describe the $J^P=\frac{1}{2}^+$ baryon multiplet. Indeed, we can achieve this fact by setting

$$T_\lambda^{\mu\nu}=\tfrac{1}{2}\sum_{\alpha,\beta=1}^{4}\epsilon^{\mu\nu\alpha\beta}\phi_{\alpha\beta\lambda}\,. \tag{4.12}$$

Conversely, we can express $\phi_{\mu\nu\lambda}$ by

$$\phi_{\mu\nu\lambda}=\tfrac{1}{2}\sum_{\alpha,\beta=1}^{4}\epsilon_{\mu\nu\alpha\beta}T_\lambda^{\alpha\beta}\,, \tag{4.13}$$

where $\epsilon_{\mu\nu\alpha\beta}$ and $\epsilon^{\mu\nu\alpha\beta}$ are completely antisymmetric tensors which are SU(4)-invariant but have signatures $(1,1,1,1)$ and $(-1,-1,-1,-1)$ respectively under the U(4) group. In terms of the quark model, $\phi_{\mu\nu\lambda}$ and $\psi_{\mu\nu\lambda}$ can be regarded as a three-quark system $q_\mu q_\nu q_\lambda$, while the tensor $T_\lambda^{\mu\nu}$ corresponds to $\overline{q}_\mu\overline{q}_\nu q_\lambda$. Therefore, we conclude that the $J^P=\frac{1}{2}^+$ baryon multiplet could be represented by either of the three-quark systems with form $q_\mu q_\nu q_\lambda$ or $\overline{q}_\mu\overline{q}_\nu q_\lambda$. We note that the latter form has been suggested by most authors[2] in the early formulation of the SU(4) theory. As far as the SU(4) symmetry is concerned, we cannot tell the difference between the two forms. However, other quantities such as electric charge and/or baryon quantum numbers depend upon the U(4) assignment rather than the SU(4), and they could lead to different consequences. In view of various successes of the $q_\mu q_\nu q_\lambda$ quark model, we take, however, the view that the baryon is really represented by the $(2,1,0,0)$ rather than $(1,0,-1,-1)$ representations of the U(4) group. We especially note that for the latter case, the $J^P=\frac{3}{2}^+$ multiplet must correspond to $(2,-1,-1,-1)$ [rather than the $(3,0,0,0)$ multiplet], which corresponds to a complicated tensor $T^{\mu\nu\lambda}_{\alpha\beta} \simeq \bar{q}_\mu \bar{q}_\nu \bar{q}_\lambda q_\alpha q_\beta$.

With respect to the $(\underline{15} \oplus \underline{1})$-dimensional $0^-$ mesons, they can be represented by a tensor $P^\mu_\nu$ which is *not* traceless. In the quark model we can write it in a form $P^\mu_\nu = \bar{q}_\mu q_\nu$. A similar remark also applies to the $1^-$ vector multiplet. Since the mass formulas based upon this identification have been investigated by various authors,[14,20,21] we will not go into detail.

## V. APPLICATIONS TO WEAK INTERACTIONS

In the model of Glashow, Iliopoulous, and Maiani[3] [hereafter referred to as (GIM)], the charged hadronic weak currents are given by

$$j_\lambda = \cos\theta[\bar{q}_1 Q_\lambda q_2 + \bar{q}_4 Q_\lambda q_3] + \sin\theta[\bar{q}_1 Q_\lambda q_3 - \bar{q}_4 Q_\lambda q_2], \tag{5.1}$$

$$\bar{j}_\lambda = \cos\theta[\bar{q}_2 Q_\lambda q_1 + \bar{q}_3 Q_\lambda q_4] + \sin\theta[\bar{q}_3 Q_\lambda q_1 - \bar{q}_2 Q_\lambda q_4], \tag{5.2}$$

$$Q_\lambda = \gamma_\lambda(1+\gamma_5)$$

in terms of four quarks $q_\mu$, where $\theta$ is the Cabibbo angle. Then, the nonleptonic weak interaction responsible for $\Delta S \neq 0$ and/or $\Delta C \neq 0$ is given by

$$H_W = \frac{1}{\sqrt{2}} G_F j_\lambda \bar{j}_\lambda . \tag{5.3}$$

Also, the neutral current $j^{(0)}_\lambda$ and electromagnetic current $j^{(\mathrm{em})}_\lambda$ are expressed by

$$j^{(0)}_\lambda = \bar{q}_1 Q_\lambda q_1 + \bar{q}_4 Q_\lambda q_4 - \bar{q}_2 Q_\lambda q_2 - \bar{q}_3 Q_\lambda q_3 , \tag{5.4}$$

$$j^{(\mathrm{em})}_\lambda = \tfrac{2}{3}(\bar{q}_1 \gamma_\lambda q_1 + \bar{q}_4 \gamma_\lambda q_4) - \tfrac{1}{3}(\bar{q}_2 \gamma_\lambda q_2 + \bar{q}_3 \gamma_\lambda q_3) . \tag{5.5}$$

Let us now consider the Weyl reflection,

$$W:\ q_1 \leftrightarrow -q_2 ,\quad q_3 \leftrightarrow q_4 , \tag{5.6}$$

which is a special finite U(4) transformation, so that it defines an inner automorphism of the U(4) group. Under this operation, we find

$$W:\ j_\lambda \leftrightarrow \bar{j}_\lambda ,\quad j^{(0)}_\lambda \leftrightarrow -j^{(0)}_\lambda . \tag{5.7}$$

However, $j^{(\mathrm{em})}_\lambda$ does not transform into itself. When we define

$$A^+ = (\tfrac{2}{3})^{1/2}(\psi_{421} - \psi_{412}) , \quad D^+ = P^2_4 ,\quad D^0_1 = \frac{1}{\sqrt{2}}(P^1_4 - P^4_1) , \tag{5.8}$$

then under $W$, these transform as

$$W:\ A^+ \leftrightarrow -\Lambda,\ p \leftrightarrow -n,\ \pi^+ \leftrightarrow \pi^- . \quad D^+ \leftrightarrow -\bar{K}^+ ,\ D^0_1 \leftrightarrow -K^0_1 ,\ \pi^0 \leftrightarrow -\pi^0 . \tag{5.9}$$

Note that $A^+$ is the particle with $I=0$ and $Y=0$ belonging to the representation $\bar{3}$, which Gaillard *et al.*[4] designated as $C^+_0$. The transformation $A^+ \leftrightarrow \Lambda$ is obvious from the explicit tensor representation (4.5).

Because of (5.7) and (5.9), we have relations

$$M(A^+ \to p\pi^0) = -M(\Lambda \to n\pi^0) , \quad M(A^+ \to n\bar{l}\nu) = M(\Lambda \to pl\bar{\nu}) \tag{5.10}$$

for the decay matrix elements of $A^+$ particles and

$$M(D^+ \to \pi^+\pi^+\pi^-) = -M(\bar{K}^+ \to \pi^-\pi^-\pi^+) ,\quad M(D^+ \to \pi^+\pi^0) = M(\bar{K}^+ \to \pi^-\pi^0) ,\quad M(D^0_1 \to \pi^+\pi^-) = -M(K^0_1 \to \pi^-\pi^+) ,\quad M(D^+ \to \pi^0 \bar{l}\nu) = M(\bar{K}^+ \to \pi^0 l\bar{\nu}) \tag{5.11}$$

for the decays of $D^{+,0}$ in the exact U(4) limit.

We can compute the decay rate of $A^+ \to p\pi^0$ as well as its asymmetry parameter $\alpha_A$ from the corresponding decay $\Lambda \to n\pi^0$ by means of (5.10). However, we do not know how good or how bad the exact U(4) will be. Here, we take a view that when we express decay matrix elements such as $M(A^+ \to p\pi^0)$ in terms of Lorentz-invariant amplitudes, then we can apply the consequences of the exact U(4) symmetry for the amplitudes if they are dimensionless quantities. For example, writing

$$M(A^+ \to p\pi^0) = \left(\frac{M_A M_p}{2E_A E_p E_\pi V^3}\right)^{1/2} \bar{u}_p (S + \gamma_5 P) u_A ,$$

we find that Lorentz-invariant amplitudes $S$ and $P$ are also dimensionless. Therefore, we can compute them from the known decay parameters[24] of $\Lambda \to n\pi^0$. Assuming $M_A \approx 2.90$ GeV, as is determined[13] from the quadratic baryon mass formula, then we estimate $\alpha_A \approx 0.5$ for the asymmetry parameter and

$$\Gamma(A^+ \to p\pi^0) \simeq 1.2 \times 10^{11}\ \mathrm{sec}^{-1} \tag{5.12}$$

for the decay rate.

Also, the same method is applicable for $D^+ \to \pi^+\pi^+\pi^-$ decay and we find

$$\Gamma(D^+ \to \pi^+\pi^+\pi^-) \simeq 3.5 \times 10^8\ \mathrm{sec}^{-1} , \tag{5.13}$$

where we assumed $M(D) \simeq 1.8$ GeV for a reason which will become apparent soon. Note that the SU(4) mass formula predicts $M(D) \simeq 2.1$ GeV. With respect to the decay $D^+ \to \pi^+\pi^0$, the Lorentz-invariant amplitude has a dimension of the mass. Hence, we assume that in applying the exact U(4), it is proportional to the mass of the decaying parent meson. Under this assumption, (5.11) enables us to compute

$$\Gamma(D^+ \to \pi^+\pi^0) \simeq 7.5 \times 10^7\ \mathrm{sec}^{-1} ,\quad \Gamma(D^0_1 \to \pi^+\pi^-) \simeq 3.4 \times 10^{10}\ \mathrm{sec}^{-1} \tag{5.14}$$

from the known decay rate of $K^+ \to \pi^+\pi^0$ and $K_1^0 \to \pi^+\pi^-$. All these rates (5.12), (5.13), and (5.14) are somewhat small, since they are really the minor decay modes obeying the $\Delta S = 0$ rule. The dominant decays in the GIM model are those[4] satisfying the $\Delta S = \pm 1$ rule, which are expected to be larger by a factor of $\cot^2\theta$ in comparison to the former.

Niu *et al.*[25] discovered a particle $X^+$ decaying into $\pi^0$ + charged. If they identify the exact decay mode as the $X^+ \to p\pi^0$ decay, then they find $M(X) = 2.95$ GeV with a lifetime of $3.6\times 10^{-14}$ sec. On the other hand, if the decay is $X^+ \to \pi^+\pi^0$, then we must have $M(X) \simeq 1.8$ GeV, with a lifetime of $2.2\times 10^{-14}$ sec. We note that the former mass is very near to the SU(4) mass value of $m(A^+) = 2.9$ GeV. Hence, we identify it with our $A^+$ particle. In that case, we can compute the decay branching ratio from (5.12) to be

$$R = \frac{\Gamma(A^+ \to p\pi^0)}{\Gamma(A^+ \to \text{all})} \simeq 0.4\times 10^{-2}, \tag{5.15}$$

which is small but not unreasonably so. On the other hand, if $X^+$ is really $D^+$, then we compute

$$R = \frac{\Gamma(D^+ \to \pi^+\pi^0)}{\Gamma(D^+ \to \text{all})} \simeq 1.6\times 10^{-6}, \tag{5.16}$$

which is too small for a single event $D^+ \to \pi^+\pi^0$ to be fortuitously discovered. Hence, the identification of the particle of Niu *et al.*, with the baryon $A^+$, is perhaps more realistic, unless it corresponds to other particles such as $V_4^2$, which is the $1^-$ counterpart of $P_4^2 = D^+$.

As another application of our $W$ symmetry, we find

$$\langle X|j_\lambda|p\rangle = \langle \tilde{X}|\bar{j}_\lambda|n\rangle,$$
$$\langle X|j_\lambda|n\rangle = \langle \tilde{X}|\bar{j}_\lambda|p\rangle \tag{5.17}$$

for any state $X$, where $\tilde{X}$ is a $W$ conjugate of $X$, i.e.,

$$\tilde{X} = WX. \tag{5.18}$$

For example, if we identify $X = K^+ = P_1^3$, then we find $\tilde{X} = -P_2^4 = -D^+$. In other words, if $X$ is a strange particle, then $\tilde{X}$ is a charmed particle and vice versa. Therefore, roughly speaking, the validity of (5.17) indicates near equality of the charmed-particle production cross section with the strangeness-particle production cross section in the neutrino reactions. This fact may be relevant to the recent experiment[26] on muon-pair production by a neutrino. For the neutral neutrino reaction, we can write more precise relations,

$$\sigma(\nu T \to \nu + Y + \text{anything}) = \sigma(\nu T \to \nu + \tilde{Y} + \text{anything}),$$
$$\sigma(\bar{\nu} T \to \bar{\nu} + Y + \text{anything}) = \sigma(\bar{\nu} T \to \bar{\nu} + \tilde{Y} + \text{anything}), \tag{5.19}$$

for isoscalar target $T$ in the exact U(4) limit, where $Y(\tilde{Y} = WY)$ denotes any particular single particle, if $j_\lambda^{(0)}$ alone is responsible.

There is another interesting Weyl symmetry operation, $W'$, defined by

$$W':\ q_4 \to q_1 \to -q_4,$$
$$q_3 \to q_2 \to -q_3. \tag{5.20}$$

Under $W'$, all currents including the electromagnetic current $j_\mu^{em}$ are now invariant, i.e.,

$$W':\ j_\lambda \to j_\lambda,\quad \bar{j}_\lambda \to \bar{j}_\lambda,\quad j_\lambda^{(0)} \to j_\lambda^{(0)},\quad j_\lambda^{(em)} \to j_\lambda^{(em)}. \tag{5.21}$$

Noting that under $W'$ we have

$$W':\ K_2^0 \to -K_2^0,\quad K_1^0 \to K_1^0,\quad D_1^0 \to D_1^0, \tag{5.22}$$

this implies that all decay matrix elements for $K_2^0 \to \gamma\gamma$ and $K_2^0 \to \mu\bar{\mu}$ as well as the $K_1^0 \to K_2^0$ transition are forbidden in the exact U(4) limit. This fact has been previously noted by Gaillard and Lee[27] on the basis of an SU(2) subgroup of the U(4). As we noted, the neutral current $j_\mu^{(0)}$ is invariant under $W'$. Conversely, if we demand that $j_\mu^{(0)}$ be $W'$-invariant with $CP = +1$ parity, then it would be easy to show that $j_\mu^{(0)}$ cannot contain any troublesome strangeness-changing component. Besides, it must be a U-spin scalar.[28] Therefore, the $W'$ invariance is perhaps one of the important ingredients of the usefulness of the GIM model, since it can effectively determine many essential features of all hadronic currents.

When we combine $W$ and $W'$ symmetries, then we can generate another Weyl symmetry,

$$W'' = W'W = -WW':\ q_1 \to q_3,\quad q_2 \to q_4, \tag{5.23}$$

which may be useful for some weak interaction processes.

Last, we briefly mention a new weak interaction scheme of Goto and Mathur,[29] who assign electric charges $\frac{2}{3}$, $-\frac{1}{3}$, $-\frac{1}{3}$, and $-\frac{4}{3}$ for four quarks, $q_1$, $q_2$, $q_3$, and $q_4$, respectively. In this theory, the weak charged current is now given by

$$j_\lambda = \cos\theta[\bar{q}_1 Q_\lambda q_2 + \bar{q}_2 Q_\lambda q_4] + \sin\theta[\bar{q}_1 Q_\lambda q_3 + \bar{q}_3 Q_\lambda q_4]$$

rather than (5.1). Unfortunately, all Weyl symmetry operations so far considered are *no longer* symmetry of the new theory, so that we *cannot* derive any analogous relations. However, the operation

$$W''': \ q_1 \leftrightarrow q_4, \ \ q_2 \leftrightarrow q_2, \ \ q_3 \leftrightarrow q_3$$

leaves

$$j_\lambda \leftrightarrow \bar{j}_\lambda, \ \ j_\lambda^{(0)} \leftrightarrow -j_\lambda^{(0)},$$

which may be useful for some problems.

*Note added in proof.* After this paper had been completed, many earlier works related to the SU (4) group in particle physics came to my attention. Following Ref. 29 I will cite these papers up to 1972 other than those to which I have already referred.

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*Work supported in part by the U. S. Atomic Energy Commission.

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