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Most-general minimality-preserving Hamiltonian

David Stoler
Department of Physics, Polytechnic Institute of New York, Brooklyn, New York 11201
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This paper is concerned with Hamiltonians that preserve the minimum value of the uncertainty product. Using the method of time-dependent unitary transformations, we derive the most general such Hamiltonian.

This paper deals with Hamiltonians which have the property of maintaining the minimum uncertainty product of states which are initially minimal. The first attempt to deal with this question was made by the present author,[1] who in fact did derive the most general result, but through an oversight omitted a term in the minimality-preserving Hamiltonian. I remedy this here, and in the process discuss Hamiltonians which preserve other eigenproperties such as coherence and "ground-stateness." The method used here has led to some useful insight into the structure of eigenproperty-preserving Hamiltonians which will be dealt with at length in a subsequent paper.

Consider a system described by a Hamiltonian H and having a state vector $|\psi(t)\rangle\!\rangle$. The Schrödinger equation for the state vector is $i(\partial/\partial t)|\psi(t)\rangle\!\rangle = H|\psi(t)\rangle\!\rangle$. The dynamical evolution of this system is unitarily equivalent to an infinite number of others. This fact is commonly exploited in the use of unitary transforms to simplify problems. Given a unitary operator $V(t)$ we can define a transformed dynamics by means of the relation $|\psi(t)\rangle\!\rangle = V(t)|\phi(t)\rangle\!\rangle$. This last equation defines $|\phi(t)\rangle\!\rangle$. The equation of motion for $|\phi(t)\rangle\!\rangle$ in the Schrödinger representation is

$$i\frac{\partial}{\partial t}|\phi(t)\rangle\!\rangle = \left(V^\dagger H V - iV^\dagger \frac{\partial V}{\partial t}\right)|\phi(t)\rangle\!\rangle \ .$$

So when the unitary transformation is time dependent the transformed Hamiltonian is not just a similarity transformation of the original one, but contains an additional piece which is analogous to a "frame energy" in classical mechanics.

The most general Hamiltonian which preserves the coherence of an arbitrary coherent state was given by Glauber[2] and Sudarshan and Mehta.[3] The Hamiltonian which they give has the form

$$H_{\text{coh}} = \omega(t)a^\dagger a + f(t)a^\dagger + f^*(t)a + \beta(t) \ , \tag{1}$$

where a and $a^\dagger$ are the creation and annihilation operators of a harmonic oscillator which satisfy the relation $[a, a^\dagger]=1$. Under the influence of this Hamiltonian a coherent state $|\alpha\rangle$ at $t=0$ will evolve into the coherent state $|\alpha(t)\rangle\!\rangle$, where $\alpha(t)$ is the complex amplitude satisfying the classical equation of motion for the oscillator and for which $\alpha(0)=\alpha$.

I have made a detailed study[1,4,5] of quantum states which minimize the position-momentum uncertainty product. The coherent states are minimum-uncertainty packets and constitute one of an infinite number of equivalence classes[1] of minimum packets. In Ref. 1, I demonstrated that all the minumum packets are unitarily equivalent to the coherent states where the equivalence is implemented by the unitary operator $U_r = \exp[\frac{1}{2}r(a^2 - a^{\dagger 2})]$. Any minimum packet may be written in the form $|r, \alpha\rangle = U_r|\alpha\rangle$, where r is a real number. In view of this explicit unitary relationship between the coherent states and the whole set of minimum packets, it becomes easy to derive the properties of the minimum packets from the analogous properties of the coherent states. Using the results of Refs. 2 and 3 for coherence, we can derive the most general minimality-preserving Hamiltonian by simply transforming H_{coh} by means of U_r. In Ref. 1, I arrived at the minimality-preserving Hamiltonian using this approach. The most general result is obtained by allowing the parameter r in U_r to be a function of time. This yields

$$H_{\min} = U_{r(t)} H_{\text{coh}} U^\dagger_{r(t)} - iU_{r(t)} \frac{\partial U^\dagger_{r(t)}}{\partial t} \ . \tag{2}$$

Equation (21) of Ref. 1 was derived this way. However, through an oversight, the frame energy term in the above (i.e., $-iU_r\,\partial U_r/\partial t$) was omitted, making the result given valid only when r is not a function of time. This omission was recently pointed out in a paper by Trifonov,[6] who writes down the correct Hamiltonian by considering the most general quadratic Hamiltonian, calculating the uncertainty product, and deriving conditions on the coefficients so that the uncertainty product remains minimal. This, however, does not constitute a proof that the result for $H_{\min}$ is the most general.

In deriving Eq. (2) by the method of time-dependent unitary transformation, I realized that one could arrive at the form of H_{coh} in the same way.

This is because the general coherent state $|\alpha(t)\rangle\!\rangle$ may be written as $D(\alpha(t))|0\rangle$, where $D(\alpha(t))$ is the unitary Weyl operator $\exp[\alpha(t)a^\dagger - \alpha^*(t)a]$ and $|0\rangle$ is the ground state of the oscillator. So the coherence-preserving Hamiltonian can be obtained from the general "ground-state-preserving" Hamiltonian by transformation under the time-dependent Weyl operator. The ground-state-preserving Hamiltonian is easily found. Under its action $|0\rangle$ evolves into $|0\rangle_t$, for which we require $a|0\rangle_t = 0$. Hence $|0\rangle_t \propto |0\rangle$, which implies, since $|0\rangle_t$ satisfies the Schrödinger equation, that $aH_g|0\rangle_t = 0$ and therefore $|0\rangle_t$ must be an eigenvector of H_g. This means that $H_g \propto a^\dagger a$. The most general expression for H_g is $\omega(t)a^\dagger a + \beta(t)$, where $\omega(t)$ and $\beta(t)$ are real functions of time. Actually this form for H_g is not the most general one which will preserve the ground state. We can add to H_g any Hermitian operator which annihilates the ground state, such as $ga^\dagger a^2 + g^* a^{\dagger 2} a$. However, Hamiltonians containing such terms yield, upon transformation under $D(\alpha(t))$, Hamiltonians which do not preserve *arbitrary* coherent states but rather only certain specific coherent states. An example of such a Hamiltonian was given by Mista,[7] who considers $H_{\rm coh}$ augmented by the terms $\mu(t)a^{\dagger 2} + K(t)a^{\dagger 2}a + \text{H.c.}$ and shows that only the coherent state $|\alpha(0)\rangle\!\rangle = |-\mu(0)/K(0)\rangle\!\rangle$ remains a coherent state under its action. This is a special case of an infinite class of such Hamiltonians which can be arrived at by the methods employed here. So we find for the general coherence-preserving Hamiltonian

$$H_{\rm coh} = D(\alpha(t))H_g D^\dagger(\alpha(t)) - iD\frac{\partial D^\dagger}{\partial t} = \omega(t)a^\dagger a + f(t)a^\dagger + f^*(t)a + \epsilon(t)\,, \tag{3}$$

where

$$f(t) = \omega(t)\alpha(t) - i\dot\alpha(t)$$

and

$$\epsilon(t) = \beta(t) + \omega(t)|\alpha(t)|^2 + \text{Im}(\dot\alpha\alpha^*)\ .$$

We see that $\alpha(t)$ satisfies the classical equation of motion for this oscillator. The function $\epsilon(t)$ is a real but otherwise arbitrary function since $\beta(t)$ is also.

Now applying the same procedure to $H_{\rm coh}$, using the relation

$$|\alpha(t)\rangle\!\rangle = U^\dagger_{r(t)}|r(t), \alpha(t)\rangle\!\rangle\,,$$

we get for $H_{\rm min}$

$$H_{\rm min} = \Omega(t)a^\dagger a + \tfrac{1}{2}\Omega(t)\tanh 2r(t)(a^2 + a^{\dagger 2}) + \frac{i}{2}\frac{dr}{dt}(a^2 - a^{\dagger 2}) + F^*(t)a + F(t)a^\dagger + B(t)\,, \tag{4}$$

where Ω, r, and $B(t)$ are real functions and $F(t)$ may be complex. The term in $H_{\rm min}$ proportional to $\dot r(t)$ was missing from my original result in Ref. 1. The calculations here have all been in the Schrödinger picture so that the operators a and $a^\dagger$ are independent of time. The results for $H_{\rm min}$ quoted in Ref. 6 are given in terms of time-dependent Heisenberg operators, which accounts for the fact that no term proportional to $(a^2 + a^{\dagger 2})$ occurs there. This means that the form of $H_{\rm min}$ given by Trifonov is valid in the Heisenberg picture but not in the Schrödinger picture. Trifonov's operators $a(\mu)$ and $a^\dagger(\mu)$ are the equivalents of S and $S^\dagger$ in Ref. 1.

The results of Refs. 2, 3, and 6 are in terms of an arbitrary number of degrees of freedom rather than just one. Our results may be easily extended to the multimode case by substituting

$$\omega(t)a^\dagger a \to \sum_{i,j}\omega_{ij}(t)a_i^\dagger a_j$$

in H_g and then using appropriate products of unitary operators to generate $H_{\rm coh}$ and $H_{\rm min}$.

Finally, it is important to note that the coherence-preserving Hamiltonian $H_{\rm coh}$ discussed here does *not* really preserve coherence as such, but rather preserves coherent states, i.e., α states. There are other states which satisfy Glauber's coherence criteria,[8] namely, the generalized coherent states (GCS) and also coherent density matrices. Using the techniques of a recent paper[9] on the GCS one can see that $H_{\rm coh}$ *does not* preserve the GCS or the coherent density matrices.

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