PHYSICAL REVIEW D VOLUME 11, NUMBER 10 15 MAY 1975

Remarks on the supersymmetry algebra

A. Chakrabarti

*Centre de Physique Théorique de l'Ecole Polytechnique, 17, rue Descartes, 75. Paris V, France**

(Received 26 November 1974)

We study certain classes of explicit realizations of the supersymmetry algebra, including internal symmetry, and some of their properties.

We propose to study certain simple explicit constructions of the supersymmetry algebra (including internal symmetry) and some of their properties. Instead of the full algebra of Wess and Zumino,[1] we will consider the subalgebra studied by Salam and Strathdee,[2] whose ideas we mainly follow. Let us first briefly fix our notations.

We start with a 2-component spinor

$$Q=(Q_1,Q_2), \quad Q^\dagger=\begin{pmatrix} Q_1^\dagger \\ Q_2^\dagger \end{pmatrix}, \tag{1}$$

such that apart from the usual Poincaré algebra of the generators $P_\mu, J_{\mu\nu}$, we have (with $M_1=J_{23}$, $N_1=J_{01}$, etc. and the Pauli matrices τ)

$$[Q^\dagger, P_\mu]=0, \tag{2}$$

$$[Q^\dagger, \vec{\mathrm{M}}]=\tfrac{1}{2}\vec{\tau}Q^\dagger, \tag{3}$$

$$[Q^\dagger, \vec{\mathrm{N}}]=\tfrac{1}{2}i\vec{\tau}Q^\dagger, \tag{4}$$

and

$$\{Q^\dagger, Q\}=\tau^0P^0+\vec{\tau}\cdot\vec{\mathrm{P}}. \tag{5}$$

To complete a 4-component Majorana spinor (Q_1,Q_2,Q_3,Q_4) we can introduce

$$Q'=(Q_3,Q_4)=(Q_2^\dagger, -Q_1^\dagger) \tag{6}$$

such that (4) and (5) are changed to

$$[Q'^\dagger, \vec{\mathrm{N}}]=-\tfrac{1}{2}i\vec{\tau}Q'^\dagger, \tag{7}$$

$$\{Q'^\dagger, Q'\}=\tau^0P^0-\vec{\tau}\cdot\vec{\mathrm{P}}. \tag{8}$$

Equations (2) and (3) hold also for Q'. For the present we will consider only Q and will introduce internal symmetry only later on.

As in Ref. 2 we introduce two fermion creation operators (a_1,a_2) so that we have

$$\{a_i,a_j\}=0=\{a_i^\dagger,a_j^\dagger\}, \quad \{a_i,a_j^\dagger\}=\delta_{ij} \qquad (i=1,2), \tag{9}$$

or

$$\{a,a^\dagger\}=\tau_0. \tag{10}$$

(N.B. Please note that, in order to follow the notation of Salam and Strathdee, contrary to conventional notation we are denoting the annihilation operator with a dagger.) Let

$$\begin{aligned}\Sigma_3^F&=\tfrac{1}{2}(a_1a_1^\dagger-a_2a_2^\dagger),\\ \Sigma_+^F&=\Sigma_1^F+i\Sigma_2^F=a_1a_2^\dagger,\\ \Sigma_-^F&=\Sigma_1^F-i\Sigma_2^F=a_2a_1^\dagger;\end{aligned} \tag{11}$$

thus with $\vec{\Sigma}^F=\frac{1}{2}(a\vec{\tau}a^\dagger)$, we have

$$[\vec{\Sigma}^F, a^\dagger]=-\tfrac{1}{2}\vec{\tau}a^\dagger, \tag{12}$$

and $\vec{\Sigma}^F$ are generators of the SU(2) algebra, acting on a space of spin $\frac{1}{2}$ given by

$$\begin{pmatrix} a_1|0\rangle \\ a_2|0\rangle \end{pmatrix}.$$

We now make the simple observation that the inverse of the 2×2 spinor matrix corresponding to any one of the well-known transformations which transform the spinor representation to a unitary one,[3-5] applied to (a_1,a_2) will lead to the anticommutators (5) [or (8) according to the type of fundamental 2-component spinor representation chosen]. Along with this we will combine a simple modification of the unitary Lorentz generators to achieve our goal.

Let us immediately give an example using the canonical case. Here the 2×2 transformation matrices are (for a space with $P^0>0$)

$$[P]_\pm=\frac{1}{[2M(P^0+M)]^{1/2}}[(P^0+M)\pm\vec{\tau}\cdot\vec{\mathrm{P}}]. \tag{13}$$

[Here $M\equiv(P^\mu P_\mu)^{1/2}$, the positive square root. To include the case $P^0<0$ we have only to write throughout ϵM where $\epsilon=P^0/|P^0|$.] Thus, defining

$$Q^\dagger=\sqrt{M}[P]_+\begin{pmatrix} a_1^\dagger \\ a_2^\dagger \end{pmatrix}, \tag{14}$$

or

$$Q_1^\dagger = [2(P^0+M)]^{-1/2}[(P^0+M+P^3)a_1^\dagger + (P^1-iP^2)a_2^\dagger],$$
$$Q_2^\dagger = [2(P^0+M)]^{-1/2}[(P^1+iP^2)a_1^\dagger + (P^0+M-P^3)a_2^\dagger], \tag{15}$$

we obtain

$$\{Q, Q^\dagger\} = \tau^0 P^0 + \vec{\tau}\cdot\vec{P}. \tag{16}$$

Let us now define a modified $J_{\mu\nu}$ as

$$\vec{M} = -i(\vec{P}\times\vec{\partial}) + (\vec{\Sigma}+\vec{\Sigma}^F),$$
$$\vec{N} = -iP^0\vec{\partial} - \frac{\vec{P}\times(\vec{\Sigma}+\vec{\Sigma}^F)}{P^0+M}. \tag{17}$$

We have simply added to the usual $(2s+1)\times(2s+1)$ spin matrix $\vec{\Sigma}$ of the canonical representation the $\vec{\Sigma}^F$, defined in (11), to constitute a total spin

$$\vec{\Sigma}_T = \vec{\Sigma} + \vec{\Sigma}^F.$$

The Poincaré algebra is evidently satisfied. From (15) and (17) one obtains (3) and (4), namely

$$[\vec{M}, Q^\dagger] = -\tfrac{1}{2}\vec{\tau}Q^\dagger,$$
$$[\vec{N}, Q^\dagger] = -\tfrac{1}{2}i\vec{\tau}Q^\dagger. \tag{18}$$

Thus, we have an explicit realization of the entire supersymmetry algebra *acting on arbitrary momentum states.*

We have, from (6),

$$\begin{pmatrix} Q_3'^\dagger \\ Q_4'^\dagger \end{pmatrix} = \begin{pmatrix} Q_2 \\ -Q_1 \end{pmatrix} = \sqrt{M}[P]_-(i\tau_2)\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \tag{19}$$

Evidently we might have interchanged the roles of $[P]_\pm$ and obtained (7) and (8) to start with by directly transforming with $[P]_-$.

We would like to emphasize the following point. If we modify the Lorentz generators of the spinor representations in an analogous fashion and write

$$\vec{M} = -i\vec{P}\times\vec{\partial} + (\vec{\Sigma}+\vec{\Sigma}^F),$$
$$\vec{N} = -iP^0\vec{\partial} \pm i(\vec{\Sigma}+\vec{\Sigma}^F), \tag{20}$$

we obtain, evidently, directly for the a's,

$$[a^\dagger, \vec{M}] = \tfrac{1}{2}\vec{\tau}a^\dagger,$$
$$[a^\dagger, \vec{N}] = \pm\tfrac{1}{2}i\vec{\tau}a^\dagger, \tag{21}$$

along with

$$\{a, a^\dagger\} = \tau_0. \tag{22}$$

Since in (20) $\vec{\Sigma}^F$ appear as a direct sum only, here the Poincaré and supersymmetry transformations remain effectively decoupled. The system (20)–(22) is *not equivalent* to the one given by (14)–(17). Considering the case $\vec{\Sigma} = \frac{1}{2}\vec{\tau}$ as an example and transforming (20) with $[P]_\pm$, respectively [for the two signs in $\vec{N}$ in (20)], we obtain

$$\vec{N} = -iP^0\vec{\partial} - \frac{1}{2}\frac{\vec{P}\times\vec{\tau}}{P^0+M} \pm i\vec{\Sigma}^F, \tag{17$'$}$$

and not (17) (with $\vec{\Sigma} = \frac{1}{2}\vec{\tau}$).

Thus the combination of the two simple *Ansätze* (14) and (17) permits us to obtain the necessary properties. One may, of course, add more sets of anticommuting Q's as an evident generalization.

As we have already mentioned, we can use the $[P]_\pm$ matrices corresponding to other representations also. We have studied elsewhere[5] many interesting properties of the null-plane (or light-cone) formalism. Let us illustrate the introduction of supersymmetry and *internal symmetry* in that context, which also leads to some special interesting features.

We have now the fermion creation operators with with an internal symmetry index α (see Ref. 2):

$$a_i^\alpha \quad (i=1,2;\quad \alpha=1,2,\ldots,n), \tag{23}$$

with

$$\{a_i^\alpha, a_j^{\beta\dagger}\} = \delta_{ij}\,\delta_{\alpha\beta}$$

and

$$\sqrt{M}[P]_+ = (P^0+P^3)^{-1/2}\begin{pmatrix} (P^0+P^3) & 0 \\ (P^1+iP^2) & M \end{pmatrix}. \tag{24}$$

Hence,

$$Q_1^{\alpha\dagger} = (P^0+P^3)^{1/2}a_1^{\alpha\dagger},$$
$$Q_2^{\alpha\dagger} = \frac{(P^1+iP^2)}{(P^0+P^3)^{1/2}}a_1^{\alpha\dagger} + \frac{M}{(P^0+P^3)^{1/2}}a_2^\dagger. \tag{25}$$

Here it is particularly evident that for $M=0$, a_2 drops out. Let

$$\vec{\Sigma}^F = \sum_\alpha (\vec{\Sigma}^F_{(\alpha)}), \tag{26}$$

where

$$\vec{\Sigma}^F_{(\alpha)} = \tfrac{1}{2}(a^\alpha\vec{\tau}a^{\alpha\dagger}). \tag{27}$$

In the notation of Ref. 5, the independent components of P_μ being p^1, p^2, and $p_n = (1/\sqrt{2})(p^0+p^3)$, the modified generators can now be written as

$$J_3 = -i(P^1\partial_2 - P^2\partial_1) + (\Sigma_3 + \Sigma_3^F),$$
$$K_3 = -iP_n\partial_n,$$
$$B_i = -iP_n\partial_i \quad (i=1,2), \tag{28}$$
$$S_1 = -i(P_n\partial_1 + P^1\partial_n) - \frac{P^2}{P_n}(\Sigma_3 + \Sigma_3^F) - \frac{M}{P_n}(\Sigma_2 + \Sigma_2^F),$$
$$S_2 = -i(P_n\partial_2 + P^2\partial_n) + \frac{P^1}{P_n}(\Sigma_3 + \Sigma_3^F) + \frac{M}{P_n}(\Sigma_1 + \Sigma_1^F).$$

It may be verified that Q_i^α satisfies

$$\{Q^{\alpha\dagger}, Q^\beta\} = (P^0\tau^0 + \vec{P}\cdot\vec{\tau})\delta_{\alpha\beta} \tag{29}$$

and

$$[Q^{\alpha\dagger}, \vec{M}] = \tfrac{1}{2}\vec{\tau}Q^{\alpha\dagger}, \quad [Q^{\alpha\dagger}, \vec{N}] = \tfrac{1}{2}i\vec{\tau}Q^{\alpha\dagger}. \tag{30}$$

Let

$$A^\alpha_{(1)\beta} = a_1^\alpha a_1^{\beta\dagger}, \quad A^\alpha_{(2)\beta} = a_2^\alpha a_2^{\beta\dagger}. \tag{31}$$

Then

$$[A^\alpha_{(i)\beta}, A^{\alpha'}_{(j)\beta'}] = \delta_{ij}(\delta_{\beta\alpha'}A^\alpha_{(i)\beta'} - \delta_{\alpha\beta'}A^{\alpha'}_{(i)\beta}) \qquad (i,j=1,2), \tag{32}$$

and we have an algebra $U(n)\otimes U(n)$ [or $SU(n)\otimes SU(n)$]. Defining

$$A^\alpha_\beta = A^\alpha_{(1)\beta} + A^\alpha_{(2)\beta}, \tag{33}$$

we have

$$[A^\alpha_\beta, \vec{\Sigma}^F] = 0. \tag{34}$$

Hence, these are the generators of the internal-symmetry group [the "diagonal $U(n)$"] commuting with the Poincaré subalgebra and transforming the supersymmetry generators as

$$[A^\alpha_\beta, Q^\nu] = \delta_{\beta\gamma}Q^\alpha, \tag{35}$$

since $[A^\alpha_{(i)\beta}, a_j^\gamma] = \delta_{ij}\delta_{\beta\gamma}a_i^\alpha$. Let us also note that

$$[\Sigma_3^F, A^\alpha_{(i)\beta}] = 0. \tag{36}$$

Hence, the entire $U(n)\otimes U(n)$ *commutes with the Galilean subgroup,*[5] or rather, with all the Poincaré generators except S_1, S_2. Since the generators K_3, B_i suffice to transform any p_μ to any other p'_μ on the same mass shell, we are far from a rest symmetry though it is not totally Poincaré invariant either. This is a special feature of the null-plane formalism, and it is not true in general. We will come back to this point at the end. Our type of introduction of internal symmetry may now be compared with that of Ref. 6. While the Q^α transform as spinors, the a^α undergo Wigner rotations under Lorentz transformations. The only nontrivial ones are, in the present case, those corresponding to transformations $e^{-i\underline{u}\cdot\underline{S}}$ and are given by (in the notation of Ref. 5)

$$\mathfrak{D}^{1/2}(R_{(B)}(u, P)). \tag{37}$$

Here the momenta are to be considered as operators as in formula (2.15) of Ref. 5. Similarly, for the generalized canonical or helicity representations the a's will undergo the corresponding well-known Wigner rotations.[7]

In fact, in many respects it is convenient to discard the Q's and consider directly as the supersymmetry generators the a_i^α which have the usual fermion anticommutation relations but undergo some suitably chosen Wigner rotations under Lorentz transformations.

They are, of course, to be combined with the usual Poincaré states defined through the corresponding boost.

Let us now consider the construction of states, using directly the a's. The usual basis of the Poincaré algebra using null-plane spin projection[5] will be written as

$$|p, \nu\rangle = e^{-i\underline{v}\cdot\underline{B}}e^{-i\omega K_3}|\vec{0}, \nu\rangle, \tag{38}$$

where

$$e^\omega = (p^0+p^3)/m, \quad \underline{v} = \underline{p}/(p^0+p^3), \qquad \underline{p} = (p_1, p_2), \underline{B} = (B_1, B_2). \tag{39}$$

Let $f(a_i^\alpha)$ be any polynomial of the creation operators (only the fully antisymmetrized component surviving). Then starting, not only from rest states, but from a state of arbitrary momentum, and defining

$$U(\Lambda) = e^{-i\underline{v}'\cdot\underline{B}}e^{-i\omega' K_3},$$

we obtain

$$U(\Lambda)f(a_i^\alpha)|p, \nu\rangle = f(a_i^\alpha)|p', \nu\rangle \qquad (\Lambda\cdot p = p'). \tag{40}$$

This "enlarged invariance" is again a special feature of the null-plane formalism. For the canonical states we have an analogous behavior for *pure* Lorentz transformations such that $\vec{p}\times\vec{p}' = 0$.

Applied on *rest states,* the Q_i^α coincide with the $\sqrt{m}a_i^\alpha$. But their transformation properties are quite different.

The technique of construction of basis vectors using fermion operators is well known and is described in detail in Ref. 8. Let us only note that because of (37), if an eigenstate of the total spin

$$(\vec{\Sigma}_T)^2 = (\vec{\Sigma} + \vec{\Sigma}^F)^2$$

is constructed by using the rotation group Clebsch-Gordan coefficients in the building of $f(a_i^\alpha)|p, \nu\rangle$, then it has indeed a Lorentz invariance significance.

Again, since we can consider the $a_i^{\alpha\dagger}$ directly as the generators of supersymmetry, their action on the states $f(a)|p, \nu\rangle$ is seen to be given simply by commuting with $f(a)$ since

$$a_i^{\alpha\dagger}|p,\nu\rangle = 0 \ .$$

Once we have constructed the eigenstates of P_μ, $(\vec{\Sigma}_T)^2$, and Σ_{T3} [the basis being spanned by the eigenvalues of $(\vec{\Sigma})^2$, $(\vec{\Sigma}^F)^2$, and other internal-symmetry quantum numbers], the reduction of the direct products of states *with respect to the Poincaré group* can be carried out using standard techniques.[5-7] We have to bear in mind the fact that the fermion operators attached to particles 1 and 2 ($a_{(1)i}^\alpha$, $a_{(2)i}^\alpha$, and their adjoints) are supposed to anticommute totally for different particle indices [(1) and (2)]. The operators such as $\vec{\Sigma}_{(1)}^F$ and $\vec{\Sigma}_{(2)}^F$ mutually commute, as they should.[2]

As for the internal-symmetry indices, it would be interesting to study the matrix elements on a suitable basis of all the generators of the *inhomogeneous symmetry group* IU(n) with "anticommuting translation generators" (formed by $A_{(i)\,\beta,a_i^\gamma,a_i^{\gamma\dagger}}^\alpha$). Elsewhere,[9] we have given a simple construction for a basis and the matrix elements for usual (commuting) translation generators. Our present case is quite different. But in practice we will be concerned only with low-dimensional representations. Direct computation for each particular case will not be too difficult. Once the matrix elements of this particular type of IU(n) are obtained, one may proceed to extract their consequences concerning the matrix elements of operators of physical interest, assuming that we can atrribute to these operators definite transformation properties under the group. One may attempt to do this through the construction of suitable Lagrangians or directly through phenomenological postulates.

Let us close our discussion with two more remarks.

Elsewhere,[10] we have constructed Poincaré generators for zero-mass continuous spin and spacelike representations which involve the generators of E_3 and O(3, 1) apart from P_μ and ∂_μ. With a finite number of fermion operators one cannot generate infinite-dimensional unitary representations of E_3 and O(3, 1).[11] But we also mentioned that an analogous construction for the timelike case is possible with O(4) generators. Thus, for example, we may write (for $M^2>0$, $P^0>0$),

$$\begin{aligned} \vec{\mathrm{M}} &= -i\,\vec{\mathrm{P}}\times\vec{\partial} + \vec{\mathrm{S}}\,, \\ \vec{\mathrm{N}} &= -i\,P^0\,\vec{\partial} + \frac{1}{(\vec{\mathrm{P}})^2}\,(P^0\vec{\mathrm{S}} + M\vec{\mathrm{K}})\times\vec{\mathrm{P}}\ , \end{aligned} \tag{41}$$

where

$$\begin{aligned} [S_1,S_2] &= iS_3\ , \\ [S_1,K_2] &= iK_3\ , \\ [K_1,K_2] &= iS_3\ ,\ \text{etc.} \end{aligned} \tag{42}$$

We have

$$\frac{1}{(\vec{\mathrm{P}})^2}\,(P^0\vec{\mathrm{S}} + M\vec{\mathrm{K}}) = \frac{1}{P^0-M}\vec{\Sigma}_{(b)} + \frac{1}{P^0+M}\vec{\Sigma}_{(a)}\ , \tag{43}$$

where

$$\begin{aligned} \vec{\Sigma}_{(b)} &= \tfrac{1}{2}(\vec{\mathrm{S}} + \vec{\mathrm{K}})\ , \\ \vec{\Sigma}_{(a)} &= \tfrac{1}{2}(\vec{\mathrm{S}} - \vec{\mathrm{K}}) \end{aligned} \tag{44}$$

are the generators of two commuting SU(2).

It may be easily verified that a generalization of the construction (14)–(17) is obtained on introducing two sets of fermion operators a_i, b_i and then defining the corresponding $\vec{\Sigma}^F$ and Q such that $Q_{(a)}^\dagger$ is given by (15) and $Q_{(b)}^\dagger$ by (15) with M replaced by $-M$. For $P^0<0$ the sign of M is reversed in $Q_{(a)}^\dagger$, $Q_{(b)}^\dagger$. The modified forms of (41) are, as before, given by the substitution

$$\begin{aligned} \vec{\Sigma}_{(a)} &\to \vec{\Sigma}_{(a)} + \vec{\Sigma}_{(a)}^F\ , \\ \vec{\Sigma}_{(b)} &\to \vec{\Sigma}_{(b)} + \vec{\Sigma}_{(b)}^F\ . \end{aligned}$$

We conserve the Poincaré algebra, and Eqs. (16) and (18) hold for both $Q_{(a)}$ and $Q_{(b)}$. Internal-symmetry indices can now be added to both as before.

Lastly, let us consider again Eq. (36). In view of such a symmetry it would be desirable to be able to construct states which exhibit representation mixing of a prescribed type with respect to the irreducible spaces spanned by the states $f(Q_i^\alpha)|p,\nu\rangle$. In this context it may be interesting to introduce operators Q_i'' ($i=1,2,3,4$), which are defined, for example, by a Melosh type of noncovariant transformation[12] instead of a 4-component version of (24) and (25). Here we note just one point. Apart from other problems there is the fact that transformations of the Foldy-Wouthuysen or Melosh type violate the Majorana constraint.[13] Thus, we will have to deal with 4-component "complex" noncovariant operators.

---

*Research team associated with C. N. R. S.

[1]J. Wess and B. Zumino, Nucl. Phys. B70, 39 (1974); B. Zumino, in *Proceedings to the XVIII International Conference on High Energy Physics, London, 1974*, edited by J. R. Smith (Rutherford Laboratory, Chilton, Didcot, Berkshire, England, 1974), p. I-254. For a formulation in the language of "graded algebra" see L. Corwin, Y. Ne'eman, and S. Sternberg, Tel-Aviv Re-

port No. TAUP-448-74 (unpublished).
[2]A. Salam and J. Strathdee, Nucl. Phys. B76, 477 (1974); B80, 499 (1974); Phys. Rev. D 11, 1521 (1975); Nucl. Phys. B84, 127 (1975).
[3]H. Joos, Forschr. Phys. 10, 65 (1962).
[4]A. Chakrabarti, J. Math. Phys. 4, 1215 (1963).
[5]A. Chakrabarti and C. Darzens, Phys. Rev. D 8, 586 (1973).
[6]P. Dondi and M. Sohninus, Nucl. Phys. B81, 317 (1974).
[7]A. Chakrabarti, J. Math. Phys. 5, 1747 (1964); P. Moussa and R. Stora, in *Methods in Subnuclear Physics, Proceedings of the 1966 International School of Elementary Particle Physics*, Herceg-Novi, Yugoslavia, edited by M. Nikolic (Gordon and Breach, New York, 1968), Vol. 2.
[8]M. Moshinsky, *Group Theory and the Many-Body Problem* (Gordon and Breach, New York, 1968).
[9]A. Chakrabarti, J. Math. Phys. 9, 2087 (1968).
[10]A. Chakrabarti, J. Math Phys. 12, 1813 (1971).
[11]See, however, Sec. 6 of Ref. 10.
[12]H. J. Melosh, Phys. Rev. D 9, 1095 (1974); J. S. Bell, in Schladming Lectures, 1974 (CERN Report No. TH. 1851) (unpublished).
[13]By this we mean that when we apply such a transformation to $(a_1^\dagger, a_2^\dagger, a_2, -a_1)$ we no longer have $Q_3''^\dagger = Q_2''$, $Q_4''^\dagger = -Q_1''$. This is yet another example (see Ref. 4) of differences between the canonical transformations and those of Foldy-Wouthuysen type.