PHYSICAL REVIEW D VOLUME 11, NUMBER 10 15 MAY 1975

Lorentz-invariant Newtonian mechanics for three or more particles

Thomas F. Jordan
Department of Physics, University of Minnesota, Duluth, Duluth, Minnesota 55812
(Received 30 January 1975)

Lorentz-invariant Newtonian equations of motion for three or more interacting particles can be constructed just as for two by assuming that the center-of-mass acceleration is zero when the center-of-mass velocity is zero. But two-particle systems are not obtained from a larger system of this kind when the other particles are widely separated in space. The same is true if the total relativistic kinematic particle momentum and its time derivative are used in place of the center-of-mass velocity and acceleration. The assumption that the center-of-mass acceleration is zero when the center-of-mass velocity is zero does not hold, in particular, for parity-conserving forces for three or more identical particles as it does for two.

A recent paper[1] showed how Lorentz-invariant Newtonian equations of motion for two interacting particles can be constructed, by making global Lorentz transformations, from a specification of the relative acceleration as a function of the relative position and relative velocity at zero center-of-mass velocity, if the center-of-mass acceleration is assumed to be zero at zero center-of mass velocity. The same can be done for three or more interacting particles, on the assumption that for the entire system of particles the center-of-mass acceleration is zero when the center-of-mass velocity is zero. The conditions to be met in specifying the relative accelerations in the center-of-mass frame are completely analogous to those for two particles. Nothing is different enough to need to be described again.

A problem arises, however, if we require separability or cluster decomposition.[2] This is the requirement that when the system of particles is split into groups that are widely separated in space, each group is a system that is independent of the others and satisfies all the postulates itself.

There is no classical-mechanical system of three or more particles in which

(i) for the entire system the center-of-mass acceleration is zero when the center-of-mass velocity is zero, and

(ii) when all particles except two are widely separated in space, their accelerations vanish and for the remaining two we have Poincaré-invariant Newtonian equations of motion with accelerations that describe an interaction in the neighborhood of zero center-of-mass velocity for the two particles and are independent of the other particles.

For under these conditions, when the center-of-mass velocity for the entire system is zero, and all particles except two are widely separated in space, the center-of-mass acceleration of the remaining two-particle system must be zero, for a range of values around zero of the center-of-mass velocity of the two-particle system depending on the masses and velocities of the other particles. It is known that there are no Poincaré-invariant Newtonian equations of motion for two interacting particles such that the center-of-mass acceleration is zero over the range of values of the velocities and relative position.[3]

Everything said here remains true if instead of the center-of-mass velocity we use the total relativistic kinematic particle momentum

$$\vec{U} = \sum_n m_n \vec{v}_n/(1 - v_n^2/c^2)^{1/2} ,$$

where m_n and $\vec{v}_n$ are the mass and velocity of the nth particle. Lorentz-invariant Newtonian equations of motion for three or more interacting particles can be constructed by assuming that for the entire system $d\vec{U}/dt$ is zero when $\vec{U}$ is zero. But two-particle or three-particle subsystems cannot be separated out of a larger system of this kind; it is known that there are no Poincaré-invariant Newtonian equations of motion for either two or three interacting particles such that $d\vec{U}/dt$ is zero over the range of values of the momenta and relative positions.[4]

For two particles there is some justification for considering those Lorentz-invariant Newtonian equations of motion for which it can be assumed that the center-of-mass acceleration is zero when the center-of-mass velocity is zero, because this assumption holds in particular for parity-conserving forces for two identical particles.[1] For three or more particles this is not so; for parity-conserving forces for three or more identical particles it is not necessary that the center-of-mass acceleration be zero when the center-of-mass velocity is zero. For example, the center-of-mass acceleration may be nonzero when the center-of-mass velocity is zero for the parity-conserving Poincaré-invariant equations of motion that we can construct for three or more identical particles on the assumption that $d\vec{U}/dt$ is zero when $\vec{U}$ is zero, and conversely $d\vec{U}/dt$ may be nonzero when $\vec{U}$ is zero for the parity-conserving Poincaré-invariant equations of motion that we can construct for three or more identical particles on the assumption that the center-of-mass acceleration is zero when the center-of-mass velocity is zero, because the center-of-mass velocity and $\vec{U}$ are not zero together for three or more identical particles as they are for two.

This point can even be seen by considering two-particle forces in a system of three or more particles. Translation, rotation, and parity invariance imply that the force on particle 1 from particle 2 produces an acceleration of the form

$$\vec{f}^{12} = A_{12}(\vec{x}^1 - \vec{x}^2) + B_{12}\vec{v}^1 + C_{12}\vec{v}^2$$

where A_{12}, B_{12}, and C_{12} are functions of $(\vec{x}^1-\vec{x}^2)^2$, $(\vec{v}^1)^2$, $(\vec{v}^2)^2$, $(\vec{x}^1-\vec{x}^2)\cdot\vec{v}^1$, $(\vec{x}^1-\vec{x}^2)\cdot\vec{v}^2$, and $\vec{v}^1\cdot\vec{v}^2$. Suppose the particles are identical. Then $\vec{f}^{12}+\vec{f}^{21}$ is zero when $\vec{v}^1+\vec{v}^2$ is zero. But when the total center-of-mass velocity for the three or more particles is zero, $\vec{v}^1+\vec{v}^2$ may be nonzero, $\vec{f}^{12}+\vec{f}^{21}$ may be nonzero, and the total center-of-mass acceleration

$$\sum_{\substack{m,n\\ m\neq n}} \vec{f}^{mn}$$

may be nonzero. Note that Galilei invariance, in contrast to Lorentz invariance, would imply that $\vec{f}^{12}$ can depend only on $\vec{x}^1-\vec{x}^2$ and $\vec{v}^1-\vec{v}^2$; then $\vec{f}^{12}+\vec{f}^{21}$ would be zero and therefore the total center-of-mass acceleration would be zero.

On the other hand, Lorentz-invariant two-particle forces do not give Lorentz-invariant equations of motion in a system of three or more particles. For equations of motion

$$\frac{d^2\vec{x}^n}{dt^2} = \vec{f}^n(\vec{x}^1, \vec{x}^2, \ldots, \vec{x}^N, \vec{v}^1, \vec{v}^2, \ldots, \vec{v}^N)$$

for N particles, $n=1,2,\ldots,N$, the conditions for Lorentz invariance[5,6] are

$$\sum_{l=1}^{3}\sum_{m=1}^{N}(x^n{}_k - x^m{}_k)\left(f^m{}_l\frac{\partial f^n{}_j}{\partial v^m{}_l} + v^m{}_l\frac{\partial f^n{}_j}{\partial x^m{}_l}\right) - \sum_{l=1}^{3}\sum_{m=1}^{N} v^m{}_k v^m{}_l \frac{\partial f^n{}_j}{\partial v^m{}_l} + \sum_{m=1}^{N}\frac{\partial f^n{}_j}{\partial v^m{}_k} + 2v^n{}_k f^n{}_j + v^n{}_j f^n{}_k = 0$$

for $j, k = 1, 2, 3$ and $n=1,2,\ldots,N$. Suppose the force on each particle is a sum of Poincaré-invariant two-particle forces, that is,

$$\vec{f}^n = \sum_{\substack{r=1\\ r\neq n}}^{N} \vec{f}^{nr}(\vec{x}^n - \vec{x}^r, \vec{v}^n, \vec{v}^r)$$

where $\vec{f}^{nr}$ and $\vec{f}^{rn}$ are rotational-vector functions that satisfy the Lorentz-invariance conditions for a system of two particles. By comparing these Lorentz-invariance conditions for the two-particle forces to those for the total $\vec{f}^n$ we find that

$$\sum_{l=1}^{3}\sum_{\substack{r=1\\ r\neq n}}^{N}\sum_{\substack{s=1\\ s\neq n\\ s\neq r}}^{N}(x^n{}_k - x^r{}_k) f^{rs}{}_l \frac{\partial f^{nr}{}_j}{\partial v^r{}_l} = 0$$

for $j, k = 1, 2, 3$ and $n = 1, 2, \ldots, N$. Since the different two-particle forces are functions of different variables, we can see there are no interesting solutions. If

$$\frac{\partial f^{nr}{}_j}{\partial v^r{}_l} = 0$$

for $j, l = 1, 2, 3$, the Lorentz-invariance conditions for the two-particle forces imply that also

$$\frac{\partial f^{nr}{}_j}{\partial x^r{}_l} = 0$$

for $j, l = 1, 2, 3$, which means the acceleration of particle n is independent of particle r.

---

[1]T. F. Jordan, this issue, Phys. Rev. D 11, 2807 (1975).
[2]L. L. Foldy, Phys. Rev. 122, 275 (1961); E. H. Wichmann and J. H. Crichton, *ibid.* 132, 2788 (1963); F. Coester, Helv. Phys. Acta 38, 7 (1965).
[3]D. G. Currie and T. F. Jordan, Phys. Rev. Lett. 16, 1210 (1966).
[4]D. G. Currie and T. F. Jordan, Phys. Rev. 167, 1178 (1968); T. F. Jordan and C. S. Shukre, Phys. Rev. D 5, 799 (1972).
[5]D. G. Currie, Phys. Rev. 142, 817 (1966).
[6]R. N. Hill, J. Math. Phys. 8, 201 (1967).