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Gyromagnetic ratio of Einstein-Maxwell fields

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It is shown that the gyromagnetic factor is $g = 2$ for electrovac solutions of Einstein's equations obtained by the prescription introduced by Ernst. This generalizes Carter's result for the Kerr-Newman metric.

One of the most striking features of the Kerr-Newman solution of Einstein's equations is that the corresponding gyromagnetic factor, g, is 2, as is the case for Dirac particles.[1] Cohen *et al.*[2] have shown that $g \to 2$ also for a slowly rotating shell approaching its Schwarzschild radius. Wald[3] has proved that $g=2$ for any slightly charged stationary axisymmetric hole.

In this paper it is shown that $g=2$ for all the stationary electrovac solutions which can be constructed by the prescription introduced by Ernst.[4]

In canonical cylindrical coordinates ρ, z, the axisymmetric line element reads

$$ds^2 = f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2] - f(dt - \omega d\varphi)^2, \tag{1}$$

where f, ω, and γ are functions of ρ and z. From the asymptotic behavior of the Newtonian potential, and using the Thirring-Lense theorem,[5] one finds that as $r \to \infty$

$$f \sim 1 - 2m/r, \tag{2}$$

$$\omega \sim -2J\sin^2\theta/r, \quad \gamma \sim 0, \tag{3}$$

where m is the mass and J is the angular momentum, both in geometrical units, and r, θ are spherical coordinates. In particular, (2) and (3) are asymptotic expressions for the Kerr metric. Different solutions are distinguished by higher multipoles in the expansion.[6]

Ernst[4] gave a prescription for constructing from any vacuum axisymmetric solution a solution of the Einstein-Maxwell equations which reduces to the vacuum solution for zero charge. The key point of our argument is that in this formalism the vector potential of the electromagnetic field is determined completely by the functions f, ω and a parameter q related to the charge e. From Eqs. (2)–(4) it follows, therefore, that the asymptotic expressions of the electromagnetic field are the same for all solutions characterized by the same parameters J, m, q, which implies that the gyromagnetic ratio is also the same.

In order to show this explicitly, we recall that in the Ernst formalism the vacuum field equations reduce to a single quasilinear equation in a complex potential ξ_0. The charged solution is described by a potential $\xi = Q\xi_0$, where Q is related to the charge parameter q by $Q = (1-q^2)^{1/2}$. The functions f, ω and the azimuthal and time components of the electromagnetic vector potential A_3, A_4 are given by

$$A_4 + iA_3' = q/(\xi+1), \tag{4}$$

$$f = 1 - 2A_4/q + (A_4{}^2 + A_3'{}^2), \tag{5}$$

$$\vec{\nabla}\omega = -(2/q)\rho f^{-1}\hat{n}\times\vec{\nabla}A_3' + \omega\vec{\nabla}A_4, \tag{6}$$

$$\vec{\nabla}A_3 = \rho f^{-2}\hat{n}\times[\vec{\nabla}A_3' + 2(A_4\vec{\nabla}A_3' - A_3'\vec{\nabla}A_4)], \tag{7}$$

where A_3' is an auxiliary function, $\hat{n}$ is the azimuthal unit vector, and $\vec{\nabla}$ is formally the gradient operator in flat space.

From the asymptotic conditions on the vector potential and Eqs. (5) and (2) one has

$$q = \frac{e}{m}. \tag{8}$$

Equations (6) and (7) reduce to the asymptotic forms

$$\vec{\nabla}\omega \sim -(2/q)r\sin\theta\hat{n}\times\vec{\nabla}A_3', \tag{6'}$$

$$\vec{\nabla}A_3 \sim r\sin\theta\hat{n}\times\vec{\nabla}A_3', \tag{7'}$$

which give

$$A_3 \sim -(q/2)\omega. \tag{9}$$

Referring to an orthonormal frame of reference, and by means of Eq. (3), we find that the dominant components of the vector potential become

$$\tilde{A}_4 \sim e/r,$$

$$\tilde{A}_3 \sim (e/m)(J\sin\theta/r^2).$$

The magnetic dipole moment is, therefore, $\mu = (e/m)J$, and the gyromagnetic factor is $g=2$.

Our results indicate that the fact that a black hole has a gyromagnetic ratio equal to that of the electron is a property common to all solutions of Einstein-Maxwell equations which can be obtained by means of the Ernst prescription. In particular, this is the case for the Kerr-Newman metric and for the charged generalization of the Tomimatsu-Sato fields.[7-9]

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