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Soliton operators for the quantized sine-Gordon equation*

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Operators for the creation and annihilation of quantum sine-Gordon solitons are constructed. The operators satisfy the anticommutation relations and field equations of the massive Thirring model. The results of Coleman are thus reestablished without the use of perturbation theory. It is hoped that the method is more generally applicable to a quantum-mechanical treatment of extended solutions of field theories.

I. INTRODUCTION

The two-dimensional classical sine-Gordon field is probably the simplest nonlinear field which possesses extended solutions of the type currently under investigation, the so-called soliton solutions.[1-3] Coleman[4] has extended the results to the quantized theory by relating the sine-Gordon field to the massive Thirring model, i.e., to a two-dimensional self-coupled Fermi field with vector interaction. It is the purpose of this note to construct operators for the creation and annihilation of bare solitons; we shall thereby obtain a simple rederivation of Coleman's results. Operators analogous to ours have been obtained for the massless Thirring model by Dell'Antonio, Frishman, and Zwanziger,[5] but they do not possess a similar physical significance. The treatment of the massive model is in some respects simpler than that of the massless model as there are no infrared divergences. We shall therefore attempt to keep our treatment self-contained, at the risk of repeating some of the analysis of Ref. 5. Our work will also be logically independent of that of Coleman, though we shall be motivated by some of his results.

The sine-Gordon field satisfies the equation

$$\ddot{\phi}(x, t) - \phi''(x, t) + (\mu^2/\beta):\sin[\beta\phi(x, t)]:=0 \ . \tag{1.1}$$

The equation is invariant under the transformation

$$\phi \rightarrow \phi + 2\pi n\beta^{-1} \ , \tag{1.2}$$

so that the vacuum possesses a discrete degeneracy, characterized by an index n which can assume any integral value (positive, negative, or zero). Solitons are solutions of the field equations where the vacuum well to the left of the disturbance is different from the vacuum well to the right. We shall define a soliton and an antisoliton by the boundary conditions

$$\begin{aligned} &\langle\phi(x)\rangle \rightarrow 0, \quad x \rightarrow +\infty \ , \\ &\langle\phi(x)\rangle \rightarrow \mp 2\pi\beta^{-1}, \quad x \rightarrow -\infty \ . \end{aligned} \tag{1.3}$$

The sign is negative for a soliton, positive for an antisoliton.

Solitons may be compared with certain types of extended solutions of classical equations in three or four dimensions, where the degenerate vacuum is characterized by a continuous parameter which varies with the direction in which one recedes from the disturbance. The two-dimensional model has only two asymptotic directions and a discretely degenerate vacuum, but it nevertheless possesses the essential features of systems such as vortices or monopoles.

Most treatments of extended solutions have been classical or semiclassical, but the work of Coleman, referred to above, is fully quantum-mechanical. Coleman showed that the sine-Gordon field is equivalent to the massive Thirring field, the solitons corresponding to the states with a fermion number of unity. He found the following relation between the constant β in (1.1) and the coupling constant q of the Thirring model:

$$g/\pi = 1 - 4\pi\beta^{-2} \ . \tag{1.4}$$

For $\beta^2 > 4\pi$ the coupling between a soliton and an antisoliton is attractive, and sine-Gordon particles probably appear as soliton-antisoliton bound states. For $\beta^2 > 4\pi$ the coupling is repulsive, and stable sine-Gordon particles probably do not exist. For $\beta^2 = 4\pi$, we are led to the remarkable result that the sine-Gordon model is equivalent to a free Fermi field.

The results of Ref. 4 were obtained by comparing two rather unconventional perturbation series, and it is doubtful whether the methods could be extended to four dimensions. We wish to show that "bare" soliton creation and annihilation operators can be constructed fairly simply from sine-Gordon operators; the construction is motivated by the physical characteristics of solitons discussed above. The operators will be shown to satisfy the commutation relations and field equations of the massive Thirring model. The relations between sine-Gordon operators and bilinear functions of the Fermi operators will agree with those found by Coleman; they are generalizations of relations for the massless case which have been used to solve the Thirring model.[6] The correspondences have been listed in a recent paper by Kogut and Susskind,[7] who suggest applications to massive quantum electrodynamics.

Since our operators are local, the bare solitons which they create will be point particles. Physical solitons become spread out by the interaction in the usual way. It is, of course, not guaranteed that there is any relation (other than that of fermion number conservation) between soliton operators and actual particles, except for the interaction-free case $\beta^2 = 4\pi$. Nevertheless, it is very plausible that such a relation exists, at any rate for a range of β around this value.

II. CONSTRUCTION OF SOLITON OPERATORS

The quantized sine-Gordon system is described by Eq. (1.1), with the ϕ's satisfying the canonical commutation relations. Since all renormalization constants are finite, we can eliminate the infinities by normal ordering with respect to bare-particle creation and annihilation operators. We therefore write

$$\phi(x, t) = \phi^+(x, t) + \phi^-(x, t) \ , \tag{2.1}$$

where ϕ^+ and ϕ^- satisfy the commutation relations

$$[\phi^+(x, t+dt), \phi^-(y, t)] = \Delta_+((x-y)^2 - (dt + i\epsilon)^2) \ . \tag{2.2}$$

For small separations

$$\Delta_+ = -(4\pi)^{-1}\ln\{c^2\mu^2[x^2 - (dt+i\epsilon)^2]\} + O(x^2). \tag{2.3}$$

The value of the constant c need not concern us. It will be convenient to use the quantity $c\mu$ as our unit of mass in defining dimensionless quantities.

An operator $\psi(x)$ which annihilates a soliton at a point x must increase the value of ϕ by $2\pi\beta^{-1}$ in regions well to the left of x, but it must have no effect in regions well to the right of x. We therefore expect the operator to satisfy the commutation relations

$$[\phi(y), \psi(x)] = 2\pi\beta^{-1}\psi(x) \quad (y < x) \ , \tag{2.4a}$$

$$[\phi(y), \psi(x)] = 0 \quad (y > x) \ . \tag{2.4b}$$

Such commutation relations will hold if ψ has the form

$$\psi(x) = :A(x)\exp\left[-2\pi i\beta^{-1}\int_{-\infty}^{x} d\xi\ \dot\phi(\xi)\right]: \ , \tag{2.5}$$

where the operator A is yet to be determined. At the moment we leave it open whether ψ is a boson or fermion operator.

We cannot take $A(x)$ equal to unity, since ψ would then be a boson operator, and the commutator between $\psi(x)$ and $\dot\psi(y)$ would not be simple when $x = y$. Taking A to be a polynomial in ϕ, $\dot\phi$, or ϕ' merely complicates matters. We therefore try modifying the exponent, and the simplest dimensionless term we can add is $C\phi(x)$. Thus,

$$\psi(x) = :\exp\left[-2\pi i\beta^{-1}\int_{-\infty}^{x} d\xi\ \dot\phi(\xi) + C\phi(x)\right]: \ . \tag{2.6}$$

From the formula

$$e^Ae^B = e^{[A,B]}e^Be^A \quad ([A, B] \text{ a } c \text{ number}) \ , \tag{2.7}$$

it follows that $\psi(x)$ and $\psi(y)$, defined by (2.6), do not in general commute or anticommute if $x \neq y$. However, if $C = Ni\beta$ they commute, while if $C = (N+\frac{1}{2})i\beta$ they anticommute. The simplest possibility is $C = \frac{1}{2}i\beta$ so that, with hindsight from the results of Refs. 4 and 7, we are led to suggest the operators

$$\psi_1(x) = (c\mu/2\pi)^{1/2} \times e^{\mu/8\epsilon} :\exp\left[-2\pi i\beta^{-1}\int_{-\infty}^{x} d\xi\ \dot\phi(\xi) - \tfrac{1}{2}i\beta\phi(x)\right]: \ , \tag{2.8a}$$

$$\psi_2(x) = -i(c\mu/2\pi)^{1/2} \times e^{\mu/8\epsilon} :\exp\left[-2\pi i\beta^{-1}\int_{-\infty}^{x} d\xi\ \dot\phi(\xi) + \tfrac{1}{2}i\beta\phi(x)\right]: \ . \tag{2.8b}$$

An adiabatic cutoff $e^{-\epsilon\xi}$ is implied in the integration. The constant factors have been inserted for convenience in our later work; in particular, the phase factor $-i$ in (2.8b) will be necessary if our operators are to correspond to the canonical γ matrices

$$\gamma^0=\sigma^1, \quad \gamma^1=i\sigma_2, \quad \gamma^5=\gamma^0\gamma^1=-\sigma_3 . \qquad (2.9)$$

We note that the reflection operator interchanges ψ_1 and ψ_2 and, at the same time, changes the vacuum index n by one unit.

It remains to find the commutation relations and field equations satisfied by the ψ's, and to show that they correspond to those of the massive Thirring model.

III. COMMUTATION RELATIONS AND CURRENT DENSITIES

We have already shown that two ψ's anticommute when $x\neq y$. When $x=y$ a more detailed investigation is necessary, both because the commutator in (2.7) is not well defined and because the product of two ψ's becomes singular as their arguments approach one another. First let us decide how to formulate the commutation relations between renormalized ψ's when Z is infinite. Formally we wish to show that

$$\{\psi_i(x), \psi_j^\dagger(y)\}=Z\delta(x-y) . \qquad (3.1)$$

Alternatively, we may define

$$j^\mu(x)=Z^{-1}\bar{\psi}(x)\gamma^\mu\psi(x)+\text{const.} \qquad (3.2)$$

We then have to verify the commutation relations

$$[j^\mu(x), \psi(y)]=-(g^{\mu 0}+\epsilon^{\mu 0}\gamma^5)\psi(x)\delta(x-y) . \qquad (3.3)$$

Our procedure will be to replace (3.2) by the equation

$$\tilde{j}^\mu(x)=\lim_{y\to x}\left[|c\mu(x-y)|^\sigma\bar{\psi}(x)\gamma^\mu\psi(y)+F(x-y)\right] , \qquad (3.4)$$

where the constant σ and the c-number function F are chosen so that the right-hand side approaches a finite limit as $y\to\infty$. The function $|x-y|^\sigma$ replaces the constant Z. We shall then verify the commutation relations (3.3), with j replaced by $\tilde{j}$.

We evaluate the product $\bar{\psi}(x)\psi(y)$ $(x\simeq y)$ using the formula

$$:e^A::e^B:=e^{[A^+,B^-]}:e^{A+B}: , \qquad (3.5)$$

which is true if $[A^+, B^-]$ is a c number. It follows by straightforward calculation from (2.2), (2.3), (2.8), and (3.5) that

$$\psi_\alpha^\dagger(x)\psi_\alpha(y)=\mp i[2\pi(x-y)]^{-1}|c\mu(x-y)|^{-\beta^2g^2/(2\pi)^3} \times:\exp\left\{-2\pi i\beta^{-1}\int_x^y d\xi\,\dot{\phi}(\xi)\mp\tfrac{1}{2}i\beta[\phi(y)-\phi(x)]+O(x-y)^2\right\}: \text{ (no sum over } \alpha) , \qquad (3.6)$$

where the $\mp$ sign is $-$ for $\alpha=1$, $+$ for $\alpha=2$. The exponential in (3.6) may be expanded up to terms linear in $x-y$, and the result compared with (3.4). We find that

$$\sigma=\beta^2g^2(2\pi)^{-3} ,$$
$$\tilde{j}_0(x)=(2\pi)^{-1}\beta\phi'(x) , \qquad (3.7)$$
$$\tilde{j}_1(x)=-2\beta^{-1}\dot{\phi}(x) .$$

On inserting (2.8) and (3.7) in (3.3), we confirm that that our operators satisfy the required commutation relations.

Equation (3.7) shows that the two operators $\tilde{j}^0$ and $\tilde{j}^1$ are not components of a vector and that they do not satisfy the equation of continuity. We therefore replace (3.4), (3.3), and (3.7) by the equations

$$j^\mu(x)=\lim_{y\to x}\left\{[\delta_0^\mu+(4\pi)^{-1}\beta^2\delta_1^\mu]|c\mu(x-y)|^\sigma\bar{\psi}(x)\gamma^\mu\psi(y)+F(x-y)\right\} , \qquad (3.8)$$

$$[j^\mu(x), \psi(y)]=-[g^{\mu 0}+(4\pi)^{-1}\beta^2\epsilon^{\mu 0}\gamma^5]\psi(x)\delta(x-y) , \qquad (3.9)$$

$$j^\mu=-(2\pi)^{-1}\beta\epsilon^{\mu\nu}\partial_\nu\phi . \qquad (3.10)$$

It is well known that a correct treatment of the infinities in the Thirring model requires the extra factor $(4\pi)^{-1}\beta^2$ in (3.8) and (3.9).

IV. FIELD EQUATIONS

Rather than expressing the Hamiltonian in terms of the ψ's we shall show that our operators do satisfy the field equations of the massive Thirring model. We thereby reduce the appearance of infinite quantities to a minimum.

We wish to establish the equations

$$(-i\gamma^\mu\partial_\mu-m_0)\psi(x)=\lim_{\delta x\to 0}\tfrac{1}{2}g\gamma^\mu[j_\mu(x+\delta x)+j_\mu(x-\delta x)]\psi(x) . \qquad (4.1)$$

The limiting procedure on the right-hand side of (4.1) is familiar in the Thirring model, and we shall find that no singular terms appear when δx approaches zero.

The only infinity in (4.1) is that associated with mass renormalization, and it will be treated in the same way as before. We write

$$-m_0\psi\sim[S,\gamma^0\psi] , \qquad (4.2)$$

where, formally,

$$S = Z m_0 \int_{-\infty}^{\infty} dx\, \bar{\psi}(x)\psi(x) \ . \tag{4.3}$$

We then replace (4.3) by the equation

$$S = \int_{-\infty}^{\infty} dx \lim_{y\to x} |c\mu(x-y)|^{-\delta} m\bar{\psi}(x)\psi(y) \ , \tag{4.4}$$

where δ is chosen so that the limit is finite. The constant m is a finite mass.

From (2.8a), we find that the time derivative of ψ_1 is given by the equation

$$\begin{aligned}\dot{\psi}_1(x) &= :\left\{\left[-2\pi i\beta^{-1}\int_{-\infty}^{x} d\xi\ \ddot{\phi}(\xi) - \tfrac{1}{2}i\beta\dot{\phi}(x)\right], \psi_1(x)\right\}: \\ &= N'\left\{\left[-2\pi i\beta^{-1}\phi'(x) - \tfrac{1}{2}i\beta\dot{\phi}(x) + 2i\pi\mu^2\beta^{-2}\int_{-\infty}^{x} d\xi\ :\sin\beta\phi(\xi):\right], \psi_1(x)\right\}\end{aligned} \tag{4.5}$$

by (1.1). Normal ordering is always understood to be with respect to sine-Gordon operators, not Thirring operators. The symbol N' indicates that the terms in the expansion of $\sin\beta\phi$ are treated as units; their individual factors are not normal ordered with respect to those of ψ. The term involving $\sin\beta\phi$ may then be written as follows:

$$N'\left\{2i\pi\mu^2\beta^{-2}\int_{-\infty}^{x} d\xi\ \sin\beta\phi(\xi), \psi_1(x)\right\} = -i\mu^2\beta^{-2}\left[\int d\xi\ :\cos\beta\phi(\xi):, \psi_1(x)\right] \ . \tag{4.6}$$

Equation (2.8a) now shows that, apart from factors due to normal ordering, the operator $\cos\beta\phi$ is just $2\pi\bar{\psi}\psi$. Taking into account the normal ordering, we find that

$$\psi_2^\dagger(x)\psi_1(y) \simeq (c\mu/2\pi)|c\mu(x-y)|^{\delta} :e^{-i\beta\phi}:, \quad x\simeq y \ , \tag{4.7a}$$

$$\psi_1^\dagger(x)\psi_2(y) \simeq (c\mu/2\pi)|c\mu(x-y)|^{\delta} :e^{i\beta\phi}:, \quad x\simeq y \ , \tag{4.7b}$$

where

$$\delta = \frac{g}{2\pi}\left(1 + \frac{\beta^2}{4\pi}\right) \ . \tag{4.7c}$$

Comparing this equation with (4.4), we may write

$$S = c\mu m \pi^{-1}\int_{-\infty}^{\infty} d\xi\ :\cos\beta\phi(\xi): \ .$$

Equation (4.6) thus becomes

$$N'\left\{2i\pi\mu^2\beta^{-2}\int_{-\infty}^{x} d\xi\ \sin\beta\phi(\xi), \psi_1\right\} = -i[S, \gamma^0\psi_2] \ , \tag{4.8}$$

with

$$m = \mu\pi/(c\beta^2) \ . \tag{4.9}$$

As we are aiming at Eq. (4.1), we calculate ψ_1' as well as $\dot{\psi}_1$:

$$\psi_1'(x) = :\{[-2\pi i\beta^{-1}\dot{\phi}(x) - \tfrac{1}{2}i\beta\phi'(x)], \psi_1(x)\}: \ . \tag{4.10}$$

From (4.5), (4.10), (4.8), and (1.4) we obtain the equation

$$\dot{\psi}_1(x) + \psi_1'(x) + i[S, \gamma^0\psi_2(x)] = -\frac{i\beta}{2\pi} g:\{[\phi'(x) - \dot{\phi}(x)], \psi_1(x)\}: \ . \tag{4.11}$$

The operators ϕ' and $\dot{\phi}$ may be taken outside the normal-ordering signs by using the formula

$$A:e^B: = :\{A + [A^+, B^-]\}e^B: \ , \tag{4.12}$$

valid when $[A^+, B^-]$ is a c number. If $A = \phi'(y) - \dot{\phi}(y)$ and $e^B = \psi_1(x)$, we find from (2.2), (2.3), and (2.8) that the commutator $[A^+, B^-]$ is an odd function of $x-y$, so that it gives no contribution to (4.12) if we take the average of terms with $y = x + \delta x$, $y = x - \delta x$. Thus, expressing the operators ϕ' and $\dot{\phi}$ in terms of current densities by (3.10), we obtain the final result

$$\dot{\psi}_1 + \psi_1' + i[S, \gamma^0\psi_2] = \lim_{\delta x\to 0} \tfrac{1}{2}ig[j^0(x+\delta x) + j^0(x-\delta x) + j^1(x+\delta x) + j^1(x-\delta x)]\psi_1(x) \ . \tag{4.13}$$

Equation (4.13) is precisely the second component of Eq. (4.1a). The first component can be obtained in a similar way, and all required properties of our soliton operators are established.

V. CONCLUDING REMARKS

It is hoped that the methods presented here can be applied to extended solutions of four-dimensional field theories. For example, one might attempt to construct operators which create bare Nielsen-Olesen vortices. Just as our soliton operators create point particles, vortex operators would create infinitely thin strings. Interactions between strings would give the vortices a finite thickness. Vortex operators would depend on the shape of an entire string rather than on a single coordinate, and one might hope to identify them with the operators of the second-quantized dual model.[8]

It may therefore be possible to establish a "duality" between quantized Nielsen-Olesen systems and dual models, analogous to the duality between the sine-Gordon field and the massive Thirring model. The Higgs scalars may cause difficulty, since one cannot construct models of strings interacting with "elementary" particles. It will probably be necessary to start from a non-Abelian Nielsen-Olesen model without a Higgs field.

The relationship between the sine-Gordon field and the massive Thirring model does not in itself provide a practical approach to quantized solitons except when β^2 is approximately equal to 4π. Nevertheless, it suggests that we might approximate a physical soliton by suitably spreading out the operators in the exponents of (2.8). A similar approximation may be possible for Nielsen-Olesen vortices, even if the limit of taking an infinitely narrow bare vortex cannot be carried through consistently.

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[1]A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin, Proc. IEEE 61, 1443 (1973).

[2]A. Barone, F. Esposito, C. J. Magee, and A. C. Scott, Riv. Nuovo Cimento 1, 227 (1971).

[3]L. D. Faddeev and L. A. Takhtajan, Dubna report (unpublished).

[4]S. Coleman, Phys. Rev. D 11, 2088 (1975).

[5]G. F. Dell'Antonio, Y. Frishman, and D. Zwanziger, Phys. Rev. D 6, 988 (1972).

[6]See, for instance, B. Klaiber, in *Lectures in Theoretical Physics*, edited by A. O. Barut and W. E. Brittin (Gordon and Breach, New York, 1968), Vol. X-A, p. 141.

[7]J. Kogut and L. Susskind, Cornell report (unpublished).

[8]M. Kaku and M. Kikkawa, Phys. Rev. D 10, 1110 (1974).