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Remarks concerning Nambu's generalized mechanics

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Hamiltonian embedding is proposed for Nambu's new mechanical equations. This embedding is studied in classical and quantized versions, utilizing Dirac's singular formalism. Nambu's canonical transformations are then compared with those of the embedding Hamiltonian.

I. INTRODUCTION

Some time ago, departing from Hamiltonian mechanics, Nambu[1] proposed in a very original paper a new mechanics in a three-dimensional phase space. After a study at a classical level of the main features of this mechanics, he investigated the problem of quantization. The purpose of this paper is twofold: 1. to prove that in the classical case this mechanics is equivalent to a singular Hamiltonian mechanics, and 2. to show, therefore, that the fact that Nambu gets more or less only the usual quantization scheme for his generalized mechanics is not so astonishing.

We therefore utilize here Dirac's[2] standard procedure for singular theories. We shall be mainly concerned with Nambu's equation:

$$\frac{d\vec{r}}{dt}=\vec{\nabla}H(\vec{r})\times\vec{\nabla}G(\vec{r}) . \tag{1}$$

We only indicate how to handle Nambu's various generalizations of Eq. (1). When faced with such classical equations the thing to do is to see if it has any connection with a variational problem. More precisely, is it possible to embed Eq. (1) in a system of $2n$ Hamiltonian equations? Here embedding implies that any solution of the $2n$ Hamiltonian equations will provide us with a solution of (1). This embedding has to be defined in a minimal way: The number of auxiliary variables ought to be minimum. A link between the gauge and canonical transformations of Eq. (1) and those of the corresponding Hamiltonian or Lagrangian should appear. We denote by $\mathcal{H}$ any Hamiltonian corresponding to such a possible embedding. In what follows we shall also denote by

$$\vec{r}=(x_1, x_2, x_3)$$

Nambu's dynamical variables and by $q=(q_1, \ldots, q_n)$, $p=(p_1, \ldots, p_n)$ the usual phase-space variables of $\mathcal{H}$. The partial derivative with respect to q_i and p_i will be denoted by D_i and D_{i+n}, respectively, where i ranges from 1 to n. The ordinary Poisson bracket of A and B will be denoted as (A, B).

We can already exclude the case $n=1$ unless G (or H) is a function of x_3 only. In this case a trivial embedding is always possible:

$$\mathcal{H}(q_1,p_1,\lambda)=\frac{dG}{d\lambda}(\lambda)H(q_1,p_1,\lambda) .$$

Here λ denotes a parameter independent of time.

At first sight $n=2$ could be a possibility: Eq. (1) would be "three quarters" of a Hamiltonian system corresponding to $\mathcal{H}(q_1, q_2, p_1, p_2)$. We formulate this possibility as follows, denoting by u, v two variables chosen among q_1, q_2, p_1, p_2 and by $\varphi(q_1, q_2, p_1, p_2)$ a function. Does there exist $\mathcal{H}$ and φ such that the following equations hold identically:

$$(\vec{\nabla}H\times\vec{\nabla}G)_1(u, v, \varphi)=(u, \mathcal{H}) ,$$
$$(\vec{\nabla}H\times\vec{\nabla}G)_2(u, v, \varphi)=(v, \mathcal{H}) ,$$
$$(\vec{\nabla}H\times\vec{\nabla}G)_3(u, v, \varphi)=(\varphi, \mathcal{H}) .$$

We assume in order to have a simple physical connection between the x_i and the q_i, p_i that φ is a linear function. We further suppose that φ is not a function of u and v only. Then, as can be easily checked, this is impossible for arbitrary H and G. It seems that this situation is typical even for a more general choice of φ.

We are thus naturally led to try $n=3$. It is not clear at all *a priori* whether Nambu's variables x_i are to be considered as the position variables q_i or as the momentum variables p_i. The subsequent discussion will motivate their choice as position variables. In both cases the embedding forces H and G to be constants of motion of the Hamiltonian system. If $x_i=q_i$, $i=1, 2, 3$, an easy computation shows that $\mathcal{H}$ is degenerate in the usual sense,

$$\det(D_{i+3}D_{j+3}\mathcal{H})=0 .$$

This is in general not true if $x_i=p_i$. The degeneracy of $\mathcal{H}$ in the first case shows that we have a singular Hamiltonian theory, i.e., a theory with constraints. This will decrease the number of independent variables of the Hamiltonian $\mathcal{H}$ which are missing in Eq. (1). In both alternatives intuitive Hamiltonians may be readily written:

$$\mathcal{H}(\vec{q},\vec{p})=\vec{p}\cdot[\vec{\nabla}H(\vec{q})\times\vec{\nabla}G(\vec{q})]\ , \tag{2}$$

$$\mathcal{H}'(\vec{q},\vec{p})=-\vec{q}\cdot[\vec{\nabla}H(\vec{p})\times\vec{\nabla}G(\vec{p})]\ . \tag{3}$$

Ignoring for the moment the degeneracy of $\mathcal{H}$, both $\mathcal{H}$ and $\mathcal{H}'$ give an obvious embedding of Eq. (1). These two Hamiltonians have a quite different structure. $\mathcal{H}'$ is in general not degenerate. Legendre's transformation may be applied to $\mathcal{H}'$, yielding a Lagrangian $\mathcal{L}'$. In the case of $\mathcal{H}$ the Lagrangian is 0 if one performs formally Legendre's transformation. Referring to the minimality requirement explained above, these consideration seem to indicate that $x_i=q_i$ with $\mathcal{H}$ given by (2) is a better choice.

Now one possible way to treat a theory with constraints is to use Dirac's method. Dirac starts with a degenerate Lagrangian and obtains a total Hamiltonian $\mathcal{H}_T$. This explains why if one uses Dirac's method it is impossible to start with the Hamiltonian (2). In Sec. II we find a suitable Lagrangian giving rise to a total Hamiltonian

$$\mathcal{H}_T(\vec{q},\vec{p})=v(t,\vec{q})\vec{p}\cdot[\vec{\nabla}H(\vec{q})\times\vec{\nabla}G(\vec{q})] \tag{4}$$

for which Dirac's singular theories *do apply*. Here v is an arbitrary function.

We finally remark that in the case of the rigid rotator, the only physical example quoted by Nambu, the variables x_i are neither position nor momentum variables. However, if one does not refer to the physical origin of the problem and considers these equations by themselves, it is not confusing to treat these variables as generalized position variables of a certain Lagrangian.

II. CLASSICAL EMBEDDING OF NAMBU'S EQUATIONS

We assume that $H(\vec{q})$ and $G(\vec{q})$ are given C^∞ functions in an open set $\Omega\subset\mathbb{R}^3$ such that

$$[\vec{\nabla}H(\vec{q})\times\vec{\nabla}G(\vec{q})]_3\neq 0\ .$$

To begin with, we exhibit a Lagrangian $L_N(\vec{q},\dot{\vec{q}})$ such that for any solution of Euler-Lagrange equations $H=\text{const}$ and $G=\text{const}$

$$L_N(\vec{q},\dot{\vec{q}})=H(\vec{q})\sum_{i=1}^{3}\dot{q}_iD_iG(\vec{q})\ , \tag{5}$$

the Euler-Lagrange equations of which may be written as

$$\frac{dG}{dt}\vec{\nabla}H=\frac{dH}{dt}\vec{\nabla}G\ . \tag{6}$$

For any solution of (6) we have $H=\text{const}$ and $G=\text{const}$, as required. We now follow Dirac's method. The rules of the game are well known: One has to be careful in working with Poisson brackets and one must not use the constraints before computing a Poisson bracket.

We obtain three primary constraints

$$\phi_i(\vec{q},\vec{p})=p_i-H(\vec{q})D_iG(\vec{q}),\quad i=1,2,3 \tag{7}$$

a Hamiltonian $\mathcal{H}_1$ which is 0, and the Hamiltonian equations of motion

$$\dot{q}_i=u_i\ ,$$
$$\dot{p}_i=\sum_{j=1}^{3}u_jD_i(HD_jG)\ . \tag{8}$$

Here u_i are unknown coefficients to be determined later. For consistency $\dot{\phi}_i$ must be 0:

$$\dot{\phi}_i=(\phi_i,\mathcal{H}_1)+\sum_{j=1}^{3}u_j(\phi_i,\phi_j)\ . \tag{9}$$

Hence

$$\sum_{j=1}^{3}u_j(D_iHD_jG-D_jHD_iG)=0\ . \tag{10}$$

Note that Eq. (10) proves that there are no secondary constraints in the sense of Dirac.[2] The most general solution of (10) is

$$u_i=v(t,\vec{q})[\vec{\nabla}H(\vec{q})\times\vec{\nabla}G(\vec{q})]_i\ , \tag{11}$$

where the function v is an arbitrary function. From Eqs. (8) and (11) one gets

$$\dot{\vec{q}}=v(t,\vec{q})[\vec{\nabla}H(\vec{q})\times\vec{\nabla}G(\vec{q})]\ . \tag{12}$$

Let us *fix* $v(t,\vec{q})$ and suppose that, for any $\vec{q}(t)\in\Omega$, $v(t,\vec{q}(t))\neq 0$. By rescaling the time axis in a position-dependent way

$$\tau=\int_0^t v(\theta,\vec{q}(\theta,\vec{q}_0))d\theta\ , \tag{13}$$

where $\vec{q}(t,\vec{q}_0)$ is the unique solution of (12) with the initial condition $\vec{q}(0)=\vec{q}_0$, (12) becomes

$$\frac{d\vec{q}}{d\tau}=\vec{\nabla}H(\vec{q})\times\vec{\nabla}G(\vec{q})\ .$$

We therefore have established for a *fixed* v a one-to-one correspondence between solutions of (12) and those of (1). Moreover, by the time rescaling (13) the correspondence becomes the identical one.

Since $v(t,\vec{q})$ is an arbitrary function in our formalism, this means that Eqs. (8) will formally have more solutions than Eq. (1). However, since the choice of the time axis is arbitrary, Eqs. (8) and (1) will contain the *same dynamical information*.

The total Hamiltonian will be

$$\mathcal{H}_T(\vec{q},\vec{p})=\sum_{i=1}^{3}u_i\phi_i(\vec{q},\vec{p})$$
$$=v(t,\vec{q})[\vec{\nabla}H(\vec{q})\times\vec{\nabla}G(\vec{q})]\cdot\vec{p}\ . \tag{14}$$

$\mathcal{H}_T$ is merely proportional to the primary first-class constraint

$$\phi(\vec{q},\vec{p}) = [\vec{\nabla}H(\vec{q})\times\vec{\nabla}G(\vec{q})]\cdot[\vec{p} - H(\vec{q})\vec{\nabla}G(\vec{q})]$$
$$= [\vec{\nabla}H(\vec{q})\times\vec{\nabla}G(\vec{q})]\cdot\vec{p} \quad (15)$$

and vanishes weakly.

We now examine the link between the gauge and canonical transformation of Nambu and those of $\mathcal{H}_T$. If one performs on H and G Nambu's transformation

$$H' = h(H,G), \quad G' = g(H,G), \quad \frac{\partial(H',G')}{\partial(H,G)} = 1 ,$$

one obtains a new Lagrangian

$$L_N' = H'\frac{dG'}{dt} .$$

The Euler-Lagrange equations of L_N'

$$\frac{dG'}{dt}\vec{\nabla}H' = \frac{dH'}{dt}\vec{\nabla}G'$$

may be written as

$$\frac{\partial(H',G')}{\partial(H,G)}\frac{dH}{dt}\vec{\nabla}G = \frac{\partial(H',G')}{\partial(H,G)}\frac{dG}{dt}\vec{\nabla}H ,$$

and the total Hamiltonian becomes

$$\mathcal{H}_T'(\vec{q},\vec{p}) = v(t,\vec{q})[\vec{\nabla}H'(\vec{q})\times\vec{\nabla}G'(\vec{q})]\cdot\vec{p}$$
$$= \frac{\partial(H',G')}{\partial(H,G)}\mathcal{H}_T(\vec{q},\vec{p}) .$$

Thus the equations of motion are invariant and the total Hamiltonian is invariant under Nambu's gauge-transformation group. We now perform Nambu's canonical transformation on the triplet $\vec{r}$:

$$x_i' = \varphi_i(\vec{r}), \quad \det[D_i\varphi_j(\vec{r})] = 1 . \quad (16)$$

To obtain a corresponding canonical transformation on the six variables, $(q_i, p_i)\to(q_i', p_i')$, such that

$$q_i' = \varphi_i(\vec{q}) ,$$

we choose as generating function

$$F(\vec{q},\vec{p}') = \sum_{i=1}^{3} p_i'\varphi_i(\vec{q}).$$

This defines the transformation completely:

$$\begin{aligned} q_i' &= D_{i+3}F(\vec{q},\vec{p}') \\ &= \varphi_i(\vec{q}) , \\ p_i &= D_iF(\vec{q},\vec{p}') \\ &= \sum_{j=1}^{3} D_i\varphi_j(\vec{q})p_j' . \end{aligned} \quad (17)$$

The generating function F being time-independent, the transformed Hamiltonian will be such that

$$\mathcal{H}_T'(\vec{q}',\vec{p}') = \mathcal{H}_T(\vec{q},\vec{p}) . \quad (18)$$

Nambu's canonical transformations are therefore a subgroup of the group of canonical transformations of the q_i, p_i. The sense in which Nambu uses the notion of form invariance is the following: Under (16) Eq. (1) goes into

$$\frac{d\vec{r}'}{dt} = \vec{\nabla}H'(\vec{r}')\times\vec{\nabla}G'(\vec{r}') ,$$

where $H'(\vec{r}') = H(\vec{r})$, $G'(\vec{r}') = G(\vec{r})$. A trivial computation shows that $\mathcal{H}_T'$ being defined by Eq. (18) may be written as

$$\mathcal{H}_T'(\vec{q}',\vec{p}') = v'(t,\vec{q}')[\vec{\nabla}H'(\vec{q}')\times\vec{\nabla}G'(\vec{q}')]\cdot\vec{p}' .$$

Here $\vec{\nabla}H'(\vec{q}') = [(\partial/\partial q_i')H'(\vec{q}')]$ and $v'(t,\vec{q}') = v(t,\vec{q})$.

We may now summarize the results of this section: Equation (1) may be embedded into a six-dimensional degenerate Hamiltonian system with three constraints, the Lagrangian of which is homogeneous of the first degree in the velocities. Modulo a time rescaling, any solution of (1) gives a unique solution of the Hamiltonian system and conversely. Nambu's gauge group leaves the Hamiltonian invariant. Nambu's canonical transformations are a subgroup of the group of canonical transformations of the Hamiltonian system. The two theories may therefore be considered to be equivalent in a certain sense.

Before comparing the quantization of both theories we briefly show how to generalize this embedding to the other cases exhibited by Nambu.

In the case of $n-1$ "Hamiltonians" the equations of motion are

$$\frac{dx_i}{dt} = \sum_{i_1,\ldots,i_{n-1}} \epsilon_{i i_1\cdots i_{n-1}} D_{i_1}H_1\cdots D_{i_{n-1}}H_{n-1} .$$

Assuming that n is odd $(n = 2s+1)$, choose

$$L_N'(q,\dot{q}) = \sum_{i=1}^{s} H_{2i-1}\frac{dH_{2i}}{dt} .$$

Then n primary constraints and a total Hamiltonian are readily obtained:

$$\phi_i(q,p) = p_i - \sum_{K=1}^{s} H_{2K-1}(q)D_iH_{2K}(q) , \quad i = 1,2,\ldots,n$$

$$\mathcal{H}_T(q,p) = v(t,q)\sum_{i,i_1,\ldots,i_{n-1}} \epsilon_{i i_1\ldots i_{n-1}}p_i \times D_{i_1}H_1\cdots D_{i_{n-1}}H_{n-1} .$$

If n is even, a similar treatment is possible.

In the case of n pairs of "Hamiltonians" Nambu's equations of motion are

$$\frac{dx_i}{dt} = \sum_{j=1}^{n}[\vec{\nabla}H_j(\vec{r})\times\vec{\nabla}G_j(\vec{r})]_i \quad i = 1, 2, 3 .$$

We choose

$$L_N''(\vec{q}, \dot{\vec{q}}) = \sum_{j=1}^{n} H_j(\vec{q}) \frac{dG_j}{dt} \quad ,$$

and obtain three primary constraints and the total Hamiltonian:

$$\phi_i(\vec{q}, \vec{p}) = p_i - \sum_{j=1}^{n} H_j(\vec{q}) D_i G_j(\vec{q}) \ ,$$

$$\mathcal{H}_T(\vec{q}, \vec{p}) = v(t, \vec{q}) \left\{ \sum_{j=1}^{n} \vec{p} \cdot [\vec{\nabla} H_j(\vec{q}) \times \vec{\nabla} G_j(\vec{q})] + \sum_{1=j<k=n} (H_j \vec{\nabla} H_k - H_k \vec{\nabla} H_j) \cdot (\vec{\nabla} G_k \times \vec{\nabla} G_j) \right\} .$$

Furthermore, Nambu's gauge transformation [Eq. (26) in Ref. 1] leaves the Euler equations of L_N'' invariant.

In the remaining case of two "Hamiltonians" in a $3N$-dimensional phase space the equations of motion are

$$\frac{dx_n}{dt} = \frac{\partial(H, G)}{\partial(y_n, z_n)} \ .$$

These equations do not seem so easy to handle lacking a suitable Lagrangian formalism, although an intuitive Hamiltonian may be easily constructed (see the Introduction):

$$\mathcal{H}(x_n, y_n, z_n) = \sum_{n=1}^{N} \left(p_{x_n} \frac{\partial(H, G)}{\partial(y_n, z_n)} + p_{y_n} \frac{\partial(H, G)}{\partial(z_n, x_n)} + p_{z_n} \frac{\partial(H, G)}{\partial(x_n, y_n)} \right) \ .$$

III. QUANTIZATION

Nambu's procedure was based upon the trilinearity of the generalized Poisson bracket. Difficulties arose when trying to respect both the alternation law and the derivation law. In order to find a solution to the problem of quantization Nambu weakens the previous laws. In the case called by him (a) + (b′), he recovers the Heisenberg equations. The proof is not rigorous since he supposes that some commutators are invertible. Furthermore, his assumption implies that $H = \alpha G + \beta$ (in such a case his classical theory is trivial). In the case (a′) + (b′) he also obtains the Heisenberg equations.

The problem of quantization of a singular theory is not yet completely solved. However, in the case of a simple Lagrangian like L_N it is easy to find a trick which explains Nambu's result. Let us first describe the Poisson-bracket Lie algebra of this singular theory. We have already found a first-class constraint ϕ given by Eq. (15). The commutation relations

$$(\phi_i, \phi_j) = \epsilon_{ijk} (\vec{\nabla} H \times \vec{\nabla} G)_k$$

show that the even number of second-class constraints is 2. We choose ϕ_1 and ϕ_2 to be those second-class constraints. Dirac's new Poisson bracket may be written as

$$(\xi, \eta)^* = (\xi, \eta) + c[(\xi, \phi_1)(\eta, \phi_2) - (\xi, \phi_2)(\eta, \phi_1)] \ ,$$

where $c = -(D_1 H D_2 G - D_2 H D_1 G)^{-1}$. The Lie-algebra structure obtained in this way is defined on the vector space of C^∞ functions in the open set

$$\Omega \times \mathbb{R}^3 \subset \mathbb{R}^6$$

in the phase space parametrized by $(\vec{q}, \vec{p})$. One can ckeck that Dirac's new Poisson bracket is a deformation of the usual Poisson bracket given by a scaling of the Lie algebra generated by the constraints.

We now consider the Hilbert space $L^2(\mathbb{R}^3)$, and we denote by $(\xi, \eta)_-$ the usual commutator of two operators ξ and η defined in $L^2(\mathbb{R}^3)$. We define between two such elements a new commutator:

$$(\xi, \eta)^*_- = (\xi, \eta)_- - ic[(\xi, D_1 + iHD_1G)_-(\eta, D_2 + iHD_2G)_- - (\xi, D_2 + iHD_2G)_-(\eta, D_1 + iHD_1G)_-] .$$

Suppose that we do not change the usual correspondence φ between the classical variables q_i, p_i and the skew-adjoint operators on $L^2(\mathbb{R}^3)$:

$$\varphi \colon \ F(\vec{q}) \to A = iF(\vec{q}) \ ,$$

$$\varphi \colon \ p_i \to P_i = -D_i \ .$$

[The corresponding observables are here as usual the self-adjoint operators $(1/i)A$ and iP_i.] Then it is easily checked that this new commutator gives rise to a representation of Dirac's Poisson bracket between the q_i and the p_j:

$$(\varphi(q_i), \varphi(p_j))^*_- = \varphi((q_i, p_j)^*) \ .$$

Thus the transition between our (singular) classical theory and a quantum theory is the ordinary one. The classical relation [for the observable $F(\vec{q})$]

$$\dot{F} = (F, \mathcal{H}_T)^*$$

becomes in our formalism

$$i\dot{F} = (F, \mathcal{H}_T)^*_- \ .$$

To conclude, it is therefore not at all surprising (from our point of view) that Nambu gets as a quantized version of his generalized mechanics the ordinary Heisenberg quantization scheme. Obviously, in full accordance with Dirac,[2] the ϕ_i's and $\mathcal{H}_T$ commute among themselves under the operation $(\, , \,)^*_-$. Therefore we can proceed as usual and find *representations* of the structure defined by $(\, , \,)^*_-$ in an algebra of differential operators under the usual commutation relations, where $\mathcal{H}_T$ will be represented by id/dt. We shall then get an integrable system of equations and a final solution to our quantization problem.

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