Generalized isobar model formalism*

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We present an isobar model formalism for analyzing the reaction $a + b \to 1 + 2 + 3$. Arbitrary spins are allowed for all the particles. Polarized particles and weak decays of an outgoing particle are discussed. We also show how to extend the formalism to allow an isobar analysis of a three-body subsystem of an n-particle final state.

INTRODUCTION

In this paper we discuss a general formalism for analyzing reactions of the form

$$a+b\to 1+2+3$$

using the isobar model. Previous work either has specialized to the case of $\pi N\to N\pi\pi$[1-9] or only covers parts of the formalism.[10-12] Our formalism is completely general in that it allows arbitrary spins for all the particles. The formalism was developed for an analysis of $\pi N\to N\pi\pi$ data[13] which appears as a companion paper.

In Sec. I we establish our notation and normalization of states, review the angular momentum decomposition of two-particle states, and develop formulas for phase space and differential cross sections. Section II deals with the T-matrix elements themselves and derives the equations for the differential and total cross sections. Section III deals with polarized particles, either incident or final, and with weak decays of an outgoing particle. Section IV treats the problem of analyzing a three-body subsystem of an n-body final state. The appendixes include a review of angular momentum, a discussion of the reaction $a+b\to c+d$ using our notation, and the details of some of the more important derivations.

I. NOTATION

In this section we establish our notation. We consider the reaction $a+b\to 1+2+3$, where a is the beam, b is the target, and 1, 2, 3 are the three outgoing particles. We let j, k, and l represent any cyclic permutation of 1, 2, and 3. The diparticle is always composed of particles k and l. All quantities pertaining to the diparticle are indexed by a subscript j. The following quantities are summarized in Fig. 1:

(a) total center-of-mass energy and angular momentum: W, J;
(b) c.m. four-momenta: $p_a p_b Q_j Q_k Q_l$;
(c) particle spins: $\sigma_a\sigma_b\sigma_j\sigma_k\sigma_l$;
(d) c.m. helicities: $\mu_a\mu_b\mu_j\mu_k\mu_l$;
(e) mass diparticle: w_j;
(f) spin and c.m. helicity of the diparticle: $j_j\lambda_j$;
(g) incident orbital angular momentum and total spin: L, S;
(h) outgoing orbital angular momentum and total spin: L_j, S_j.

In the diparticle rest frame we have the quantities
(i) four-momenta of the decay particles: $q_k q_l$;
(j) helicities of the decay particles: $\nu_k\nu_l$;
(k) orbital angular momentum and total spin of decay particles: l_j, s_j.

Angular momenta are coupled in the following manner:

$$\vec{S}=\vec{\sigma}_a+\vec{\sigma}_b\ ,$$
$$\vec{J}=\vec{L}+\vec{S}\ ,$$
$$\vec{s}_j=\vec{\sigma}_k+\vec{\sigma}_l\ ,$$
$$\vec{j}_j=\vec{l}_j+\vec{s}_j\ ,$$
$$\vec{S}_j=\vec{\sigma}_j+\vec{j}_j\ ,$$
$$\vec{J}=\vec{L}_j+\vec{S}_j\ .$$

We assume that L, L_j, and l_j are chosen so as to conserve parity. We use μ to represent a fixed set $(\mu_a\mu_b\mu_j\mu_k\mu_l)$ of all five helicities. For simplification in later sections, n represents the set of quantities

$$n\equiv(j;J;LS;L_jS_j;j_jl_js_j)\ , \qquad (1)$$

where j specifies the grouping of the final-state particles into a single one (j) and the pair (kl).

We use the helicity formalism with the phase convention of Jacob and Wick[14] (hereafter called JW).

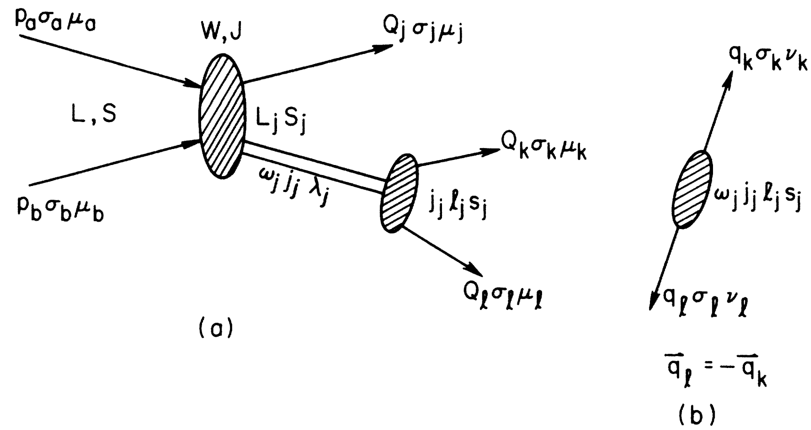

FIG. 1. Notation for the reaction $a + b \to j + k + l$. (a) Quantities in the center-of-mass rest frame. (b) Quantities in the diparticle rest frame.

A. Particle states, phase spaces, and cross sections

One-particle states are defined with the phase convention of JW although the normalization is different. If $\Psi_{p\lambda}$ represents a state with momentum p along the z axis and helicity λ, then the general state is defined by [cf. JW Eq. (6)]

$$|\vec{p}\lambda\rangle = |p\,\theta\phi, \lambda\rangle = R(\phi, \theta, -\phi)\Psi_{p\lambda} . \tag{2}$$

We choose the normalization to be

$$\langle p'\theta'\phi', \lambda' | p\,\theta\phi, \lambda\rangle = 2E\delta^3(\vec{p}\,' - \vec{p})\delta_{\lambda\lambda'} , \tag{3}$$

which differs from JW by the factor $2E/(2\pi)^3$. We also define states $\chi_{p\lambda}$ by

$$\chi_{p\lambda} = (-1)^{s-\lambda} R(0, \pi, 0)\Psi_{p\lambda} = (-1)^{s-\lambda}\Psi_{-p\lambda} . \tag{4}$$

The general χ state is given by

$$|-\vec{p}\lambda\rangle = |-p\,\theta\phi, \lambda\rangle = R(\phi, \theta, -\phi)\chi_{p\lambda} . \tag{5}$$

We shall denote these states by the minus sign on p. Thus

$$|-p\,\theta\phi, \lambda\rangle = (-1)^{s-\lambda}|p\,\pi - \theta\,\phi + \pi, \lambda\rangle . \tag{6}$$

Clearly these states have the same normalization as $|p\,\theta\phi, \lambda\rangle$.

We also need to know how the states $|p\,\theta\phi, \lambda\rangle$ transform under Lorentz transformations. Let the Lorentz transformation be l, where $p' = lp$ and let $U(l)$ be the unitary operator for l. Wick[10] has shown that

$$U(l)|p\,\theta\phi, \lambda\rangle = \sum_\nu D^s_{\nu\lambda}(\Omega\hat{n})|p'\theta'\phi', \nu\rangle , \tag{7}$$

where Ω is called the Wigner angle and $\hat{n}$ is a unit vector along $\vec{p}\,'\times\vec{p}$ if Ω is always taken to be positive. This is clear, since in the transformation the momentum vector makes a positive rotation around the direction $\vec{p}\times\vec{p}\,'$ and the spin lags behind, thus making a negative rotation with respect to the momentum vector. We discuss Ω in detail in a later section.

Multiparticle states are defined as the direct product of one-particle states. Thus

$$|\vec{p}_1\lambda_1, \vec{p}_2\lambda_2, \ldots, \vec{p}_n\lambda_n\rangle = |p_1\theta_1\phi_1, \lambda_1\rangle\,|p_2\theta_2\phi_2, \lambda_2\rangle \cdots |p_n\theta_n\phi_n, \lambda_n\rangle \tag{8}$$

and

$$\langle \vec{p}_1'\lambda_1', \vec{p}_2'\lambda_2', \ldots, \vec{p}_n'\lambda_n' | \vec{p}_1\lambda_1, \vec{p}_2\lambda_2, \ldots, \vec{p}_n\lambda_n\rangle = \prod_i (2E_i)\delta^3(\vec{p}_i' - \vec{p}_i)\delta_{\lambda_i\lambda_i'} . \tag{9}$$

For two-body states it is sometimes more convenient to use the variables

$$\vec{P} = \vec{p}_1 + \vec{p}_2 , \qquad \vec{p} = \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2) . \tag{10}$$

Letting $(p\,\theta\phi)$ be the polar coordinates of $\vec{p}$, we have

$$|\vec{P}, p\,\theta\phi, \lambda_1\lambda_2\rangle = |\vec{p}_1\lambda_1\rangle\,|\vec{p}_2\lambda_2\rangle , \tag{11}$$

with the states on the right-hand side either Ψ or χ states. These states are normalized such that

$$\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'} = \int \langle \vec{P}', p'\theta'\phi', \lambda_1'\lambda_2' | \vec{P}, p\,\theta\phi, \lambda_1\lambda_2\rangle \times \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2} . \tag{12}$$

Now

$$d^3p_1 d^3p_2 = d^3P d^3p = d^3P p^2 dp\, d^2\omega ,$$

where $d^2\omega = d\cos\theta d\phi$. If W is the total energy, $W = E_1 + E_2$, then

$$dW = \left[W + \vec{P}\cdot\vec{p}\left(\frac{E_2 - E_1}{2p^2}\right)\right]\frac{p\,dp}{E_1E_2} . \tag{13}$$

With these two relations it is easy to show that the normalization is

$$\langle \vec{P}', \vec{p}'\theta'\phi', \lambda_1'\lambda_2' | \vec{P}, p\theta\phi, \lambda_1\lambda_2\rangle = \frac{4}{p}\left[W + \vec{P}\cdot\vec{p}\left(\frac{E_2 - E_1}{2p^2}\right)\right]\delta(W' - W)\delta^3(\vec{P}' - \vec{P}) \times \delta^2(\omega' - \omega)\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'} . \tag{14}$$

In the center-of-mass system, $\vec{P} = 0$, so this reduces to

$$\langle \vec{P}' = 0, p'\theta'\phi', \lambda_1'\lambda_2' | \vec{P} = 0, p\theta\phi, \lambda_1\lambda_2\rangle = \frac{4W}{p}\delta(W' - W)\delta^3(\vec{P}' - \vec{P})\delta^2(\omega' - \omega)\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'} . \tag{15}$$

To discuss the decomposition of the two-particle states into angular momentum states, we work in the two-particle center-of-mass system and assume particle 2 to be in χ state. For this case, using Eq. (11), we have

$$|\vec{P} = 0, p\theta\phi, \lambda_1\lambda_2\rangle = |\vec{p}\lambda_1\rangle\,|-\vec{p}\lambda_2\rangle = R(\phi, \theta, -\phi)\Psi_{p\lambda_1}\chi_{p\lambda_2} . \tag{16}$$

We now define a state of total angular momentum J and z component M by

$$|\vec{P} = 0, pJM, \lambda_1\lambda_2\rangle = N_J \int D^{J*}_{M\lambda}(\phi, \theta, -\phi)|\vec{P} = 0, p\theta\phi, \lambda_1\lambda_2\rangle d^2 w , \tag{17}$$

where $\lambda = \lambda_1 - \lambda_2$. Using Eq. (15) and the normalization properties of the D functions, we have

$$\langle \vec{P}' = 0, p'J'M', \lambda_1'\lambda_2' | \vec{P} = 0, pJM, \lambda_1\lambda_2\rangle = N_J N_J^* \frac{4\pi}{2J+1}\frac{4W}{P}\delta_{JJ'}\delta_{MM'}\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'} \times \delta^3(\vec{P}' - \vec{P})\delta(W' - W) , \tag{18}$$

where W is the total c.m. energy. Thus we choose

$$N_J = \left(\frac{2J+1}{4\pi}\right)^{1/2}\left(\frac{p}{4W}\right)^{1/2} . \tag{19}$$

The factor $(p/4W)^{1/2}$ [cf. JW Eq. (22)] comes from our choice of normalization for the one-particle states. Using Eq. (17), the transformation matrix is

$$\langle \vec{P}' = 0, p'\theta\phi, \lambda_1'\lambda_2' | \vec{P} = 0, pJM, \lambda_1\lambda_2\rangle = \left(\frac{2J+1}{4\pi}\right)^{1/2}\left(\frac{p}{4W}\right)^{-1/2} D^{J*}_{M\lambda}(\phi, \theta, -\phi)\delta^3(\vec{P}' - \vec{P}) \times \delta(W' - W)\delta_{\lambda_1\lambda_1'}\delta_{\lambda_2\lambda_2'} . \tag{20}$$

In terms of the orbital and spin angular momentum, L and S, we have the standard expansion

$$|\vec{P} = 0, pJM, LS\rangle = \sum_{\lambda_1\lambda_2}\left(\frac{2L+1}{2J+1}\right)^{1/2} C(S_1, S_2, S|\lambda_1, -\lambda_2) \times C(L, S, J|0, \lambda_1 - \lambda_2)|\vec{P} = 0, pJM, \lambda_1\lambda_2\rangle , \tag{21}$$

with the normalization

$$\langle \vec{P}' = 0, p'J'M', L'S' | \vec{P} = 0, pJM, LS\rangle = \delta_{JJ'}\delta_{MM'}\delta_{LL'}\delta_{SS'}\delta^3(\vec{P}' - \vec{P})\delta(W' - W) . \tag{22}$$

If particle 1 is a photon, one instead usually uses the multipole expansion

$$|\vec{P} = 0, pJM, j\pi\rangle = \sum_{\lambda_1\lambda_2}\left[\frac{2j+1}{2(2J+1)}\right]^{1/2}(-1)^e C(j, S_2, J|\lambda_1, -\lambda_2) \times |\vec{P} = 0, pJM, \lambda_1\lambda_2\rangle , \tag{21a}$$

where the total (spin plus orbital) angular momentum and parity of the photon are j and $\pi = (-1)^{j+e}$, respectively. For $e = 0$, we have the electric 2^j pole and for $e = 1$, we have the magnetic 2^j pole. These states are normalized such that

$$\langle \vec{P}' = 0, p'J'M', j'\pi' | \vec{P} = 0, pJM, j\pi\rangle = \delta_{JJ'}\delta_{MM'}\delta_{jj'}\delta_{\pi\pi'}\delta^3(\vec{P}' - \vec{P})\delta(W' - W) . \tag{22a}$$

In the rest of this section we deal in detail with the numerical factors appearing in cross-section formulas. The normalization, Eqs. (3) and (10), is such that the number of particles of type i in a volume V is $2E_i V/(2\pi)^3$. In a volume V the total number of states available is $Vd^3p_i/(2\pi)^3$, so that the density of final states per particle is $d^3p_i/2E_i$. Thus the number of three-particle final states available, $d\rho_F$, is given by

$$d\rho_F = \left(\frac{d^3p_1}{2E_1}\right)\left(\frac{d^3p_2}{2E_2}\right)\left(\frac{d^3p_3}{2E_3}\right) . \tag{23}$$

The probability of transition (in all space and time, $Vt \to \infty$) is given by

$$|\delta^4(P_{out} - P_{in})M|^2 d\rho_F , \tag{24}$$

where M is the transition matrix element with our normalization of states, i.e., $M = \langle out|T|in\rangle$. The relation with the S matrix is

$$\langle out|S|in\rangle = \delta_{out,\,in} + i\delta^4(P_{out} - P_{in})\langle out|T|in\rangle . \tag{25}$$

Equation (24) then gives

$$|\delta^4(P_{out} - P_{in})M|^2 = \lim_{Vt\to\infty}\frac{Vt}{(2\pi)^4}\delta^4(P_{out} - P_{in})|M|^2 \tag{26}$$

so that the transition probability per unit volume per unit time is

$$(2\pi)^{-4}\delta^4(P_{\text{out}}-P_{\text{in}})|M|^2 d\rho_F \, . \tag{27}$$

With the normalization of Eq. (3), the incident flux is

$$\left[\frac{2E_a}{(2\pi)^3}\right]\left[\frac{2E_b}{(2\pi)^3}\right]\left(\frac{P_a}{E_a}+\frac{P_b}{E_b}\right)=\frac{4F}{(2\pi)^6} \, , \tag{28}$$

where F is the invariant flux factor and $F=[(p_a\cdot p_b)^2-m_a{}^2m_b{}^2]^{1/2}$. In the c.m. system, $F=pW$.

The density of final states $d\rho_F$ together with the δ^4 function of Eq. (27) gives

$$d\rho=\delta^4(P_{\text{out}}-P_{\text{in}})d\rho_F \, . \tag{29}$$

Berman and Jacob[15] have discussed this phase space and reduced it to

$$d\rho=\tfrac{1}{8}\,dE_1dE_2d\cos\Theta d\Phi d\alpha \, , \tag{30}$$

where Θ, Φ, and α are the Euler angles specifying the orientation of the final three-particle state with respect to the incident system. Equation (30) can be further manipulated to give

$$d\rho=\frac{1}{8}\frac{q_1Q_1}{2Ww_1}\,dw_1{}^2d\cos\theta_1 d\cos\Theta d\Phi d\alpha \tag{31a}$$

$$=\frac{1}{8}\,\frac{q_1Q_1}{W}\,dw_1d\cos\theta_1 d\cos\Theta d\Phi d\alpha \tag{31b}$$

$$=\tfrac{1}{8}(4W^2)^{-1}dw_1{}^2dw_2{}^2d\cos\Theta d\Phi d\alpha, \tag{31c}$$

where w_1 is the invariant mass of particles 2 and 3, and θ_1 is the angle between particles 1 and 2 in the (23) c.m. system, Q_1 is the momentum of particle 1 in the (123) c.m. system, and q_1 is the momentum of particles 2 or 3 in the (23) rest frame.

The differential cross section for the case of spinless particles is

$$d\sigma=\frac{\pi^2}{F}\,|M|^2d\rho \, , \tag{32}$$

which is the basic expression we use in the calculation of our formulas.

II. T-MATRIX ELEMENTS

Initially we discuss the reaction proceeding through just one intermediate isobar, i.e., j is always fixed at a certain value of 1, 2, or 3. Later we treat the case of more than one type of diparticle.

In terms of the transition operator, T, the matrix elements in the center of mass are

$$f_\mu=\langle\vec{Q}_j\mu_j,\vec{Q}_k\mu_k,\vec{Q}_l\mu_l|T|\vec{p}_a\mu_a,\vec{p}_b\mu_b\rangle \, . \tag{33}$$

The operator T is assumed to be the product of a production operator T_p and a decay operator T_d. Assuming that only two-body intermediate states are produced, we have

$$f_\mu=\sum_{\mu_m\mu_n}\int\frac{d^3Q_m}{2E_m}\,\frac{d^3Q_n}{2E_n}\,2E_n\delta^3(\vec{Q}_m+\vec{Q}_n)\langle\vec{Q}_j\mu_j,\vec{Q}_k\mu_k,\vec{Q}_l\mu_l|T_d|\vec{Q}_m\mu_m,\vec{Q}_n\mu_n\rangle\langle\vec{Q}_m\mu_m,\vec{Q}_n\mu_n|T_p|\vec{p}_a\mu_a,\vec{p}_b\mu_b\rangle \, . \tag{34}$$

Within the isobar model one assumes that the intermediate state consists of an isobar state recoiling against a single particle and that T_d operates only on the isobar state; therefore

$$\langle\vec{Q}_j\mu_j,\vec{Q}_k\mu_k,\vec{Q}_l\mu_l|T_d|\vec{Q}_m\mu_m,\vec{Q}_n\mu_n\rangle =2E_m\delta^3(\vec{Q}_j-\vec{Q}_m)\delta_{\mu_j\mu_m}\langle\vec{Q}_k\mu_k,\vec{Q}_l\mu_l|T_d|\vec{Q}_n\mu_n\rangle \, , \tag{35}$$

so that Eq. (33) becomes

$$f_\mu=\sum_{\lambda_j}\langle\vec{Q}_k\mu_k,\vec{Q}_l\mu_l|T_d|-\vec{Q}_j\lambda_j\rangle \times\langle\vec{Q}_j\mu_j,-\vec{Q}_j\lambda_j|T_p|\vec{p}_a\mu_a,\vec{p}_b\mu_b\rangle \, . \tag{36}$$

We have assumed the isobar to be in a χ state [cf. Eq. (5)] and have changed the isobar helicity to λ_j. One term represents the production of the isobar and particle j; the other term represents the decay of the isobar into particles k and l. We now discuss each term separately.

A. Production amplitudes

Since we have assumed the diparticle to be in a χ state, we use Eq. (20) to decompose both $|\vec{p}_a\mu_a,\vec{p}_b\mu_b\rangle$ and $\langle\vec{Q}_j\mu_j,-\vec{Q}_j\lambda_j|$ into angular momentum states. We have

$$\langle\vec{Q}_j\mu_j,-\vec{Q}_j\lambda_j|T_p|\vec{p}_a\mu_a,p_b\mu_b\rangle =\sum_{JM}\frac{2J+1}{4\pi}\left(\frac{4W}{p}\right)^{1/2}\left(\frac{4W}{Q_j}\right)^{1/2}D^{J*}_{M\mu_j-\lambda_j}(j)D^J_{M\mu_a-\mu_b}(\text{beam})\langle\vec{Q}=0,\,Q_jJM,\,\mu_j\lambda_j|T_p|\vec{P}=0,pJM,\,\mu_a\mu_b\rangle \, . \tag{37}$$

Since we have not as yet specified a coordinate system, we do not give angles as arguments of the D functions. We will return to this point later. Using Eq. (7) from Appendix A and converting from helicity states to LS states, we have

$$\langle \vec{Q}_j\mu_j, -\vec{Q}_j\lambda_j | T_p | \vec{p}_a\mu_a, \vec{p}_b\mu_b\rangle = \frac{W}{\pi}(pQ_j)^{-1/2} \sum_{\substack{JLS\\L_jS_j}} [(2L+1)(2L_j+1)]^{1/2} \times C(\sigma_a,\sigma_b,S|\mu_a,-\mu_b)C(L,S,J|0,\mu_a-\mu_b) \times C(\sigma_j,j_j,S_j|\mu_j,-\lambda_j)C(L_j,S_j,J|0,\mu_j-\lambda_j) \times D^J_{\mu_j-\lambda_j\,\mu_a-\mu_b}(j^{-1}\text{beam}) \times \langle \vec{Q}=0, Q_jJM, L_jS_j|T_p|\vec{P}=0,pJM, LS\rangle \ . \quad (38)$$

If particle a is a photon, instead of converting to LS states, one would prefer to couple to the multipole states defined in Sec. I. We may now use rotational invariance to write the reduced partialwave production matrix element as

$$\langle \vec{Q}=0, Q_jJM, L_jS_j|T_p|\vec{P}=0,pJM, LS\rangle = T^{Jj_j}_{LSL_jS_j}(W,w_j) \ . \quad (39)$$

B. Decay amplitude

The decay amplitude is most easily evaluated in the rest frame of the diparticle. We use Eq. (7) to transform the states. Recalling that the diparticle is in a χ state, its transformation is quite simple. (While the χ state reduces to a simple form in its rest frame, it also implies a fixed direction for the z axis, along the direction $\vec{Q}_j$ in this frame. The decay angles θ_j and ϕ_j are then the angles of $\vec{q}_k$ in this co ordinate system; i.e., only ϕ_j is unspecified, since the x and y axes are not yet defined.) Thus

$$\langle \vec{Q}_k\mu_k, \vec{Q}_l\mu_l|T_d|-\vec{Q}_j\lambda_j\rangle = \sum_{\nu_k\nu_l} D^{\sigma_k *}_{\nu_k\mu_k}(\theta^k_j\hat{n}_k) D^{\sigma_l *}_{\nu_l\mu_l}(\theta^l_j\hat{n}_l)\langle \vec{q}_k\nu_k, \vec{q}_l\nu_l|T_d|j_j-\lambda_j\rangle \ . \quad (40)$$

Using Eq. (4) to convert the states of particle l to χ states, we can then insert an angular momentum decomposition. Converting to an LS representation, we have

$$\langle \vec{Q}_k\mu_k, \vec{Q}_l\mu_l|T_d|-\vec{Q}_j\lambda_j\rangle = \left(\frac{w_j}{\pi q_k}\right)^{1/2} \sum_{\nu_k\nu_l} (2l_j+1)^{1/2} C(\sigma_k,\sigma_l,s_j|\nu_k,-\nu_l)C(l_j,s_j,j_j|0,\nu_k-\nu_l) \times D^{j_j *}_{-\lambda_j\,\nu_k-\nu_l}(\text{decay}) D^{\sigma_k *}_{\nu_k\mu_k}(\theta^k_j\hat{n}_k) D^{\sigma_l *}_{\nu_l\mu_l}(\theta^l_j\hat{n}_l)(-1)^{\sigma_l-\nu_l} \times \langle \vec{q}=0, q_kj_j-\lambda_j, l_js_j|T_d|j_j-\lambda_j\rangle \ . \quad (41)$$

The reduced decay matrix element is then just a function of w_j,

$$\langle \vec{Q}=0, q_kj_j-\lambda_j, l_js_j|T_d|j_j-\lambda_j\rangle = B^{j_j}_{l_js_j}(w_j) \ . \quad (42)$$

Recalling the definition of n in Eq. (1) and combining Eqs. (38), (39), (41), and (42) (remember that μ stands for the set $\mu_a\mu_b\mu_j\mu_k\mu_l$), we see that f_μ can be written as

$$f_\mu = \sum_n g^\mu_n(j)T_n(W,w_j) \ , \quad (43)$$

where

$$T_n(W,w_j) = T^{Jj_j}_{LSL_jS_j}(W,w_j)B^{j_j}_{l_js_j}(w_j) \quad (44)$$

and

$$g^\mu_n(j) = \frac{W}{\pi}\left(\frac{w_j}{\pi p Q_j q_k}\right)^{1/2} [(2L+1)(2L_j+1)(2l_j+1)]^{1/2} C(\sigma_a,\sigma_b,S|\mu_a,-\mu_b)C(L,S,J|0,\mu_a-\mu_b) \times \sum_{\lambda_j} C(\sigma_j,j_j,S_j|\mu_j,-\lambda_j)C(L_j,S_j,J|0,\mu_j-\lambda_j)D^J_{\mu_j-\lambda_j\,\mu_a-\mu_b}(j^{-1}\text{beam}) \times \sum_{\nu_k\nu_l} C(\sigma_k,\sigma_l,s_j|\nu_k,-\nu_l)C(l_j,s_j,j_j|0,\nu_k-\nu_l)D^{j_j *}_{-\lambda_j\,\nu_k-\nu_l}(\text{decay}) \times D^{\sigma_k *}_{\nu_k\mu_k}(\theta^k_j\hat{n}_k) D^{\sigma_l *}_{\nu_l\mu_l}(\theta^l_j\hat{n}_l)(-1)^{\sigma_l-\nu_l} \ . \quad (45)$$

Although it is included in n, we have explicitly written the type of isobar with g_n^μ. Since the isobar quantum numbers are included in n, Eq. (43) is valid when there is more than one j-type isobar.

Up to this point we have made no mention of a coordinate system and our formulas are completely general. Some simplification occurs with various choices of axes. We choose the Y axis to be the normal to the three-particle plane (another common choice is to take the Z axis as the normal to the three-particle plane):

$$\vec{Y} = \vec{Q}_j \times \vec{Q}_k = \vec{Q}_k \times \vec{Q}_l = \vec{Q}_l \times \vec{Q}_j \ . \tag{46}$$

In the case of the isobar model it is then convenient to choose the Z axis as a polar vector in the three-particle plane. We choose $\hat{Z}$ along $\vec{Q}_j$. The polar angles of the beam are Θ and Φ, while the particles j, k, and l have polar angles (Θ_j, Φ_j), and (Θ_k, Φ_k), and (Θ_l, Φ_l), respectively, in the c.m. system. With our choice of axes it is clear Φ_j, Φ_k, Φ_l are either 0 or π and $\Theta_j = 0$. These angles are summarized in Fig. 2. In this case we also have $\hat{n}_k = -\hat{n}_l = \vec{Y}$. For convenience we introduce the angles $\alpha_j, \beta_j, \gamma_j$, where

$$\begin{aligned} R(\alpha_j, \beta_j, \gamma_j) &= R(j^{-1}\text{beam}) \\ &= R(\Phi_j, -\Theta_j, -\Phi_j) R(\Phi, \Theta, -\Phi) \ . \end{aligned} \tag{47}$$

We then have the following simplifications in the expression for g_n^μ:

$$\begin{aligned} &D^J_{\mu_j-\lambda_j\,\mu_a-\mu_b}(j^{-1}\text{beam}) \to D^J_{\mu_j-\lambda_j\,\mu_a-\mu_b}(\alpha_j, \beta_j, \gamma_j) \ , \\ &D^{j_j *}_{-\lambda_j\,\nu_k-\nu_l}(\text{decay}) \to d^{j_j}_{-\lambda_j\,\nu_k-\nu_l}(\theta_j) \ , \\ &D^{\sigma_k *}_{\nu_k\mu_k}(\theta_j^k \hat{n}_k) \to d^{\sigma_k}_{\nu_k\mu_k}(\theta_j^k) \ , \\ &D^{\sigma_l *}_{\nu_l\mu_l}(\theta_j^l \hat{n}_l) \to d^{\sigma_l}_{\nu_l\mu_l}(-\theta_j^l) \ . \end{aligned} \tag{48}$$

At this time we can now consider the angles θ_j^k and θ_j^l. Wick[10] discusses these angles in detail and shows that

$$\begin{aligned} \cos\theta_j^k &= \frac{(\cosh\rho - \cosh\sigma_k \cosh\sigma_k')}{(\sinh\sigma_k \sinh\sigma_k')} \ , \\ \cos\theta_j^l &= \frac{(\cosh\rho - \cosh\sigma_l \cosh\sigma_l')}{(\sinh\sigma_l \sinh\sigma_l')} \ , \end{aligned} \tag{49}$$

where

$\tanh\rho = v_j$ = velocity of j in the c.m. system ,

$\tanh\sigma_k = v_k$ = velocity of k in the c.m. system,

$\tanh\sigma_k' = v_k'$ = velocity of k in the (kl) rest frame ,

with similar equations for l. We want to further clarify the sign of the rotation angles. Figure 3 illustrates the effects of the Lorentz transformation in a non-Euclidean plane. Remembering that the spin lags behind the momentum during a Lorentz transformation, one sees that for particle k a positive rotation about the Y axis is needed, and for particle l a negative rotation about the Y axis (corresponding to $\hat{n}_l = -\vec{Y}$ above). We understand θ_j^k and θ_j^l are always positive in Eqs. (48) and (49). In terms of the Stapp[16] angle Ω the Wigner angles are

$$\begin{aligned} \theta_j^k &= \Theta_{kj} - \theta_j - \Omega_{kj} \ , \\ \theta_j^l &= \Theta_{lj} + \theta_j - \Omega_{lj} - \pi \ , \end{aligned} \tag{50}$$

where Θ_{kj} and Θ_{lj} are the c.m. angles between $\vec{Q}_j$ and $\vec{Q}_k$, $\vec{Q}_l$ respectively.

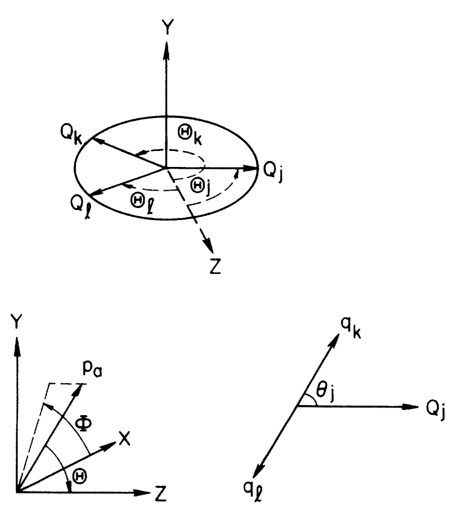

FIG. 2. Definition of angles in our coordinate system. (a) Beam angles in the center-of-mass rest-frame. (b) Angles of particles j, k, l in the center-of-mass rest-frame. (c) Angles of particles k, l in the diparticle rest-frame.

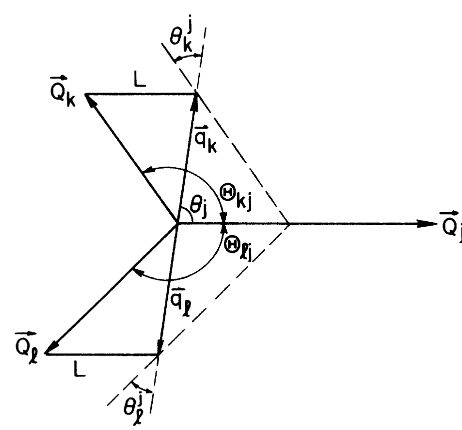

FIG. 3. Symbolic diagram of the effect of the Lorentz transformation L on the momentum vectors. Although the diagram is not quantitative, it does show the correct direction for the various angles.

C. The reduced production and decay transition matrix elements

We now look at the function T_n in more detail. T_n as defined in Eq. (44) is composed of two factors and we consider each separately.

1. Production matrix element

The first factor $T^{Jj_j}_{LSL_js_j}(W, w_j)$ is the production amplitude. For convenience near threshold, one can explicitly write the barrier penetration factors[17]

$$(4W)^{-1/2}p^{L+(1/2)}(4W)^{-1/2}Q_j^{L_j+(1/2)} \quad . \tag{51}$$

The charge dependence is also removed by including the isospin vector addition coefficients. Thus

$$T^{Jj_j}_{LSL_js_j}(W, w_j) = C(I^a, I^b, I|I_z^a, I_z^b)C(I^D, I^j, I|I_z^D, I_z^j) \times \frac{p^{L+(1/2)}Q_j^{L_j+(1/2)}}{4W}\tau^{Jj_j}_{LSL_js_j}(W, w_j) , \tag{52}$$

where $\tau^{Jj_j}_{LSL_js_j}(W, w_j)$ is a function slowly varying in w_j, I^a and I_z^a are the isospin and z component of isospin for a, I^b and I_z^b are the isospin and z component of isospin for b, I^j and I_z^j are the isospin and z component of isospin for j, I^D and I_z^D are the isospin and z component of isospin for the isobar, and I is the total isospin.

Further explicit dependence on W or w_j can be introduced as factors in $\tau^{Jj_j}_{LSL_js_j}(W, w_j)$. One popular choice is a form factor of the form

$$(1+R^2Q_j^{\,2})^{-L_j/2} , \tag{53}$$

which includes a radius of interaction R.

2. Decay matrix element

Taking the charge dependence out of the decay term we have

$$B^{j_j}_{l_js_j}(w_j) = C(I^k, I^l, I^D|I_z^k, I_z^l)A^{j_j}_{l_js_j}(w_j) , \tag{54}$$

where we have used the same notation as before.

To evaluate $A^{j_j}_{l_js_j}(w_j)$ one uses either the Watson final-state interaction theorem or a modified Breit-Wigner function. Using the Watson theorem, one takes

$$A^{j_j}_{l_js_j} \propto \frac{e^{i\delta}\sin\delta}{(q_k)^{l_j+1}}\left(\frac{q_k}{4w_j}\right)^{1/2} , \tag{55}$$

where δ is the elastic scattering phase shift at the mass w_j. We have added the extra factor $(q_k/4w_j)^{1/2}$ to ensure the proper threshold behavior in our normalization. With Breit-Wigner functions one may choose either the relativistic or nonrelativistic form. For the relativistic case one uses

$$A^{j_j}_{l_js_j} = (\pi)^{-1/2}\frac{[w_0\Gamma_j(w_j)]^{1/2}}{(w_0^{\,2}-w_j^{\,2})-iw_0\Gamma_j(w_j)} , \tag{56}$$

where

$$\Gamma_j(w_j) = \Gamma_j(w_0)\left[\frac{q_k(w_j)}{q_k(w_0)}\right]^{2l_j+1}\frac{\rho(w_j)}{\rho(w_0)} \tag{57}$$

and w_0 is the resonance mass. Jackson[11] has given a discussion of the different forms for $\rho(w)$. For the nonrelativistic case one uses

$$A^{j_j}_{l_js_j} = (2\pi w_0)^{-1/2}\frac{[\Gamma_j(w_j)/2]^{1/2}}{(w_0-w_j)-i\Gamma_j(w_j)/2} , \tag{58}$$

where $\Gamma_j(w_j)$ is defined as before. Both of these forms are defined such that in the limit of zero width we have

$$\lim_{\Gamma_j\to 0}|A^{j_j}_{l_js_j}(w_j)|^2 = \delta(w_0^{\,2}-w_j^{\,2}) . \tag{59}$$

D. Cross sections and threshold dependence

We are still considering just one diparticle pair (kl), but there may still be multiple isobars in this system. From Eq. (32) the differential cross section is, for unpolarized incident particles and without observing the polarizations of the final particles,

$$d\sigma = \frac{\pi^2}{F}\overline{\sum_\mu}|f_\mu|^2 d\rho , \tag{60}$$

where

$$\overline{\sum_\mu} = [(2\sigma_a+1)(2\sigma_b+1)]^{-1}\sum_\mu . \tag{61}$$

Since we are concerned with unpolarized cross sections, we may integrate over α (the angle of rotation about the incident beam) in Eq. (31a) to give

$$d\rho = \frac{\pi q_k Q_j}{8Ww_j}dw_j^{\,2}d\cos\theta_j d\cos\Theta d\Phi . \tag{62}$$

The total cross section then becomes

$$\sigma = \int\frac{\pi^2}{Wp}\overline{\sum_\mu}\sum_{nm}g_n^\mu g_m^{\mu *}T_n(W, w_j)T_m^*(W, w_j) \times\frac{\pi q_k Q_j}{8Ww_j}dw_j^{\,2}d\cos\theta_j d\cos\Theta d\Phi . \tag{63}$$

This expression can then be reduced (as in Appendix C) to give

$$\sigma = \frac{\pi}{p^2}\sum_n\frac{(2J+1)}{(2\sigma_a+1)(2\sigma_b+1)}\int|T_n(W, w_j)|^2 dw_j^{\,2} . \tag{64}$$

We note that isobars of different quantum numbers in the (kl) subsystem do not interfere.

If we now use a Breit-Wigner form for $A^{j_j}_{l_js_j}(w_j)$

and take the limit as $\Gamma_j(w_j) \to 0$, the cross section reduces to

$$\sigma = \frac{\pi}{p^2} \sum_n \frac{(2J+1)}{(2\sigma_a+1)(2\sigma_b+1)} |T^{Jj_j}_{LSL_js_j}(W, w_0)|^2 . \tag{65}$$

Since in this limit the diparticle has become a stable particle, this equation should be the same as that for the reaction $a+b \to c+d$. Comparing with Eq. (B7) of Appendix B, we do have agreement.

E. Other isobars

Up to this point we have been dealing with j-type isobars only. Unfortunately one usually must include k- and l-type isobars as well. Since we have included the type of isobar in the index n, Eq. (43) is still valid. For k-type isobars we have

$$T_n(W, w_k) = T^{Jj_k}_{LSL_ks_k}(W, w_k) B^{j_k}_{l_ks_k}(w_k) \tag{66}$$

and

$$g_n^\mu(k) = \frac{W}{\pi}\left(\frac{w_k}{\pi p Q_k q_l}\right)^{1/2} [(2L+1)(2L_k+1)(2l_k+1)]^{1/2} C(\sigma_a, \sigma_b, S|\mu_a, -\mu_b) C(L, S, J|0, \mu_a-\mu_b)$$
$$\times \sum_{\lambda_k} C(\sigma_k, j_k, S_k|\mu_k, -\lambda_k) C(L_k, S_k, J|0, \mu_k-\lambda_k) D^J_{\mu_k-\lambda_k \mu_a-\mu_b}(\alpha_k, \beta_k, \gamma_k)$$
$$\times \sum_{\nu_l \nu_j} C(\sigma_l, \sigma_j, s_k|\nu_l, -\nu_j) C(l_k, s_k, j_k|0, \nu_l-\nu_j) d^{j_k}_{-\lambda_k \nu_l-\nu_j}(\theta_k) d^{\sigma_l}_{\nu_l\mu_l}(\theta_k^l) d^{\sigma_j}_{\nu_j\mu_j}(-\theta_k^j)(-1)^{\sigma_j-\nu_j} . \tag{67}$$

For l-type isobars we have the equations

$$T_n(W, w_l) = T^{Jj_l}_{LSL_ls_l}(W, w_l) B^{j_l}_{l_ls_l}(w_l) \tag{68}$$

and

$$g_n^\mu(l) = \frac{W}{\pi}\left(\frac{w_l}{\pi p Q_l q_j}\right)^{1/2} [(2L+1)(2L_l+1)(2l_l+1)]^{1/2} C(\sigma_a, \sigma_b, S|\mu_a, -\mu_b) C(L, S, J|0, \mu_a-\mu_b)$$
$$\times \sum_{\lambda_l} C(\sigma_l, j_l, S_l|\mu_l, -\lambda_l) C(L_l, S_l, J|0, \mu_l-\lambda_l) D^J_{\mu_l-\lambda_l \mu_a-\mu_b}(\alpha_l, \beta_l, \gamma_l)$$
$$\times \sum_{\nu_j \nu_k} C(\sigma_j, \sigma_k, s_l|\nu_j, -\nu_k) C(l_l, s_l, j_l|0, \nu_l-\nu_j) d^{j_l}_{-\lambda_l \nu_j-\nu_k}(\theta_l) d^{\sigma_j}_{\nu_j\mu_j}(\theta_l^j) d^{\sigma_k}_{\nu_k\mu_k}(-\theta_l^k)(-1)^{\sigma_k-\nu_k} . \tag{69}$$

In each case we have preserved the cyclic order of j, k, and l. The total transition amplitude in the case of more than one subsystem containing isobars is then written as before:

$$f_\mu = \sum_{jn} g_n^\mu(j) T_n(W, w_j) . \tag{43}$$

This coherent addition implies some double counting of the amplitudes which has, in practical situations, been shown to be small.[18]

In the case when there are identical particles present, care has to be taken to ensure that one uses a correctly symmetrized combination in Eq. (43). Our cyclic ordering of the particles j, k, and l will not necessarily ensure this and this has to be explicitly introduced.

F. Symmetry properties of the amplitudes

We next discuss the symmetry properties of g_n^μ under certain circumstances.

1. Parity

Consider the case of $\mu \to -\mu$, the result which occurs under the operation of parity. In Appendix D we show that

$$g_n^{-\mu} = \eta(-1)^{\sigma_a+\mu_a}(-1)^{\sigma_b-\mu_b}(-1)^{\sigma_j+\mu_j} \times (-1)^{\sigma_k+\mu_k}(-1)^{\sigma_l+\mu_l} g_n^{\mu *} , \tag{70}$$

where η is the product of all five parities. For any specific problem this reduces the number of independent g_n^μ. For the case of $\pi N \to N\pi\pi$, $\eta = -1$ and we have

$$g_n^{-\mu} = (-1)^{\mu_f - \mu_i} g_n^{\mu *} , \tag{71}$$

where μ_i is the incident nucleon helicity and μ_f is the final nucleon helicity. Since T_n is independent of μ, we have

$$f_{-\mu} = \sum_n g_n^{-\mu} T_n = \sum_n g_n^{\mu *} T_n . \tag{72}$$

2. Interchange of two particles

We may also discuss the properties of our amplitudes $g_n^\mu(w_j^2, w_k^2, w_l^2)$ under the interchange of two particles k and l. Such a change is relevant for discussion of symmetry properties in the presence of two identical particles.

In our formalism a cyclic order is always preserved and thus interchanging k and l leads to a change in the coordinate system, $\vec{Y} \to -\vec{Y}$. Associated with this, we have the changes

$$
\begin{aligned}
&W \to W\,,\\
&w_j{}^2 \to w_j{}^2,\; w_k{}^2 \to w_l{}^2,\; w_l{}^2 \to w_k{}^2\,,\\
&\Theta \to \Theta,\; \Phi \to \Phi + \pi\,,\\
&\Theta_j \to \Theta_j,\; \Theta_k \to \Theta_l,\; \Theta_l \to \Theta_k\,,\\
&\Phi_j = 0 \to \Phi_j = 0,\; \Phi_k = 0 \to \Phi_k = 0,\; \Phi_l = \pi \to \Phi_l = \pi\,,\\
&\theta_j \to \pi - \theta_j,\; \theta_k \to \pi - \theta_l,\; \theta_l \to \pi - \theta_k \qquad (73)\\
&\theta_j^k \to \theta_j^l,\; \theta_j^l \to \theta_j^k\,,\\
&\theta_k^l \to \theta_l^k,\; \theta_l^k \to \theta_k^l\,,\\
&\theta_k^j \to \theta_l^j,\; \theta_l^j \to \theta_k^j\,,\\
&\mu_a \to \mu_a,\; \mu_b \to \mu_b\,,\\
&\mu_j \to \mu_j,\; \mu_k \to \mu_l,\; \mu_l \to \mu_k\,.
\end{aligned}
$$

We find that (see Appendix E) for j-type isobars,

$$
\begin{aligned}
g_{nj}^{\mu_a\mu_b\mu_j\mu_l\mu_k}&(w_j{}^2, w_l{}^2, w_k{}^2)\\
&= (-1)^{\mu_a-\mu_b-\mu_j-\mu_k-\mu_l}(-1)^{s_j+\sigma_k+\sigma_l}\\
&\quad\times(-1)^{l_j} g_{nj}^{\mu_a\mu_b\mu_j\mu_k\mu_l}(w_j{}^2, w_k{}^2, w_l{}^2)\;; \qquad (74)
\end{aligned}
$$

for k-type isobars,

$$
\begin{aligned}
g_{nk}^{\mu_a\mu_b\mu_j\mu_l\mu_k}&(w_j{}^2, w_l{}^2, w_k{}^2)\\
&= (-1)^{\mu_a-\mu_b-\mu_j-\mu_k-\mu_l}(-1)^{s_l+\sigma_j+\sigma_k}\\
&\quad\times(-1)^{l_l} g_{nl}^{\mu_a\mu_b\mu_j\mu_k\mu_l}(w_j{}^2, w_k{}^2, w_l{}^2)\;; \qquad (75)
\end{aligned}
$$

and for l-type isobars,

$$
\begin{aligned}
g_{nl}^{\mu_a\mu_b\mu_j\mu_l\mu_k}&(w_j{}^2, w_l{}^2, w_k{}^2)\\
&= (-1)^{\mu_a-\mu_b-\mu_j-\mu_k-\mu_l}(-1)^{s_k+\sigma_l+\sigma_j}\\
&\quad\times(-1)^{l_k} g_{nk}^{\mu_a\mu_b\mu_j\mu_k\mu_l}(w_j{}^2, w_k{}^2, w_l{}^2)\,. \qquad (76)
\end{aligned}
$$

III. SCATTERING FROM POLARIZED TARGETS AND THE MEASUREMENTS OF FINAL PARTICLE POLARIZATIONS

The formalism we have developed can be used to discuss polarization experiments when the particles have arbitrary spin. However, this becomes involved and for the sake of simplicity we consider the case $M_1B_1 \to M_2M_3B_2$, where M is a 0^- meson and B is a $\frac{1}{2}^+$ baryon.[19]

We use helicity states for the incident and final particles. The reference coordinate system we use in all our calculations is $OXYZ$, where OY is perpendicular to the three-particle decay plane and OZ lies in the three-particle plane (see Fig. 2). We have used the prescription of Jacob and Wick for constructing general states, i.e.,

$$|p\theta\phi,\lambda\rangle = R(\phi,\theta,-\phi)|p00,\lambda\rangle\,. \qquad (77)$$

Now Eq. (77) can be viewed in a passive sense; i.e., it gives the orientation of the rest frame with respect to $OXYZ$ in which the spin components λ are defined. This rest frame is obtained from $OXYZ$ by the operation $R(\phi,\theta,-\phi)$. These final coordinate axes are then the helicity frame axes. These are described in Fig. 4 and we see that the particle has spin component λ along OZ''' in the coordinate system $OX'''Y'''Z'''$.

A. Final particle coordinate systems

For our final particles the helicity frame axes are defined by

$$
\begin{aligned}
&OZ' = \vec{p}_j / |\vec{p}_j|\,,\\
&OY' = \vec{p}_j \times \vec{p}_k / |\vec{p}_j \times \vec{p}_k|\,, \qquad (78)\\
&OX' = OY' \times OZ'
\end{aligned}
$$

and are demonstrated in Fig. 5.

B. Initial-state coordinate system

In this case the helicities are defined in a rest frame $OX_1Y_1Z_1$, which is obtained from $OXYZ$ by rotation through the Euler angles $\Phi, \Theta, -\Phi$; thus, OZ_1 is along the incident momentum $\vec{p}_a$. Now, if we use a polarized target, then we define a very specific initial coordinate system. Let this coordinate system be $Oxyz$ with Oz along $\vec{p}_a$. Then $Oxyz$ is related to $OX_1Y_1Z_1$ by a rotation α around the OZ_1 axis. We have the following relations be-

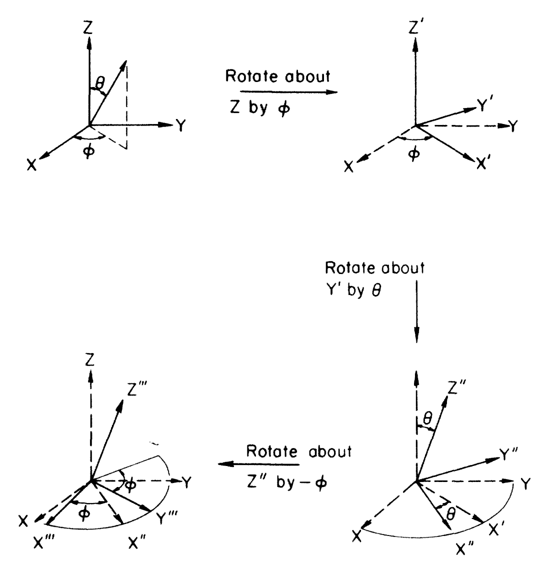

FIG. 4. Illustration of the effect of the rotation $R(-\phi,\Theta,\phi)$ on the axes $OXYZ$.

tween coordinate frames:

$$OXYZ \to OX_1Y_1Z_1, \quad \text{Euler angles } \Phi, \Theta, -\Phi\ ,$$
$$OX_1Y_1Z_1 \to Oxyz, \quad \text{Euler angles } 0, 0, \alpha\ , \qquad (79)$$
$$OXYZ \to Oxyz, \quad \text{Euler angles } \Phi, \Theta, \alpha-\Phi\ .$$

C. Transition matrix elements

We have calculated transition matrix elements from initial states defined in the frame $OX_1Y_1Z_1$, whereas we require transitions from states defined in $Oxyz$ to discuss scattering from polarized targets. If A_μ is the amplitude for transition from $OX_1Y_1Z_1$ and A'_μ is the transition amplitude from $Oxyz$, then

$$A'_\mu = A_\mu e^{-i(\mu_a-\mu_b)\alpha}\ . \qquad (80)$$

If we consider only the reactions of the type $\pi N \to N\pi\pi$, then Eq. (80) reduces to

$$A'_\mu = A_\mu e^{i\mu_i\alpha}\ . \qquad (81)$$

D. Polarization experiments

We assume we have a coordinate system $Oxyz$ in which the initial polarization is specified and the final baryon polarization is described in the helicity frame.

1. Unpolarized cross sections

The initial density matrix is $\vec{\rho}^{\,i} = \frac{1}{2}\vec{1}$. The differential cross section is then written as

$$I_0 = \text{Trace}(\vec{A}'\vec{\rho}^{\,i}\vec{A}'^{+}) = \frac{1}{2}\sum_\mu |A'_\mu|^2 = \frac{1}{2}\sum_\mu |A_\mu|^2\ . \qquad (82)$$

2. Polarized target

The initial density matrix is now $\vec{\rho}^{\,i} = \frac{1}{2}(\vec{1} + \vec{P}_b\vec{\sigma}_b)$. We then have a differential cross section

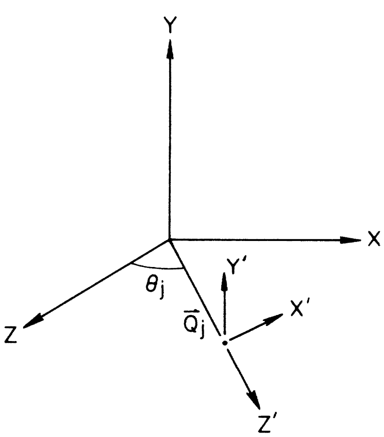

FIG. 5. Final-state helicity-frame axes for particle j.

$$I_p = \text{Trace}(\vec{A}'\vec{\rho}^{\,i}\vec{A}'^{+}) = I_0(1 + \vec{P}_b\cdot\vec{\alpha})\ ,$$
$$I_0\vec{\alpha} = \tfrac{1}{2}\,\text{Trace}(\vec{A}'\vec{\sigma}_b\vec{A}'^{+})\ , \qquad (83)$$

where $\vec{P}_b$ is the polarization vector of particle b in the $Oxyz$ frame.

3. Final polarization (of particle l)

Here $I_0\vec{\rho}^{\,f} = \frac{1}{2}\vec{A}'\vec{A}'^{+}$, where Trace $(\vec{\rho}^{\,f}) = 1$. The final baryon polarization is given by

$$I_0 P_l = \tfrac{1}{2}\,\text{Trace}(\vec{A}'\vec{A}'^{+}\vec{\sigma}_l)\ . \qquad (84)$$

4. Depolarization tensor

For final polarization from a polarized target we have

$$I_p\,\vec{\rho}^{\,f} = \vec{A}'\vec{\rho}^{\,i}\vec{A}'^{+}\ , \qquad (85)$$

where Trace $(\vec{\rho}^{\,f}) = 1$. The component of spin of particle l along an axis M $(= X, Y,$ or $Z)$ is P_{lM} and is given by

$$P_{lM} = \text{Trace}(\vec{\rho}^{\,f}\sigma_{lM})\ . \qquad (86)$$

Then

$$I_p P_{lM} = \text{Trace}(\vec{A}'\vec{\rho}^{\,i}\vec{A}'^{+}\sigma_{lM}) = I_0\Big(P_{lM} + \sum_i P_{bi}D_{bi,lM}\Big) \qquad (87)$$

and

$$I_0 D_{bi,lM} = \tfrac{1}{2}\,\text{Trace}(\vec{A}'\sigma_{bi}\vec{A}'^{+}\sigma_{lM})\ . \qquad (88)$$

These results are summarized in Table I.

TABLE I. Expressions for all observable quantities in the reaction $MB \to BMM$. Amplitudes $A_{\mu_f\mu_i}$ with $\mu_f\,\mu_i = \pm\frac{1}{2}$ are written as $\mu_f\mu_i = +-$.

$I_0 = \frac{1}{2}(\lvert A_{++}\rvert^2 + \lvert A_{+-}\rvert^2 + \lvert A_{-+}\rvert^2 + \lvert A_{--}\rvert^2)$
$I_0 A_x = \text{Re}(A_{++}A^*_{+-}e^{i\alpha}) + \text{Re}(A_{-+}A^*_{--}e^{i\alpha})$
$I_0 A_y = \text{Im}(A_{++}A^*_{+-}e^{i\alpha}) + \text{Im}(A_{-+}A^*_{--}e^{i\alpha})$
$I_0 A_z = \frac{1}{2}(\lvert A_{++}\rvert^2 + \lvert A_{-+}\rvert^2 - \lvert A_{+-}\rvert^2 - \lvert A_{--}\rvert^2)$
$I_0 P^{(0)}_X = \text{Re}(A_{++}A^*_{-+}) + \text{Re}(A_{+-}A^*_{--})$
$I_0 P^{(0)}_Y = -\text{Im}(A_{++}A^*_{-+}) - \text{Im}(A_{+-}A^*_{--})$
$I_0 P^{(0)}_Z = \frac{1}{2}(\lvert A_{++}\rvert^2 + \lvert A_{+-}\rvert^2 - \lvert A_{-+}\rvert^2 - \lvert A_{--}\rvert^2)$
$I_0 D_{xX} = \text{Re}(A_{+-}A^*_{-+}e^{-i\alpha}) + \text{Re}(A_{++}A^*_{--}e^{i\alpha})$
$I_0 D_{xY} = -\text{Im}(A_{+-}A^*_{-+}e^{-i\alpha}) - \text{Im}(A_{++}A^*_{--}e^{i\alpha})$
$I_0 D_{xZ} = \text{Re}(A_{++}A^*_{+-}e^{i\alpha}) - \text{Re}(A_{-+}A^*_{--}e^{i\alpha})$
$I_0 D_{yX} = -\text{Im}(A_{+-}A^*_{-+}e^{-i\alpha}) + \text{Im}(A_{++}A^*_{--}e^{i\alpha})$
$I_0 D_{yY} = \text{Re}(A_{++}A^*_{--}e^{i\alpha}) - \text{Re}(A_{+-}A^*_{-+}e^{-i\alpha})$
$I_0 D_{yZ} = \text{Im}(A_{++}A^*_{+-}e^{i\alpha}) - \text{Im}(A_{-+}A^*_{--}e^{i\alpha})$
$I_0 D_{zX} = \text{Re}(A_{++}A^*_{-+}) - \text{Re}(A_{+-}A^*_{--})$
$I_0 D_{zY} = -\text{Im}(A_{++}A^*_{-+}) + \text{Im}(A_{+-}A^*_{--})$
$I_0 D_{zZ} = \frac{1}{2}(\lvert A_{++}\rvert^2 + \lvert A_{--}\rvert^2 - \lvert A_{-+}\rvert^2 - \lvert A_{+-}\rvert^2)$

5. Decay of final-state baryon

If the decay is weak, e.g., $\Lambda \to p\pi^-$, then this decay angular distribution will analyze the parent baryon polarization, and thus this is an appropriate place for the discussion of such situations. We introduce the decay amplitude directly into the transition amplitude.

We have shown previously [Eq. (43)] that the transition amplitude for the process $a+b \to j+k+l$ can be written as

$$f_\mu = \sum_n g_n^\mu T_n \ ,$$

where n is summarized in Eq. (2). Now suppose that we consider particle j undergoing weak decay to two other particles and we define their spin states with respect to the helicity-frame axes of the parent particle. Then the amplitude for this decay is

$$\begin{aligned} B^D_{m_1 m_2} &= \langle \sigma_1 m_1 \sigma_2 m_2 | T^D | \sigma_j \mu_j \rangle \\ &= \sum_{L_d S_d} B^{L_d S_d} C(\sigma_1, \sigma_2, S_d | m_1, m-m_1) C(L_d, S_d, \sigma_j | \mu_j - m, m) Y_{L_d}^{\mu_j - m}(\theta_d, \phi_d) \ , \end{aligned} \tag{89}$$

where $B^{L_d S_d}$ is the partial-wave amplitude for the decay:

$$B^{L_d S_d} = \langle \sigma_1 \sigma_2 L_d S_d | T^D | \sigma_j \mu_j \rangle \ . \tag{90}$$

Thus we find that the final amplitude for producing particles k and l in states $|Q_k \mu_k\rangle\, |Q_l \mu_l\rangle$ together with the decay products in states $|\sigma_1 m_2\rangle\, |\sigma_2 m_2\rangle$ with respect to the helicity-frame axes of particle j is

$$\begin{aligned} f_{\mu_a \mu_b m_1 m_2 \mu_k \mu_l} = \sum_n \sum_{\mu_j} \sum_{L_d S_d} \{ & B^{L_d S_d} C(\sigma_1, \sigma_2, S_d | m_1, m-m_1) \\ & \times C(L_d, S_d, \sigma_j | \mu_j - m, m) Y_{L_d}^{\mu_j - m}(\theta_d, \phi_d) g_n^{\mu_a \mu_b \mu_j \mu_k \mu_l} T_n \} \ . \end{aligned} \tag{91}$$

In the case of Λ decay obtained, for instance, in the reaction

$$K^- p \to \Lambda \pi^+ \pi^- \,, \quad \Lambda \to p\pi^- \ ,$$

many simplifications result, $\sigma_2 = 0$, $m_2 = 0$, and we have

$$f_{\mu_a \mu_b m_1 \mu_k \mu_l} = \sum_n \sum_{\mu_j} \sum_{L_d = 0,1} [B^{L_d} C(L_d, \tfrac{1}{2}, \tfrac{1}{2} | \mu_j - m_1, m_1) Y_{L_d}^{\mu_j - m_1}(\theta_d, \phi_d) g_n^{\mu_a \mu_b \mu_j \mu_k \mu_l} T_n] \ . \tag{92}$$

Further, if we perform the reactions from polarized targets, defining a specific initial coordinate system, then

$$f'_{\mu_a \mu_b m_1 m_2 \mu_k \mu_l} = f_{\mu_a \mu_b m_1 m_2 \mu_k \mu_l} e^{-i(\mu_a - \mu_b)\alpha} \ . \tag{93}$$

IV. ANALYSIS OF THREE-BODY STATES OBTAINED IN PRODUCTION REACTIONS

Another fruitful area for application of the formalism we have developed is in the study of three-particle states formed in production experiments. We are particularly concerned with reactions of the type

$$a + b \to c + X \,, \quad X \to j + k + l \ , \tag{94}$$

of which there are many examples being studied at present, e.g.,

$$\begin{aligned} &\pi + p \to p + (A_1, A_2, A_3) \ , \\ &k + p \to p + (Q, L) \ , \\ &\pi(k) + p \to \pi(k) + N^* \ . \end{aligned} \tag{95}$$

We now develop the slight changes necessary to deal with these reactions. We use a notation essentially the same as that described in Sec. I. The only modifications are the following:

(a) We have to define the quantities pertaining to the extra particle c. We use

σ_c: intrinsic spin of c ,

μ_c: helicity of c ,

p_c: four-momentum of c .

(b) All quantities referring to particles a, b, and c are measured in the total c.m. system.

(c) All quantities pertaining to particles j, k, and l are measured in the (jkl) c.m. system. This includes variables used in the development of the formulas for the decay of the three-particle state.

(d) We do not make a spin-parity decomposition of the incident state, so that L and S are not needed. Further, J will represent the total spin of the (jkl) system and not the over-all angular momentum in the process.

(e) We use two coordinate systems, S and S', both in the (jkl) rest frame. S is used to describe the decay of $X \to jkl$. This system is the one defined with respect to the final state for the discussion of $2 \to 3$ particle processes in Sec. II. On the other hand, S' is that particular coordinate frame, in the rest system of particle X, in which we choose to describe the spin (or helicity) state $|JM\rangle$ of X. Thus the intermediate particle X has spin projection M with respect to the Z' axis of S'. The choice of S' will reflect our prejudices about the type of production process occurring, since one will try to choose S' in such a way as to make the spin (or helicity) density matrix of X, $\rho_{MM'}$, as simple as possible. Thus one would choose, e.g., (1) S' as the Gottfried-Jackson system if one is interested in one-particle exchange, or generally if one expects a simple t-channel spin structure; (2) S' as the helicity frame (defined from the s-channel for the reaction $a+b \to c+X$) if one is concerned with s-channel helicity conservation.

The intermediate state $c+X$ will be characterized by a wave function of the form

$$\psi = \sum_{nM\mu_c} f^{nM}_{\mu_a\mu_b\mu_c}(\mathfrak{M}, s, t)|nM\rangle\,|p_c\mu_c\rangle \ , \qquad (96)$$

where $f^{nM}_{\mu_a\mu_b\mu_c}(\mathfrak{M}, s, t)$ is the amplitude to produce in the reaction $a+b \to c+X$ a state X with quantum numbers n, i.e., the set $(j; J; L_j, S_j; j_j, l_j, s_j)$, and spin projection M in the coordinate frame S'. This amplitude depends upon $\mu_a\mu_b\mu_c$, the c.m. helicities of a, b, and c; s and t, the Mandelstam invariants for $a+b \to c+X$; and $\mathfrak{M}$, the mass of X.

For the decay of X we use the coordinate system S, which we have used earlier in Sec. II:

$$\begin{aligned} \hat{Z} &= \vec{Q}_j/|\vec{Q}_j| \ , \\ \hat{Y} &= \vec{Q}_j \times \vec{Q}_k/|\vec{Q}_j \times \vec{Q}_k| \ , \\ \hat{X} &= \hat{Y} \times \hat{Z} \ . \end{aligned} \qquad (97)$$

We require the following transition matrix elements for the decay of $X \to jkl$:

$$\begin{aligned} {}_S\langle \vec{Q}_j\mu_j, \vec{Q}_k\mu_k, \vec{Q}_l\mu_l | T | nM\rangle_{S'} &= \sum_m {}_S\langle \vec{Q}_j\mu_j, \vec{Q}_k\mu_k, \vec{Q}_l\mu_l | T | nm\rangle_{S\,S}\langle nm|nM\rangle_{S'} \\ &= \sum_m {}_S\langle \vec{Q}_j\mu_j, \vec{Q}_k\mu_k, \vec{Q}_l\mu_l | T | nm\rangle_S D^J_{mM}(\alpha, \beta, \gamma) \ , \end{aligned} \qquad (98)$$

where α, β, γ are the Euler angles defining the transformation from S to S'. This matrix element depends on all the quantum numbers n, M of the jkl state, as well as on the helicities $\mu_j\mu_k\mu_l$ and the continuous variables describing the jkl state, $\mathfrak{M}$, α, β, γ, w_j^2, and w_k^2. We will write briefly $G^{\mu_j\mu_k\mu_l}_{nM}$ for this decay matrix element. Its calculation involves the evaluation of $\langle \vec{Q}_j\mu_j, \vec{Q}_k\mu_k, \vec{Q}_l\mu_l | T | nm\rangle$, which is just the transition matrix element calculated in Sec. II, provided that the factors associated with the partial-wave decomposition of the incident beam are ignored. From the results of Sec. II we have

$$\begin{aligned} G^{\mu_j\mu_k\mu_l}_{nM} = \sum_m D^J_{mM}(\alpha, \beta, \gamma) \Bigg\{ \left[\frac{\mathfrak{M} w_j}{\pi^2 Q_j q_k}\right]^{1/2} [(2L_j+1)(2l_j+1)]^{1/2} \\ \times \sum_{\lambda_j} C(\sigma_j, j_j, S_j|\mu_j, -\lambda_j) C(L_j, S_j, J|0, \mu_j-\lambda_j) D^{J*}_{m,\mu_j-\lambda_j}(\Phi_j, \Theta_j, -\Phi_j) \qquad (99a) \\ \times \sum_{\nu_k\nu_l} C(\sigma_k, \sigma_l, s_j|\nu_k, -\nu_l) C(l_j, s_j, j_j|0, \nu_k-\nu_l) d^{j_j}_{-\lambda_j, \nu_k-\nu_l}(\theta_j) \\ \times d^{\sigma_k}_{\nu_k\mu_k}(\theta^k_j) d^{\sigma_l}_{\nu_l\mu_l}(-\theta^l_j)(-1)^{\sigma_l-\nu_l} \Bigg\} T_n(\mathfrak{M}, w_j) \\ = g^{\mu_j\mu_k\mu_l}_{nm} T_n(\mathfrak{M}, w_j) \ , \qquad (99b) \end{aligned}$$

where

$$T_n(\mathfrak{M}, w_j) = T^{Jj_j}_{L_jS_j}(\mathfrak{M}, w_j) B^{j_j}_{l_js_j}(w_j) \ . \qquad (100)$$

The forms and amplitudes we introduce into T_n were discussed in Sec. II.

The amplitude for a final state derived from an intermediate state X of quantum numbers n, M is then represented by

$$f^{nM}_{\mu_a\mu_b\mu_c} G^{\mu_j\mu_k\mu_l}_{nM} \qquad (101)$$

and the differential cross section for the process is given by

$$d\sigma(\mu_a\mu_b\mu_c\mu_j\mu_k\mu_l) \propto \left|\sum_{nm} f^{nM}_{\mu_a\mu_b\mu_c}(\mathfrak{M}, s, t) G^{\mu_j\mu_k\mu_l}_{nM}\right|^2 \ . \qquad (102)$$

A. Symmetry properties due to parity conservation

If a conventional choice for S' is made with the Z' axis a polar vector and the Y' axis an axial vector as in the Gottfried-Jackson frame, then a familiar result is obtained:

$$\begin{aligned} f^{nM}_{\mu_a\mu_b\mu_c} &= f^{n-M}_{-\mu_a-\mu_b-\mu_c} \eta_J\eta_a\eta_b\eta_c(-1)^{\sigma_a-\mu_a} \\ &\times (-1)^{\sigma_b-\mu_b}(-1)^{\sigma_c-\mu_c}(-1)^{J-M} \ , \end{aligned} \qquad (103)$$

where η_J is the parity of the intermediate state X. Similar calculations as those in Appendix D result in

$$g_{nM}^{\mu_j\mu_k\mu_l}=(-1)^{J-M}(-1)^{L_j+l_j}(-1)^{\sigma_j+\mu_j}(-1)^{\sigma_k+\mu_k}\times(-1)^{\sigma_l+\mu_l}g_{n-M}^{-\mu_j-\mu_k-\mu_l\,*}\,. \quad (104)$$

B. Differential cross section

In general we write the unpolarized cross section as

$$d\sigma\propto\sum_{\mu_a\mu_b\mu_c}\sum_{\mu_j\mu_k\mu_l}\left(\sum_{nM}f^{nM}_{\mu_a\mu_b\mu_c}G_{nM}^{\mu_j\mu_k\mu_l}\right)\times\left(\sum_{n'M'}f^{n'M'}_{\mu_a\mu_b\mu_c}G_{n'M'}^{\mu_j\mu_k\mu_l}\right)^*, \quad (105)$$

where $d\sigma$ is the differential cross section over seven variables which we take to be $\mathfrak{M}$, t, α, β, γ, w_j^2, w_k^2. We can also write $d\sigma$ in the form

$$d\sigma\propto\sum_{nM}\sum_{n'M'}\rho^{nn'}_{MM'}\left(\sum_{\mu_j\mu_k\mu_l}G_{nM}^{\mu_j\mu_k\mu_l}G_{n'M'}^{\mu_j\mu_k\mu_l\,*}\right), \quad (106)$$

where we have defined an unnormalized density matrix

$$\rho^{nn'}_{MM'}=\sum_{\mu_a\mu_b\mu_c}f^{nM}_{\mu_a\mu_b\mu_c}f^{n'M'*}_{\mu_a\mu_b\mu_c}\,. \quad (107)$$

This matrix has the properties

$$\rho^{nn'}_{MM'}=(\rho^{n'n}_{M'M})^*=\eta_J\eta_{J'}(-1)^{J-M}(-1)^{J'-M'}\rho^{nn'}_{-M-M'}\,. \quad (108)$$

Integration over $\alpha\beta\gamma$ leads to the well-known result that the Dalitz plot distribution is independent of the magnetic quantum number M with which X is produced. Careful manipulation of Eqs. (99) and (106) leads to the other well-known result that waves of opposite parity do not interfere in the Dalitz plot.

Use of Eq. (106) allows the measurement of the following parameters of interest:

(a) $\rho^{nn'}_{MM'}$, the production density matrix,

(b) $T^{Jj_j}_{L_js_j}$ the coupling of the intermediate state to the various decay channels.

In the case in which the intermediate state is composed of three pseudoscalar mesons, the expressions for $G_{nM}^{\mu_j\mu_k\mu_l}$ are simplified since $\sigma_j=\sigma_k=\sigma_l=0$. In this case we have

$$G_{nM}^{\mu_j\mu_k\mu_l}=\sum_m D^J_{mM}(\alpha,\beta,\gamma)\left\{\left(\frac{\mathfrak{M}w_j}{\pi^2Q_jq_k}\right)^{1/2}[(2L_j+1)(2l_j+1)]^{1/2}\times\sum_{\lambda_j}C(L_j,j_j,J|0,-\lambda_j)D^{J*}_{m,-\lambda_j}(\Phi_j,\Theta_j,-\Phi_j)d^{j_j}_{-\lambda_j,0}(\theta_j)\right\}T_n(\mathfrak{M},w_j)\,. \quad (109)$$

An analysis of A_1, A_2 production using a formalism similar to this has been performed by Ascoli *et al.*[20]

Clearly we can extend this formalism with only slight modification to the case of a group of particles recoiling against the particle X instead of just one particle c. The internal variables describing this group of particles enter in the function $f^{nM}_{\mu_a\mu_b\mu_c}(s,t,\mathfrak{M}\cdots)$.

APPENDIX A

In this appendix we review some of the properties of rotations and their representations. Most of the material should be familiar, but we wish to restate all the properties used in the text using our notation. All sign conventions are those of Rose.[21]

Since a given rotation may be expressed in a number of different ways, convenience is usually the deciding factor. We shall use either of two methods. A given rotation will be specified either by its Euler angles, $\alpha\beta\gamma$, or by the angle and axis of rotation, $\theta\hat{n}$. In terms of the angular momentum operator J, the rotation operator R is

$$R(\alpha,\beta,\gamma)=e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z}=e^{-i\theta\hat{n}\cdot\vec{J}}\,. \quad (A1)$$

If in some coordinate system, $\hat{n}$ can be expressed by $(-\sin\phi,\ \cos\phi,\ 0)$ then

$$R(\theta\hat{n})=R(\phi,\theta,-\phi)\,. \quad (A2)$$

One other equality we use is

$$R(0,\theta,0)=R(2\pi,\theta,-2\pi)=R(-2\pi,\theta,2\pi)\,. \quad (A3)$$

Since the product of two rotations is again a rotation, we have

$$R(\alpha,\beta,\gamma)=R(\alpha'',\beta'',\gamma'')R(\alpha',\beta',\gamma')\,. \quad (A4)$$

To discuss a matrix representation of the rotations R, we consider the vector space spanned by the basis vectors $|jm\rangle$, where

$$J^2|jm\rangle=j(j+1)|jm\rangle\,,\qquad J|jm\rangle=m|jm\rangle\,, \quad (A5)$$

with $J=aJ_x+bJ_y+cJ_z$. (The usual choice for J is J_z. This choice makes evaluating the matrix elements much easier but is not necessary.) The elements of the matrix corresponding to R are then given by

$$D^j_{mn}(R) = \langle jm|R|jn\rangle \ . \quad \text{(A6)}$$

In terms of the matrices, Eq. (A4) is written as

$$D^j_{mn}(\alpha,\beta,\gamma) = \sum_p D^j_{mp}(\alpha'',\beta'',\gamma'')D^j_{pn}(\alpha',\beta',\gamma') \ . \quad \text{(A7)}$$

Expressing R in terms of the Euler angles and making the usual choice of $J = J_z$, the matrix elements simplify to

$$D^j_{mn}(\alpha,\beta,\gamma) = e^{-i(m\alpha+n\gamma)}\, d^j_{mn}(\beta) \ , \quad \text{(A8)}$$

where the functions $d^j_{mn}(\beta)$ are real. These functions satisfy the general relations

$$\begin{aligned} d^j_{mn}(\beta) &= (-1)^{m-n} d^j_{nm}(\beta) = (-1)^{m-n} d^j_{-m\,-n}(\beta) \ , \\ d^j_{mn}(\pi-\beta) &= (-1)^{j-n} d^j_{-mn}(\beta) = (-1)^{j+m} d^j_{m\,-n}(\beta) \ , \\ d^j_{mn}(\beta+2\pi) &= (-1)^{2j} d^j_{mn}(\beta) \ , \\ d^j_{mn}(-\beta) &= d^j_{nm}(\beta) \ . \end{aligned} \quad \text{(A9)}$$

The normalization integrals are

$$\begin{aligned} &\int d^j_{mn}(\beta) d^{j'}_{mn}(\beta)\, d\cos\beta = \frac{2}{(2j+1)}\,\delta_{jj'} \ , \\ &\int D^j_{mn}(\alpha,\beta,\gamma) D^{j'*}_{m'n'}(\alpha,\beta,\gamma)\, d\alpha\, d\cos\beta\, d\gamma \\ &\qquad = \frac{8\pi^2}{(2j+1)}\,\delta_{jj'}\delta_{mm'}\delta_{nn'} \ . \end{aligned} \quad \text{(A10)}$$

We use the same conventions as the Particle Data Group for the vector addition coefficients[22]:

$$C(j_1,j_2,j\,|m_1,m_2) = \langle j_1 j_2 m_1 m_2 | j_1 j_2 j m\rangle \ . \quad \text{(A11)}$$

We have the following relations for these coefficients:

$$\begin{aligned} C(j_1,j_2,j\,|m_1,m_2) &= (-1)^{j_1+j_2-j} C(j_2,j_1,j\,|m_2,m_1) \\ &= (-1)^{j_1+j_2-j} C(j_1,j_2,j\,|-m_1,m_2) \ . \end{aligned} \quad \text{(A12)}$$

APPENDIX B

We consider the case of $a+b \to c+d$ using our normalization of states. From Eqs. (29) and (32) we have the differential cross section

$$d\sigma = \frac{\pi^2}{F}\overline{\sum_\mu} |f_\mu|^2\, \delta^4(p_a+p_b-p_c-p_d)\,\frac{d^3p_c}{2E_c}\,\frac{d^3p_d}{2E_d} \quad \text{(B1)}$$

and

$$f_\mu = \langle \vec{p}_c\mu_c, \vec{p}_d\mu_d\,|\,T\,|\,\vec{p}_a\mu_a, \vec{p}_b\mu_b\rangle \ . \quad \text{(B2)}$$

Assuming that both b and d are in χ states, we have from Eq. (38)

$$\begin{aligned} f_\mu = \frac{W}{\pi}(pq)^{-1/2}\, 2 \sum_{JLSL'S'} &[(2L+1)(2L'+1)]^{1/2} C(\sigma_a,\sigma_b,S\,|\,\mu_a,-\mu_b) \\ &\times C(L,S,J\,|\,0,\mu_a-\mu_b) C(\sigma_c,\sigma_d,S'\,|\,\mu_c,-\mu_d) C(L',S',J\,|\,0,\mu_c-\mu_d) \\ &\times D^J_{\mu_c-\mu_d\,\mu_a-\mu_b}(c^{-1}\text{ beam})\,\langle \vec{Q}=0,qJM,L'S'\,|\,T\,|\,\vec{P}=0,pJM,LS\rangle \ . \end{aligned} \quad \text{(B3)}$$

For simplicity we take the beam to be along the z axis, in which case

$$D^J_{\mu_c-\mu_d\,\mu_a-\mu_b}(c^{-1}\text{ beam}) = D^j_{\mu_c-\mu_d\,\mu_a-\mu_b}(c^{-1}). \quad \text{(B4)}$$

We now have

$$\frac{d^3p_c}{2E_c}\,\frac{d^3p_d}{2E_d} = \frac{q}{4W}\, d^3Q\, dW\, d^2\omega \ , \quad \text{(B5)}$$

where ω represents the polar angles of c in the c.m. system. Using conservation of energy and c.m. momentum together with $F = pW$, we have

$$\begin{aligned} d\sigma = (p^2)^{-1}\overline{\sum_\mu}\Bigg|\sum_{JLSL'S'} &[(2L+1)(2L'+1)]^{1/2} C(\sigma_a,\sigma_b,S\,|\,\mu_a,-\mu_b) \\ &\times C(L,S,J\,|\,0,\mu_a-\mu_b) C(\sigma_c,\sigma_d,S'\,|\,\mu_c,-\mu_d) C(L',S',J\,|\,0,\mu_c-\mu_d) \\ &\times D^J_{\mu_c-\mu_d\,\mu_a-\mu_b}(\omega^{-1})\,\langle \vec{Q}=0,qJM,L'S'\,|\,T\,|\,\vec{P}=0,pJM,LS\rangle\Bigg|^2 d^2\omega \ . \end{aligned} \quad \text{(B6)}$$

Using Eq. (A11) and integrating over d^2w, using the normalization of the vector addition coefficients, we see that the cross section becomes

$$\sigma = \frac{\pi}{p^2}\sum_J \frac{2J+1}{(2\sigma_a+1)(2\sigma_b+1)} \sum_{LSL'S} |\langle \vec{Q}=0,qJM,L'S'\,|\,T\,|\,\vec{P}=0,pJM,LS\rangle|^2 \ . \quad \text{(B7)}$$

For the case of $\pi N \to \pi N$, L and L' are determined by parity, $S = S'$, and we have

$$\sigma^{J^P} = \frac{\pi}{p^2}(J+\tfrac{1}{2})\,|\langle J^p | T | J^p \rangle|^2 . \tag{B8}$$

Thus we see that our equations reduce to the usual equations for the two-body process.

APPENDIX C

This appendix, along with the following two appendixes, details derivations of text equations. Here we derive Eq. (64). The total cross section is given by

$$\sigma = \int \frac{\pi^2}{Wp} \overline{\sum_{\mu}} \sum_{nm} g_n^{\mu} g_m^{\mu *} T_n(W, w_j) T_m^*(W, w_j) \frac{\pi q_k Q_j}{8Ww_j} dw_j{}^2\, d\cos\theta_j\, d\cos\Theta\, d\Phi . \tag{63}$$

From Eq. (45), with our choice of axes, we have

$$g_n^{\mu} g_m^{\mu *} = \frac{W^2 w_j}{\pi^3 p Q_j q_k}[(2L+1)(2L'+1)(2L_j+1)(2L_j'+1)(2l_j+1)(2l_j'+1)]^{1/2} \tag{C1a}$$

$$\times C(\sigma_a, \sigma_b, S \mid \mu_a, -\mu_b) C(L, S, J \mid 0, \mu_a - \mu_b) C(\sigma_a, \sigma_b, S' \mid \mu_a, -\mu_b) C(L', S', J' \mid 0, \mu_a - \mu_b) \tag{C1b}$$

$$\times \sum_{\lambda_j \lambda_j'} C(\sigma_j, j_j, S_j \mid \mu_j, -\lambda_j) C(L_j, S_j, J \mid 0, \mu_j - \lambda_j) C(\sigma_j, j_j', S_j' \mid \mu_j, -\lambda_j') C(L_j', S_j', J' \mid 0, \mu_j - \lambda_j') \tag{C1c}$$

$$\times D^{J}_{\mu_j - \lambda_j \mu_a - \mu_b}(\alpha_j, \beta_j, \gamma_j) D^{J' *}_{\mu_j - \lambda_j' \mu_a - \mu_b}(\alpha_j, \beta_j, \gamma_j) \tag{C1d}$$

$$\times \sum_{\nu_k \nu_l \nu_k' \nu_l'} C(\sigma_k, \sigma_l, s_j \mid \nu_k, -\nu_l) C(l_j, s_j, j_j \mid 0, \nu_k - \nu_l) C(\sigma_k, \sigma_l, s_j' \mid \nu_k', -\nu_l') \tag{C1e}$$

$$\times C(l_j', s_j', j_j' \mid 0, \nu_k' - \nu_l') d^{j_j}_{-\lambda_j \nu_k - \nu_l}(\theta_j) d^{j_j'}_{-\lambda_j' \nu_k' - \nu_l'}(\theta_j) \tag{C1f}$$

$$\times d^{\sigma_k}_{\nu_k \mu_k}(\theta_j^k) d^{\sigma_l}_{\nu_l \mu_l}(-\theta_j^l) d^{\sigma_k}_{\nu_k' \mu_k}(\theta_j^k) d^{\sigma_l}_{\nu_l' \mu_l}(-\theta_j^l)\, (-1)^{\sigma_l - \nu_l} (-1)^{\sigma_l - \nu_l'} . \tag{C1g}$$

To evaluate the total cross section, we will discuss each of the parts separately. Using Eqs. (A7) and (A9) and summing over μ_k and μ_l, line (C1g) becomes

$$\sum_{\mu_k \mu_l} d^{\sigma_k}_{\nu_k \mu_k}(\theta_j^k) d^{\sigma_k}_{\nu_k' \mu_k}(\theta_j^k) d^{\sigma_l}_{\nu_l \mu_l}(-\theta_j^l) d^{\sigma_l}_{\nu_l' \mu_l}(-\theta_j^l)\, (-1)^{\sigma_l \nu_k} (-1)^{\sigma_l \nu_l'} = d^{\sigma_k}_{\nu_k \nu_k'}(0) d^{\sigma_l}_{\nu_l \nu_l'}(0) (-1)^{\sigma_l - \nu_l} (-1)^{\sigma_l - \nu_l'}$$

$$= \delta_{\nu_k \nu_k'} \delta_{\nu_l \nu_l'} . \tag{C2}$$

In (C1d) we have

$$D^{J}_{\mu_j - \lambda_j \mu_a - \mu_b}(\alpha_j, \beta_j, \gamma_j) D^{J' *}_{\mu_j - \lambda_j' \mu_a - \nu_b}(\alpha_j, \beta_j, \gamma_j)$$

$$= \sum_{MM'} D^{J}_{\mu_j - \lambda_j M}(j^{-1}) D^{J' *}_{\mu_j - \lambda_j' M'}(j^{-1}) D^{J}_{M \mu_a - \mu_b}(\Phi, \Theta, -\Phi) D^{J' *}_{M' \mu_a - \mu_b}(\Phi, \Theta, -\Phi) . \tag{C3}$$

Using the normalization Eq. (A11) and integrating over $d\cos\Theta d\Phi$ gives

$$\int (\mathrm{C1d})\, d\cos\Theta\, d\Phi = \sum_{MM'} D^{J}_{\mu_j - \lambda_j M}(j^{-1}) D^{J' *}_{\mu_j - \lambda_j' M'}(j^{-1}) \frac{4\pi}{2J+1} \delta_{JJ' MM'}$$

$$= D^{J}_{\mu_j - \lambda_j \mu_j - \lambda_j'}(j^{-1} j) \frac{4\pi}{2J+1} \delta_{JJ'}$$

$$= \frac{4\pi}{2J+1} \delta_{JJ'} \delta_{\lambda_j \lambda_j'} . \tag{C4}$$

With the δ functions from Eqs. (C2) and (C4), the integration of line (C1f) over $d\cos\theta_j$ yields

$$\int d^{j_j}_{-\lambda_j \nu_k - \nu_l}(\theta_j) d^{j_j'}_{-\lambda_j \nu_k - \nu_l}(\theta_j)\, d\cos\theta_j = \frac{2}{2j_j+1} \delta_{j_j j_j'} . \tag{C5}$$

With this we see that isobars with different total spin, j_j, do not interfere in the total cross section.

With the δ functions and the orthogonality of the vector addition coefficients, line (C1e) reduces to

$$\sum_{\nu_k\nu_l} C(\sigma_k,\sigma_l,s_j\,|\,\nu_k,-\nu_l)C(l_j,s_j,j_j\,|\,0,\nu_k-\nu_l)C(\sigma_k,\sigma_l,s_j'\,|\,\nu_k,-\nu_l)C(l_j',s_j',j_j\,|\,0,\nu_k-\nu_l)=\frac{2j_j+1}{2l_j+1}\,\delta_{l_jl_j'}\,\delta_{s_js_j'}\,. \tag{C6}$$

Similiarly,

$$\sum_{\mu_a\mu_b}(\text{C1b})=\frac{2J+1}{2L+1}\,\delta_{LL'}\,\delta_{SS'}\,,$$
$$\sum_{\mu_j\lambda_j}(\text{C1c})=\frac{2J+1}{2L_j+1}\,\delta_{L_jL_j'}\,\delta_{S_jS_j'}\,. \tag{C7}$$

With these intermediate results the total cross section is given by

$$\sigma=\frac{1}{8p^2}\frac{1}{(2\sigma_a+1)(2\sigma_b+1)}\int\sum_{nm}(2L+1)(2L_j+1)(2l_j+1)\frac{2J+1}{2L+1}\frac{(2J+1)}{(2L_j+1)}\frac{(2j_j+1)}{(2l_j+1)}$$
$$\times\delta_{JJ'}\delta_{LL'}\delta_{L_jL_j'}\delta_{SS'}\delta_{S_jS_j'}\delta_{l_jl_j'}\delta_{s_js_j'}$$
$$\times\frac{4\pi}{2J+1}\frac{2}{2j_j+1}\,T_n(W,w_j)T_m^*(W,w_j)dw_j^{\,2}$$
$$=\frac{\pi}{p^2}\sum_n\frac{2J+1}{(2\sigma_a+1)(2\sigma_b+1)}\int|T_n(W,w_j)|^2dw_j^{\,2}\,. \tag{C8}$$

APPENDIX D

Here we derive Eq. (70) of the text. From Eqs. (45) and (48) we have

$$g_n^{-\mu}=\frac{W}{\pi}\left(\frac{w_j}{\pi pQ_jq_k}\right)^{1/2}[(2L+1)(2L_j+1)(2l_j+1)]^{1/2}C(\sigma_a,\sigma_b,s\,|-\mu_a,\mu_b)C(L,S,J|0,-\mu_a+\mu_b) \tag{D1a}$$

$$\times\sum_{\lambda_j}C(\sigma_j,j_j,S_j|-\mu_j,-\lambda_j)C(L_j,S_j,J|0,-\mu_j-\lambda_j)D^J_{-\mu_j-\lambda_j-\mu_a+\mu_b}(\alpha_j,\beta_j,\gamma_j) \tag{D1b}$$

$$\times\sum_{\nu_k\nu_l}C(\sigma_k,\sigma_l,s_j|\nu_k,-\nu_l)C(l_j,s_j,j_j|0,\nu_k-\nu_l)d^{j_j}_{-\lambda_j\nu_k-\nu_l}(\theta_j) \tag{D1c}$$

$$\times d^{\sigma_k}_{\nu_k-\mu_k}(\theta_j^k)d^{\sigma_l}_{\nu_l-\mu_l}(-\theta_j^l)(-1)^{\sigma_l-\nu_l}\,. \tag{D1d}$$

Using Eq. (A9), line (D1d) becomes

$$d^{\sigma_k}_{\nu_k-\mu_k}(\theta_j^k)d^{\sigma_l}_{\nu_l-\mu_l}(-\theta_j^l)(-1)^{\sigma_l-\nu_l}=(-1)^{\nu_k+\mu_k}(-1)^{\nu_l+\mu_l}d^{\sigma_k}_{-\nu_a\mu_k}(\theta_j^k)d^{\sigma_l}_{-\nu_l\mu_l}(-\theta_j^l)(-1)^{\sigma_l-\nu_l}$$
$$=(-1)^{\sigma_l+\mu_l}(-1)^{\nu_k+\mu_k}d^{\sigma_k}_{-\nu_k\mu_k}(\theta_j^k)d^{\sigma_l}_{-\nu_l\mu_l}(-\theta_j^l)\,, \tag{D2}$$

and line (D1c) becomes

$$d^{j_j}_{-\lambda_j\nu_k-\nu_l}(\theta_j)=(-1)^{\lambda_j-\nu_k+\nu_l}d^{j_j}_{\lambda_j-\nu_k+\nu_l}(\theta_j)\,. \tag{D3}$$

Since

$$D^J_{ab}(\alpha,\beta,\gamma)=(-1)^{a-b}D^{J*}_{a-b}(\alpha,\beta,\gamma)\,, \tag{D4}$$

line (D1b) reduces to

$$D^J_{-\mu_j-\lambda_j-\mu_a+\mu_b}(\alpha_j,\beta_j,\gamma_j)=(-1)^{-\mu_j-\lambda_j+\mu_a-\mu_b}D^{J*}_{\mu_j+\lambda_j\mu_a-\mu_b}(\alpha_j,\beta_j,\gamma_j)$$
$$=(-1)^{2J}(-1)^{\mu_j+\lambda_j+\mu_a-\mu_b}D^{J*}_{\mu_j+\lambda_j\mu_a-\mu_b}(\alpha_j,\beta_j,\gamma_j)\,. \tag{D5}$$

Using Eq. (A10) and making all the substitutions, we have

$$g_n^{-\mu}=\frac{W}{\pi}\left(\frac{w_j}{\pi p Q_j q_k}\right)^{1/2}[(2L+1)(2L_j+1)(2l_j+1)]^{1/2}(-1)^{L+S-J}(-1)^{\sigma_a+\sigma_b-s}C(\sigma_a,\sigma_b,s|\mu_a,-\mu_b)C(L,S,J|0,\mu_a-\mu_b)$$
$$\times\sum_{\lambda_j}(-1)^{L_j+S_j-J}(-1)^{\sigma_j+j_j-S_j}C(\sigma_j,j_j,S_j|\mu_j,+\lambda_j)C(L_j,S_j,J|0,\mu_j+\lambda_j)$$
$$\times(-1)^{2J}(-1)^{\mu_j+\lambda_j+\mu_a-\mu_b}D^{J*}_{\mu_j+\lambda_j\,\mu_a-\mu_b}(\alpha_j,\beta_j,\gamma_j)$$
$$\times\sum_{\nu_k\nu_l}(-1)^{l_j+s_j-j_j}(-1)^{\sigma_k+\sigma_l-s_j}C(\sigma_k,\sigma_l,s_j|-\nu_k,\nu_l)C(l_j,s_j,j_j|0,-\nu_k+\nu_l)$$
$$\times(-1)^{-\lambda_j-\nu_k+\nu_l}d^{j_j}_{\lambda_j-\nu_k+\nu_l}(\theta_j)(-1)^{\sigma_l+\mu_l}(-1)^{\nu_k+\mu_k}d^{\sigma_k}_{-\nu_k\,\mu_k}(\theta_j^k)d^{\sigma_l}_{-\nu_l\,\mu_l}(-\theta_j^l)\,. \tag{D6}$$

Since λ_j, ν_k, and ν_l are just dummy variables, we can make the change $\lambda_j\to-\lambda_j$, $\nu_k\to-\nu_k$, and $\nu_l\to-\nu_l$. Thus

$$g_n^{-\mu}=(-1)^{L+L_j+l_j}(-1)^{\sigma_a+\mu_a}(-1)^{\sigma_b-\mu_b}(-1)^{\sigma_j+\mu_j}(-1)^{\sigma_k+\mu_k}(-1)^{\sigma_k+\mu_l}g_n^{\mu\,*}\,. \tag{D7}$$

Since we have assumed that L, L_j, and l_j are chosen to conserve parity, we have that

$$1=\eta_a\eta_b\eta_j\eta_k\eta_l(-1)^{L+L_j+l_j}$$
$$=\eta(-1)^{L+L_j+l_j}, \tag{D8}$$

where η is the product of all five parities. Finally then

$$g_n^{-\mu}=\eta(-1)^{\sigma_a+\mu_a}(-1)^{\sigma_b-\mu_b}(-1)^{\sigma_j+\mu_j}$$
$$\times(-1)^{\sigma_k+\mu_k}(-1)^{\sigma_l+\mu_l}g_n^{\mu\,*}\,. \tag{D9}$$

APPENDIX E

In this appendix we derive Eqs. (74)–(76) for the interchange of particles k and l. From Eq. (73) we have the following changes under interchange:

$$W\to W,$$
$$w_j^2\to w_j^2,\quad w_k^2\to w_l^2,\quad w_l^2\to w_k^2,$$
$$\Theta\to\Theta,\quad \Phi\to\Phi+\pi,$$
$$\Theta_j\to\Theta_j,\quad \Theta_k\to\Theta_l,\quad \Theta_l\to\Theta_k,$$
$$\Phi_j=0\to\Phi_j=0,\quad \Phi_k=0\to\Phi_k=0,\quad \Phi_l=\pi\to\Phi_l=\pi,$$
$$\theta_j\to\pi-\theta_j,\quad \theta_k\to\pi-\theta_l,\quad \theta_l\to\pi-\theta_k,$$
$$\theta_j^k\to\theta_j^l,\quad \theta_j^l\to\theta_j^k,$$
$$\theta_k^l\to\theta_l^k,\quad \theta_l^k\to\theta_k^l,$$
$$\theta_k^j\to\theta_l^j,\quad \theta_l^j\to\theta_k^j,$$
$$\mu_a\to\mu_a,\quad \mu_b\to\mu_b,$$
$$\mu_j\to\mu_j,\quad \mu_k\to\mu_l,\quad \mu_l\to\mu_k\,.$$

Most of the changes are obvious. For convenience we shall let $\mu'=(\mu_a\,\mu_b\,\mu_j\,\mu_l\,\mu_k)$. We also indicate the type of isobar with an additional subscription on g.

For j-type isobars, we have

$$g_{nj}^{\mu'}(w_j^2w_l^2w_k^2)=\frac{W}{\pi}\left(\frac{w_j}{\pi p Q_j q_k}\right)^{1/2}[(2L+1)(2L_j+1)(2l_j+1)]^{1/2}C(\sigma_a,\sigma_b,s|\mu_a,-\mu_b)C(L,S,J|0,\mu_a-\mu_b)$$
$$\times\sum_{\lambda_j}C(\sigma_j,j_j,S_j|\mu_j,-\lambda_j)C(L_j,S_j,J|0,\mu_j-\lambda_j)D^J_{\mu_j-\lambda_j\,\mu_a-\mu_b}(\alpha_j',\beta_j',\gamma_j')$$
$$\times\sum_{\nu_k\nu_l}C(\sigma_l,\sigma_k,s_j|\nu_k,-\nu_l)C(l_j,s_j,j_j|0,\nu_k-\nu_l)d^{j_j}_{-\lambda_j\,\nu_k-\nu_l}(\pi-\theta_j)$$
$$\times d^{\sigma_l}_{\nu_k\,\mu_l}(\theta_j^l)d^{\sigma_k}_{\nu_l\,\mu_k}(-\theta_j^k)(-1)^{\sigma_k-\nu_l}\,. \tag{E1}$$

Now since $\Theta_j=\Phi_j=0$, we have $\alpha_j=\Phi$, $\beta_j=\Theta$, $\gamma_j=-\Phi$, thus

$$\alpha_j'=\Phi+\pi=\alpha_j+\pi,\quad \beta_j'=\beta_j,\quad \gamma_j'=-\Phi-\pi=\gamma_j-\pi,$$

and

$$D^J_{\mu_j-\lambda_j\,\mu_a-\mu_b}(\alpha_j',\beta_j',\gamma_j')=e^{-i\pi(\mu_j-\lambda_j)}e^{i\pi(\mu_a-\mu_b)}D^J_{\mu_j-\lambda_j\,\mu_a-\mu_b}(\alpha_j,\beta_j,\gamma_j)\,. \tag{E2}$$

From Eq. (A9) we have

$$\begin{aligned} d^{j_j}_{-\lambda_j \nu_k - \nu_l}(\pi - \theta_j) &= (-1)^{j_j - \lambda_j} d^{j_j}_{-\lambda_j \nu_l - \nu_k}(\theta_j), \\ d^{\sigma_l}_{\nu_k \mu_l}(\theta^l_j) &= (-1)^{\nu_k - \mu_l} d^{\sigma_l}_{\nu_k \mu_l}(-\theta^l_j), \\ d^{\sigma_k}_{\nu_l \mu_k}(-\theta^k_j) &= (-1)^{\nu_l - \mu_k} d^{\sigma_k}_{\nu_l \mu_k}(\theta^k_j) . \end{aligned} \tag{E3}$$

Since ν_k and ν_l are just dummy indices, we let $\nu_k \to -\nu_l$ and $\nu_l \to -\nu_k$. Thus

$$\begin{aligned} g^{\mu'}_{nj}(w_j{}^2 w_l{}^2 w_k{}^2) = \frac{W}{\pi}\left(\frac{w_j}{\pi p Q_j q_k}\right)^{1/2} & [(2L+1)(2L_j+1)(2l_j+1)]^{1/2} C(\sigma_a, \sigma_b, s|\mu_a, -\mu_b) C(L, S, J|0, \mu_a - \mu_b) \\ & \times \sum_{\lambda_j} C(\sigma_j, j_j, S_j|\mu_j, -\lambda_j) C(L_j, S_j, J|0, \mu_j - \lambda_j)(-1)^{\mu_a - \mu_b - \mu_j + \lambda_j} D^J_{\mu_j - \lambda_j\, \mu_a - \mu_b}(\alpha_j, \beta_j, \gamma_j) \\ & \quad \times \sum_{\nu_k \nu_l} C(\sigma_k, \sigma_l, s_j|\nu_k, -\nu_l) C(l_j, s_j, l_j|0, \nu_k - \nu_l)(-1)^{l_j + s_j - j_j}(-1)^{j_j - \lambda_j} \\ & \quad\quad \times d^{j_j}_{-\lambda_j \nu_k - \nu_l}(\theta_j) d^{\sigma_k}_{\nu_k \mu_k}(\theta^k_j) d^{\sigma_l}_{\nu_l \mu_l}(-\theta^l_j)(-1)^{\sigma_k - \nu_k}(-1)^{\nu_k - \mu_l}(-1)^{\nu_l - \mu_k} \end{aligned} \tag{E4}$$

$$= (-1)^{l_j}(-1)^{s_j + \sigma_k + \sigma_l}(-1)^{\mu_a - \mu_b - \mu_j - \mu_k - \mu_l} g^{\mu}_{nj}(w_j{}^2 w_k{}^2 w_l{}^2) . \tag{74}$$

For k- or l-type isobars, the interchange is only meaningful when k and l are the same type of particle. In this case similar calculations give for k-type isobars

$$g^{\mu'}_{nk}(w_j{}^2 w_l{}^2 w_k{}^2) = (-1)^{l_k}(-1)^{s_k + \sigma_j + \sigma_l}(-1)^{\mu_a - \mu_b - \mu_j - \mu_k - \mu_l} g^{\mu}_{nl}(w_j{}^2 w_k{}^2 w_l{}^2) \tag{75}$$

and for l-type isobars

$$g^{\mu'}_{nl}(w_j{}^2 w_l{}^2 w_k{}^2) = (-1)^{l_l}(-1)^{s_l + \sigma_j + \sigma_k}(-1)^{\mu_a - \mu_b - \mu_j - \mu_k - \mu_l} g^{\mu}_{nk}(w_j{}^2 w_k{}^2 w_l{}^2) . \tag{76}$$

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