PHYSICAL REVIEW D VOLUME 11, NUMBER 2 15 JANUARY 1975

Bose condensation in supercritical external fields*

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(Received 12 August 1974)

We study the relativistic field theory of a charged spin-zero boson field in the presence of an external field, such as the Coulomb field of a prescribed charge distribution. It is shown that for a sufficiently intense field the ground state is unstable against the formation of a Bose-Einstein condensate of charged boson pairs. A consistent quantum theory can be formulated when known nonlinear couplings such as the Coulomb interactions of the bosons are properly included in the Hamiltonian. Speculations are offered concerning the possible stability of nuclei with charge number $Z \gtrsim 10^3$.

I. INTRODUCTION

In this paper, we study the problem of the quantization of a charged spin-zero field in the presence of a localized positive-charge distribution of arbitrary strength ("Coulomb field of a nucleus").

When the nuclear charge number Z exceeds a certain critical value, Z_c, solutions of the linear Klein-Gordon (KG) equation in the external field no longer form a complete set and standard quantization of the quadratic Hamiltonian operator fails. With conventional nuclear charge densities, the fact that this phenomenon occurs for $Z \gtrsim 10^3$ may explain the paucity of attention which it has so far received in the literature.[1-4] It turns out that when one examines the usual quantum theory one notices that at $Z = Z_c$ the vacuum becomes unstable against the production of an indefinite number of charged boson pairs. To guarantee a stable ground state for $Z > Z_c$ one must, at the very least, take account of the self-Coulomb interaction of the boson field. The basic physics of the resulting nonlinear field theory as well as some of the mathematics are given in Migdal's paper,[3] which is, however, technically incomplete. The core of our work, Secs. III and IV, thus has minimal overlap with the former's work.

Thus, at the very least, we exhibit in the material of Secs. II–IV an amusing example of a spatially inhomogeneous Bose-Einstein (BE) condensation phenomenon whose occurrence is in a basic sense attributable to the special properties of a relativistic many-body theory.

In the last section Sec. V, we speculate that ultimate connection with physical reality may not be beyond the possible. For nuclei with $Z > Z_c$ to be stable we argue that two conditions are necessary. The first is that ordinary $N \cong Z$ nuclei contain a neutral BE condensate of equal mixtures of neutral and charged pions. (Bounds on the density of such a condensate set by present experiments will be the subject of separate investigations.) In the domain of large $Z > Z_c$, the considerations of Sec. IV indicate that the Coulomb energy will be reduced in the presence of the pion condensate. Physically this can only happen if the charge density sits well outside the matter density, the net charge residing on the condensate. This leads to our second condition, which is that this effect be sufficiently pronounced to prevent nuclear fission of such superheavy nuclei. Though we are somewhat skeptical concerning the possible concurrence of these two conditions, such a possibility probably merits further investigation.

The behavior of pions in the Coulomb field of a "heavy" nucleus is in marked contrast with the behavior of electrons, a subject which has been thoroughly investigated.[5] These are so far the only cases for which the theory has been worked out.

II. KLEIN-GORDON EQUATION IN A STRONG COULOMB FIELD. INSTABILITY AGAINST CHARGED PAIR CONDENSATION

A. Forms of the KG equation.

We study (with $\hbar = c = 1$) the equation

$$(E - V)^2 \psi(\vec{r}) = (p^2 + m^2)\psi(\vec{r}), \qquad (2.1)$$

where $\psi(\vec{r})$ is a complex scalar function and $V(r)$ is the potential energy of a negatively charged scalar particle of mass m (henceforth called a pion) in the field of a fixed extended charged distribution. For example, we may take

$$V(r) = -\frac{Ze^2}{r} f(r), \qquad (2.2)$$

$$f(r) = 1 - e^{-\kappa r} \qquad (2.3)$$

where $\kappa^{-1} \cong R$, the nuclear radius.

The formal properties of the equation are more easily studied if we introduce a formalism of first order in the energy (time derivative).[6] In terms of the two-component vector

$$\vec{\phi} = \begin{pmatrix} \psi \\ (E-V)\psi \end{pmatrix} , \quad (2.4)$$

Eq. (2.1) becomes

$$E\tau_1\vec{\phi} = \mathcal{H}\vec{\phi} , \quad (2.5)$$

$$\mathcal{H} = \tfrac{1}{2}(1+\tau_3)(p^2+m^2) + \tfrac{1}{2}(1-\tau_3) + \tau_1 V . \quad (2.6)$$

Here τ_i $(i=1,2,3)$ are the usual Pauli spin matrices. Since $\mathcal{H}$ is Hermitian, we recognize that the fundamental scalar product is the integral of the density

$$d_{ab}(\vec{r}) = \vec{\phi}_a^\dagger(\vec{r})\,\tau_1\,\vec{\phi}_b(\vec{r}) , \quad (2.7)$$

providing an orthogonality theorem for two solutions of (2.5) belonging to different energies.

For the norm of any given solution, we compute

$$\int \vec{\phi}^\dagger\,\tau_1\vec{\phi} = 2\int \psi^*(E-V)\psi . \quad (2.8)$$

Remembering the sign of V, Eq. (2.2), we see that (2.8) is certainly positive for $E \geq 0$. On the other hand, for negative-energy continuum solutions, the norm must be negative, by analytic continuation from the limit $V=0$. Thus if we choose the original KG scalar function to be normalized according to

$$\int |\psi|^2 = [2|E - \langle V\rangle|]^{-1} , \quad (2.9)$$

where $\langle V\rangle = \int\psi^* V\psi / \int\psi^*\psi$, we see that, in general, the solution set for given V divides into two subsets ϕ_p and ϕ_n characterized as follows:

$$\int \vec{\phi}_p^\dagger\,\tau_1\,\vec{\phi}_p = 1, \quad E_p - \langle V\rangle_p > 0 ,$$
$$\int \vec{\phi}_n^\dagger\,\tau_1\,\vec{\phi}_n = -1, \quad E_n - \langle V\rangle_n < 0 . \quad (2.10)$$

The considerations above would obviously fail should there occur an eigenvalue for which the inequalities (2.10) are replaced by an equality. The resolution of this difficulty, carried out below, represents the basic goal of this part of our work.

For the normally occurring situations for which (2.10) applies, the completeness relation for the solution of (2.5) takes the dyadic form

$$\sum_p \vec{\phi}_p(\vec{r})[\vec{\phi}_p^\dagger(\vec{r}')\,\tau_1] - \sum_n \vec{\phi}_n(\vec{r})[\vec{\phi}_n^\dagger(\vec{r}')\tau_1] = I\delta(\vec{r}-\vec{r}') , \quad (2.11)$$

and I is the unit two-by-two matrix.

A third form of the KG equation is sometimes useful because of its analogy with the nonrelativistic Schrödinger equation. For this form, we write

$$\epsilon_{\text{eff}}\,\psi = (p^2/2m)\,\psi + V_{\text{eff}}\psi , \quad (2.12)$$

where

$$\epsilon_{\text{eff}} = E'[1+(E'/2m)] ,$$
$$V_{\text{eff}} = V[1+(E'/m)-(V/2m)] , \quad (2.13)$$
$$E = E' + m .$$

B. Bound-state spectrum and approach to the critical point.

For full understanding of the situation when Z becomes very large, we consider together the solutions for both negative and positive pions ($\pi^\pm$). The behavior of the most deeply bound orbits is illustrated schematically in Fig. 1. We shall first describe the results and then indicate how they follow from the KG equation. For a potential energy of the form (2.2), the curve marked E_- represents in its solid part the lowest bound state of a π^-. Two special values of Z are to be noted. For $Z > Z_0$ there emerges from the negative-energy continuum a new bound-state branch for π^- which meets the branch E_- at a point of vertical tangency, $Z = Z_c$. For $Z > Z_c$ there is simply no bound-state solution corresponding to this branch. The curve marked E_+ is a reflection of E_- with respect to the abscissa $E=0$ and represents a solution branch for π^+.

We shall now indicate the derivation of these results from the KG equation. To understand their physical implication we must resort to the quantum field theory described in the next section.

It follows most easily from (2.1) that

$$\psi(\vec{r}; E, e) = \psi(\vec{r}; -E, -e) . \quad (2.14)$$

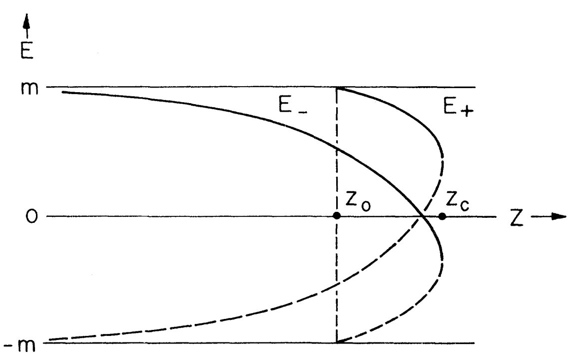

FIG. 1. Schematic representation of the lowest bound-state branch for a negative pion in the Coulomb field of a superheavy nucleus (neglecting strong interactions). The branch for the positive pion related by charge conjugation is also shown.

This shows that if branch E_- occurs for π^-, branch E_+ occurs for π^+. Next consider the equation for π^+ near zero binding energy. From (2.13), remembering to change the sign of the charge, we have

$$V_{\text{eff}}(\pi^+, E' \sim 0) \cong |V(r)| - (V^2/2m) . \tag{2.15}$$

For Z large enough, the second term must dominate and give $(Z > Z_0)$ a π^+ bound state—even in the field of a positively charged nucleus—as shown in the solid part of the branch E_+. The combination of (2.14) and (2.15) establishes the qualitative character of Fig. 1.

One naturally inquires after the physical picture which explains how a π^+ can be bound in the field of a positively charged nucleus. Here, the fact that the charge density is not positive-definite plays an essential role. Though the total charge is one unit, this is obtained by adding individually large contributions of negative and positive charge. If the negative charge is on the average closer to the nuclear charge, we obtain the required binding force.

It is in fact easy to see from the considerations of Sec. II A that points on the solid parts of curves $E_\pm$ correspond to solutions with positive norm whereas dashed portions represent solutions with negative norm. The meeting point must therefore be one at which $E = \langle V \rangle$. Now consider Eq. (2.1) for $V \to \lambda V$ and form the expectation value,

$$\langle (E - \lambda V)^2 \rangle = \langle p^2 \rangle + m^2 . \tag{2.16}$$

With the help of the KG equation, the first derivative of (2.16) at $\lambda = 1$ becomes

$$(E - \langle V \rangle) \frac{dE}{d\lambda} = \langle EV \rangle - \langle V^2 \rangle . \tag{2.17}$$

At the point $E = \langle V \rangle$, the right-hand side cannot vanish. Therefore we must have $dE/d\lambda \to \infty$, a point of vertical tangency.

Let $E_- = -\mu$ represent this point occurring at $Z = Z_c$. Then we see that for the branch E_+, we have $E_+ = \mu$. Therefore in a quantum field theory we could produce an indefinite number of $\pi^\pm$ pairs without energy cost (in the absence of other interactions). This apparent instability will be dealt with in the next section.

Let us estimate the value of Z_c predicted by our model, using the pion mass. This can be done with the help of a virial theorem. From the statement

$$\langle [\vec{r} \cdot \vec{p}, (E - V)^2 - p^2 - m^2] \rangle = 0 , \tag{2.18}$$

we derive for bound states

$$\langle p^2 \rangle = \langle (\vec{r} \cdot \nabla V)(E - V) \rangle . \tag{2.19}$$

Combining this with Eqs. (2.16) ($\lambda = 1$) and (2.2) and (2.3) we derive ($\alpha = e^2/\hbar c$)

$$E(E - \langle V \rangle) = m^2 - Z\alpha E \langle f'(r) \rangle + Z\alpha \langle f'(r) V \rangle . \tag{2.20}$$

The correct order of magnitude for Z_c should be obtained if we examine (2.20) for $E = 0$. Thus we have

$$m^2 = (Z\alpha)^2 \kappa \langle e^{-\kappa r}/r \rangle \sim \frac{(Z\alpha)^2}{R^2} , \tag{2.21}$$

using the essential fact that the pion wave function is largely confined to the nuclear interior. If we set $R^2 = (10^2/m^2)$ we find $Z\alpha \sim 10$, which is roughly consistent but appears to render our problem somewhat academic, at least under presently known conditions. (If we had a pion of electronic mass, Z_c would be reduced by an order of magnitude.)

Returning now to the technical aspects of our problem, we notice that at $Z = Z_c$ there is only one solution remaining of the two we had for $Z < Z_c$. To understand the approach to the limit, we define in terms of the two solutions of interest $\vec{\phi}_\pm$ (± referring to norm) two new *unchanged* linear combinations:

$$\begin{aligned} \vec{\phi}_e &= (1/\sqrt{2})(\vec{\phi}_+ + \vec{\phi}_-) , \\ \vec{\phi}_o &= (1/\sqrt{2})(\vec{\phi}_+ - \vec{\phi}_-) . \end{aligned} \tag{2.22}$$

With

$$d_e(r) = \vec{\phi}_e^\dagger(r) \tau_1 \vec{\phi}_e(r) , \tag{2.23}$$

etc., we have straightforwardly

$$\int d_e = \int d_o = 0 , \tag{2.24}$$

but

$$\int d_{eo} = \int d_{oe} = 1 . \tag{2.25}$$

It may be instructive to exhibit expressions for these densities. In the following we make use of the orthogonality between $\vec{\phi}_+$ and $\vec{\phi}_-$, which if we assume real wave functions can be expressed in the form

$$E_+ + E_- = 2V_{+-} , \tag{2.26}$$

where

$$V_{ab} = (\psi_a, V\psi_b)/(\psi_a, \psi_b) \tag{2.27}$$

and (as will be needed below) $V_{aa} = V_a$. We also record our expressions using unit normalization for ψ_a. We then have

$$d_e(\vec{r}) = \frac{1}{2}\left\{\psi_+^2 - \psi_-^2 + \frac{[V_+ - V(r)]\psi_+^2}{|E_+ - V_+|} + \frac{(V_- - V)\psi_-^2}{|E_- - V_-|} + \frac{2(V_{+-} - V)\psi_+\psi_-}{(|E_+ - V_+||E_- - V_-|)^{1/2}}\right\}, \tag{2.28}$$

$$d_{oe}(\vec{r}) = \frac{1}{2}\left[\psi_+^2 + \psi_-^2 + \frac{(V_- - V)\psi_-^2}{|E_- - V_-|} - \frac{(V_+ - V)\psi_+^2}{|E_+ - V_+|}\right], \tag{2.29}$$

$$d_o(\vec{r}) = \frac{1}{2}\left[\psi_+^2 - \psi_-^2 + \frac{(V_+ - V)\psi_+^2}{|E_+ - V_+|} - \frac{(V_- - V)\psi_-^2}{|E_- - V_-|}\right]. \tag{2.30}$$

From these expressions, we verify (2.24) and (2.25). From (2.28) we see that the limit $Z \to Z_c$ is a delicate one. If we define

$$\alpha = \tfrac{1}{2}(|E_+ - V_+| + |E_- - V_-|), \tag{2.31}$$

it can be shown that d_e behaves like $O(\alpha^{-1})$. By contrast, d_{oe} and d_o remain finite and vanish respectively as $\alpha \to 0$.

III. QUANTIZATION OF THE KLEIN-GORDON EQUATION IN A COULOMB FIELD. STABILIZATION OF THE VACUUM BY NONLINEAR INTERACTIONS

A. Quantization below the critical point.

This is achieved most elegantly in an external field by use of the first-order formalism. We adopt the Lagrangian

$$L(t) = \quad d^3r\,[i\phi^\dagger_{op}(\vec{r},t)\tau_1\phi_{op}(\vec{r},t) - \phi^\dagger_{op}\mathcal{H}\phi_{op}]. \tag{3.1}$$

This yields the canonical momentum

$$\pi_{op}(\vec{r},t) = i\phi^\dagger_{op}(\vec{r},t)\tau_1. \tag{3.2}$$

Thus the required commutation relations are

$$[\phi_{op}(\vec{r},t), \phi^\dagger_{op}(\vec{r}\,',t)] = \tau_1\delta(\vec{r} - \vec{r}\,'). \tag{3.3}$$

The Hamiltonian which follows from (3.1) and (3.2) is

$$H = \int \phi^\dagger_{op}\mathcal{H}\phi_{op}. \tag{3.4}$$

To this we adjoin our candidate for total charge operator, namely

$$Q = -|e|\int \phi^\dagger_{op}\tau_1\phi_{op}. \tag{3.5}$$

It is understood that H and Q are to be taken in normal form with respect to the vacuum state to be defined below.

The completeness relation (2.11) will guarantee a satisfactory quantum theory when used in conjunction with the expansion

$$\phi_{op}(\vec{r}) = \sum_p a_p\vec{\phi}_p(\vec{r}) + \sum_n b_n^\dagger\vec{\phi}_n(\vec{r}), \tag{3.6}$$

if we assume that

$$[a_p, a^\dagger_{p'}] = \delta_{pp'}, \quad [b_n, b^\dagger_{n'}] = \delta_{nn'} \tag{3.7}$$

(and of course if the a's and b's commute, etc.). Thus (3.6) and (3.7) satisfy (3.3). Furthermore, if the vacuum is annihilated by a_p and b_n, we find for (3.4) and (3.5)

$$H = \sum_p E_p a_p^\dagger a_p + \sum_n |E_n| b_n^\dagger b_n, \tag{3.8}$$

$$Q = -|e|\left(\sum_p a_p^\dagger a_p - \sum_n b_n^\dagger b_n\right). \tag{3.9}$$

As long as $Z < Z_c$, this represents an unequivocally satisfactory theory of noninteracting bosons of either charge in the external field of a positively charged nucleus. As we approach the point $Z = Z_c$, we arrive, now on a proper quantum basis, at the situation already described: The vacuum state is no longer unique, but approaches degeneracy with an infinite set of other states containing various numbers of $\pi^\pm$ of energy $\pm\mu$ each.

To remedy this situation, it is sufficient to include in the quantum theory the mutual Coulomb interaction of any produced pions. This will be demonstrated with sufficient rigor in the next two sections, but the results can be anticipated on physical grounds. Thus, as we approach $Z = Z_c$, including the Coulomb interaction of the pions, multipion states of small excitation energy can now mix with the previously defined vacuum. But the expectation value of the Coulomb interaction of any assembly of charges is positive, and since it is quartic in the amplitudes it must ultimately dominate and prevent collapse. Moreover, if we consider the equation of motion for excitations of negative charge, which is the appropriate generalization of the unquantized KG equation, we must find that the charge distribution of the other pions screens the nuclear field on the average. The net result is an effective nuclear field which remains subcritical. A vacuum state is thus stabilized, but in terms of the eigenstates of the Hilbert space defined by the expansion (3.6) its description will involve many components.

B. Quantization beyond the critical point. The reduced Hamiltonian.

Formally the quantization can be carried out precisely as above. We need only replace (3.4) by the Hamiltonian

$$H=\int \phi^{\dagger}_{\rm op}\mathcal{H}\phi_{\rm op} + \tfrac{1}{2}e^2\int \frac{[\phi^{\dagger}_{\rm op}(\vec{r})\tau_1\phi_{\rm op}(\vec{r})][\phi^{\dagger}_{\rm op}(\vec{r}\,')\tau_1\phi_{\rm op}(\vec{r}\,')]}{|\vec{r}-\vec{r}\,'|}\,. \tag{3.10}$$

Since V in $\mathcal{H}$ is now, by assumption, beyond criticality, $|V|>|V_c|$, we choose V_0 such that $|V_0|<|V_c|$ and $|V_c-V_0|/|V_c|\ll 1$ and write

$$\mathcal{H}=\mathcal{H}_0-\tau_1\Delta\,,\qquad V-V_0=-\Delta,\quad \Delta>0\,. \tag{3.11}$$

We also define

$$\epsilon_{\pm}\tau_1\vec{\phi}_{\pm}=\mathcal{H}_0\vec{\phi}_{\pm}\,, \tag{3.12}$$

$$\delta=\tfrac{1}{2}(\epsilon_{+}-\epsilon_{-})>0\,, \tag{3.13}$$

$$\mu=\tfrac{1}{2}(\epsilon_{-}+\epsilon_{+})<0\,, \tag{3.14}$$

where these all refer to the near-critical eigenvalues.

To establish the stability of the Hamiltonian (3.10), we write

$$\phi_{\rm op}(\vec{r})=a\vec{\phi}_{+}(\vec{r})+b^{\dagger}\vec{\phi}_{-}(\vec{r})+\chi_{\rm op}(\vec{r})\,. \tag{3.15}$$

For the remainder of the present section we drop the terms arising from $\chi_{\rm op}$, since these make reference to the nondangerous levels; a full discussion including these terms will be carried out in Sec. IV. We also rewrite the first terms of (3.15) by means of the operators

$$q=\tfrac{1}{2}(a+b^{\dagger})\,,\qquad p=\tfrac{1}{2}i(a^{\dagger}-b)\,, \tag{3.16}$$

yielding

$$\phi_{\rm op}(\vec{r})=q\vec{\phi}_e(\vec{r})+ip^{\dagger}\vec{\phi}_o(\vec{r})\,, \tag{3.17}$$

where $\vec{\phi}_e$ and $\vec{\phi}_o$ are the combinations defined in (2.22). With the aid of definitions (3.11)–(3.14) and the further definitions

$$\Delta_{ab}=\int\vec{\phi}_a^{\dagger}\tau_1\Delta\vec{\phi}_b\,, \tag{3.18}$$

$$U_{ab,cd}=e^2\int(\vec{\phi}_a^{\dagger}\tau_1\vec{\phi}_b)(\vec{\phi}_c^{\dagger}\tau_1\vec{\phi}_d)'/|\vec{r}-\vec{r}\,'|\,, \tag{3.19}$$

$$L=b^{\dagger}b-a^{\dagger}a=:i(q^{\dagger}p^{\dagger}-pq):\,, \tag{3.20}$$

we obtain by straightforward transcription (where $U_{eeee}\equiv U_e$, etc.)

$$\begin{aligned}H_{\rm red}=&(\delta-\Delta_{oo})p^{\dagger}p+(\delta-\Delta_{ee})q^{\dagger}q+(-\mu+\Delta_{oe})L\\&+\tfrac{1}{2}U_e(q^{\dagger}q)^2+\tfrac{1}{2}U_{e,o}\{q^{\dagger}q,p^{\dagger}p\}+\tfrac{1}{2}U_o(p^{\dagger}p)^2\\&+\tfrac{1}{2}\{q^{\dagger}q,L\}U_{e,oe}-\tfrac{1}{2}\{p^{\dagger}p,L\}U_{o,oe}+\tfrac{1}{2}L^2U_{oe,oe}\,.\end{aligned} \tag{3.21}$$

The subscript on H reminds us that we are describing only two degrees of freedom.

For $Z-Z_c\cong 1$, the basis defined by (3.11) and (3.12) should be very satisfactory for the diagonalization of the Hamiltonian (3.21). The unperturbed vacuum (we consider the subspace $L=0$) will mix with states with at most a few pion pairs to yield a new stable vacuum. This is guaranteed by the positive-definite character of the quartic terms of H.

For $Z-Z_c\equiv Z'\gg 1$, the mixing becomes large and the treatment by means of the basis defined above becomes cumbersome. In this case the stability of the ground state may be inferred from a classical approximation for which

$$\langle(q^{\dagger}q)^2\rangle\cong\langle q^{\dagger}q\rangle\langle q^{\dagger}q\rangle=(q^2)^2\,, \tag{3.22a}$$

$$\langle q^{\dagger}qp^{\dagger}p\rangle=\langle q^{\dagger}q\rangle\langle p^{\dagger}p\rangle=q^2p^2\,, \tag{3.22b}$$

etc.

With

$$\delta-\Delta_{oo}=-B,\quad \delta-\Delta_{ee}=-A\,, \tag{3.23}$$

we have ($L=0$)

$$\begin{aligned}W(q^2,p^2)&=\langle H_{\rm red}\rangle\\&=-Aq^2-Bp^2+\tfrac{1}{2}U_e(q^2)^2+U_{eo}q^2p^2+\tfrac{1}{2}U_o(p^2)^2\,.\end{aligned} \tag{3.24}$$

As will be seen below, we can, without loss of generality, choose $p^2=0$. The variation of W with respect to q^2 then yields the minimum at

$$q^2=A/U_e\,, \tag{3.25}$$

and

$$W=-\tfrac{1}{2}A^2/U_e\,, \tag{3.26}$$

which is proportional to Z'^2.

Further progress and substantiation of the above simplification depends on recognizing that the variational expression (3.24) may be derived as the expectation value of $H_{\rm red}$ with respect to a coherent trial function

$$|a',b',\theta\rangle=\exp(a^{\dagger}a'+b^{\dagger}b'e^{-i\theta})|{\rm vac}\rangle\,. \tag{3.27}$$

[This is true insofar as $|a'|^2\gg 1$, $|b'|^2\gg 1$, and requires remembering that (3.27) is not normalized.] With the identifications

$$q^2=|q_0|^2,\quad p^2=|p_0|^2\,, \tag{3.28}$$

we have (immediately dropping the subscript zero)

$$\sqrt{2}\,q=a'+b'e^{-i\theta}\,, \tag{3.29}$$

$$\sqrt{2}\,p=i(a'-b'e^{-i\theta})\,. \tag{3.30}$$

Average charge neutrality is assured by choosing $a'=b'$. The choice $\theta=0$ for the arbitrary phase furthermore guarantees $p=0$. Both here and below, however, it is useful to retain this phase until the final stages of the formulation.

The observations just made suggest that the previous classical approximation can be improved by also varying the operators $a^\dagger$ and $b^\dagger$, i.e., by varying the functions $\vec{\phi}_e(\vec{r})$ and $\vec{\phi}_o(\vec{r})$ in (3.17). The expectation value of H_{red} with respect to a general trial state (3.27) again has the form (3.24) with the more concise definitions of A and B,

$$A = -\int \vec{\phi}_e^\dagger \mathcal{H} \vec{\phi}_e , \tag{3.31}$$

$$B = -\int \vec{\phi}_o^\dagger \mathcal{H} \vec{\phi}_o . \tag{3.32}$$

It is important in the following to notice that (3.24) is a functional of the quantities $q\vec{\phi}_e$ and $p\vec{\phi}_o$ (and their Hermitian conjugates).

The variation is to be carried out subject to the constraints that $\vec{\phi}_\pm$ given by

$$\vec{\phi}_\pm \equiv (2)^{-1/2}(\vec{\phi}_e \pm \vec{\phi}_o) \tag{3.33}$$

have positive and negative norm, respectively, as required by (2.10). From the variational expression (3.24), we thus subtract

$$-\epsilon_+ |a'|^2 \int \vec{\phi}_+^\dagger \tau_1 \vec{\phi}_+ - \epsilon_- |b'|^2 \int \vec{\phi}_-^\dagger \tau_1 \vec{\phi}_- , \tag{3.34}$$

the coefficients of the norms representing a definition of Lagrange multipliers. This is transformed by means of (3.29), (3.30), and (3.33) into the expression

$$-\mu(|q|^2+|p|^2)\left(\int \vec{\phi}_e^\dagger \tau_1 \vec{\phi}_e + \int \vec{\phi}_o^\dagger \tau_1 \vec{\phi}_o\right) - \delta(|q|^2+|p|^2)\left(\int \vec{\phi}_e^\dagger \tau_1 \vec{\phi}_o + \int \vec{\phi}_o^\dagger \tau_1 \vec{\phi}_e\right) , \tag{3.35}$$

where we have also assumed charge neutrality and used the definitions (3.13) and (3.14). Variation of the sum of (3.24) and (3.35) with respect to $|q|^2 \vec{\phi}_e^\dagger$ and $|p|^2 \vec{\phi}_o^\dagger$ yields the equations

$$\mathcal{H}_{\text{eff}} \vec{\phi}_e = \mu \tau_1 \vec{\phi}_e + \delta \tau_1 \vec{\phi}_o , \tag{3.36a}$$

$$\mathcal{H}_{\text{eff}} \vec{\phi}_o = \mu \tau_1 \vec{\phi}_o + \delta \tau_1 \vec{\phi}_e , \tag{3.36b}$$

or equivalently

$$\mathcal{H}_{\text{eff}} \vec{\phi}_\pm = \epsilon_\pm \tau_1 \vec{\phi}_\pm , \tag{3.37}$$

where

$$\mathcal{H}_{\text{eff}} = \mathcal{H} + v_{\text{eff}} , \tag{3.38}$$

$$v_{\text{eff}}(\vec{r}) = e\tau_1 \int \rho_{\text{eff}}(\vec{r}\,')|\vec{r} - \vec{r}\,'|^{-1} , \tag{3.39}$$

and

$$\rho_{\text{eff}}(\vec{r}) = |q|^2 \vec{\phi}_e^\dagger(\vec{r})\tau_1 \vec{\phi}_e(\vec{r}) + |p|^2 \vec{\phi}_o^\dagger(\vec{r})\tau_1 \vec{\phi}_o(\vec{r}) = |q|^2 \rho_e + |p|^2 \rho_o . \tag{3.40}$$

An understanding of the structure of the nonlinear problem posed by (3.36)–(3.40) requires further discussion. For instance, what happens when we set $|p|^2=0$? Since this corresponds to setting the phase $\theta=0$, we are led to study the equivalent transformation

$$\vec{\phi}_-' = e^{i\theta} \vec{\phi}_- , \quad \vec{\phi}_+' = \vec{\phi}_+ , \tag{3.41}$$

or in terms of $\vec{\phi}_e$ and $\vec{\phi}_o$

$$\vec{\phi}_e' = \tfrac{1}{2}(1+e^{i\theta})\vec{\phi}_e + \tfrac{1}{2}(1-e^{i\theta})\vec{\phi}_o , \tag{3.42a}$$

$$\vec{\phi}_o' = \tfrac{1}{2}(1-e^{i\theta})\vec{\phi}_e + \tfrac{1}{2}(1+e^{i\theta})\vec{\phi}_o . \tag{3.42b}$$

With the help of (3.29) and (3.30), we can furthermore write

$$|q|^2 = \chi n , \quad |p|^2 = (1-\chi)n , \tag{3.43}$$

where

$$n = |a'|^2 + |b'|^2 , \quad \chi = \tfrac{1}{2}(1+\cos\theta) . \tag{3.44}$$

By means of trivial algebra we now find as expected that (3.36) or (3.37) is invariant in form under the transformation (3.42), but because

$$\rho_e' = \chi \rho_e + (1-\chi)\rho_o \tag{3.45}$$

the definition (3.40) is simplified to

$$\rho_{\text{eff}}(\vec{r}) = n\rho_e'(\vec{r}) \equiv \rho(\vec{r}) . \tag{3.40$'$}$$

Thus we have shown that the solution for $|p|^2 \neq 0$ can be transformed into one with $|p|^2=0$ without affecting any physical results. But to obtain a general formulation of the problem this should be done after the variation has been carried out.

If we apply (3.42) to the energy (3.24) (or equivalently set $|p|^2=0$), we find

$$W(n^2, 0) = -A'n + \tfrac{1}{2}U_e' n^2 , \tag{3.46}$$

which is a minimum for [cf. (3.25)]

$$n = A/U_e , \tag{3.47}$$

where we henceforth drop the primes. Remembering the definitions (3.19) and (3.23), we see that $\rho_{\text{eff}}(r)$, (3.40$'$), is independent of the scale of $\vec{\phi}_e$.

The last observation is the essential one to understand: the limit $\delta \to 0$. In this limit ρ_{eff} will remain finite, but of the separate factors $n=q^2 \to 0$ and $\rho_e \to \infty$. It is strongly suggested that we rescale the functions $\vec{\phi}_e$, $\vec{\phi}_o$. We therefore define

$$\vec{\phi} = q\vec{\phi}_e , \tag{3.48a}$$

$$\vec{\psi} = (\vec{\phi}_o/q) , \tag{3.48b}$$

thus retaining the scalar product between them [Eq. (3.53) below]. With

$$\lambda \equiv (\delta/q^2)\,, \tag{3.49}$$

Eqs. (3.36) become

$$(\mathcal{H}_{\rm eff} - \mu\tau_1)\vec{\phi} = q^4\lambda\tau_1\vec{\psi}\,, \tag{3.50a}$$

$$(\mathcal{H}_{\rm eff} - \mu\tau_1)\vec{\psi} = \lambda\tau_1\vec{\phi}\,, \tag{3.50b}$$

where

$$\mathcal{H}_{\rm eff} = \mathcal{H} + v\,, \tag{3.51}$$

$$v = e\tau_1\int \rho(\vec{r}\,')|\vec{r} - \vec{r}\,'|^{-1}\,, \tag{3.52}$$

$$\int \bar{\vec{\psi}}\vec{\phi} = 1,\ \int \bar{\vec{\psi}}\vec{\psi} = \int \bar{\vec{\phi}}\vec{\phi} = 0\,, \tag{3.53}$$

$$\bar{\vec{\psi}} = \vec{\psi}^{\dagger}\tau_1\,, \text{etc.}, \tag{3.54}$$

$$\rho(\vec{r}) = \bar{\vec{\phi}}\vec{\phi}\,. \tag{3.55}$$

In this new form the structure of the self-consistency problem is completely clear. The set (3.50)–(3.55) will have a self-consistent solution for any (reasonable) $q \geq 0$. [At self-consistency q^2 will, in fact, be given by (3.47).] The limit $q = 0$ may be taken with inpunity. In this limit, the self-consistency problem becomes independent of $\vec{\psi}$ and of its Eq. (3.50b). This means, as we know, that we have lost one solution of our self-consistent KG equation. *Equation (3.50b) now defines a nonvanishing function suitably orthogonal to all solutions of the KG equation, thus providing a missing function to make up a complete set.*

Let $\vec{\phi}_a$ be all solutions of

$$\mathcal{H}_{\rm eff}\vec{\phi}_a = \epsilon_a\tau_1\vec{\phi}_a\,, \tag{3.56}$$

other than ϕ itself. As described in Sec. II, these will divide neatly into two sets $\vec{\phi}_p$ and $\vec{\phi}_n$, with positive and negative norm. For an arbitrary vector

$$\vec{f}(\vec{r}) = \begin{pmatrix} f_1(\vec{r}) \\ f_2(\vec{r}) \end{pmatrix}\,, \tag{3.57}$$

we have (including the limit $\delta = 0$, but not confined to it) an expansion theorem

$$\vec{f} = f_0\vec{\phi} + f_1\vec{\psi} + \sum_p f_p\vec{\phi}_p + \sum_n f_n\vec{\phi}_n\,, \tag{3.58}$$

where

$$f_0 = \int \bar{\vec{\psi}}\vec{f}\,,\quad f_1 = \int \bar{\vec{\phi}}\vec{f}\,,\quad f_p = \int \bar{\vec{\phi}}_p\vec{f},\ \ f_n = -\int \bar{\vec{\phi}}_n\vec{f}\,. \tag{3.59}$$

The discussion just completed provides us with the basic machinery needed to carry out a complete quantization in the supercritical domain $Z > Z_c$. This is done below.

IV. CONSISTENT QUANTIZATION FOR SUPERCRITICAL FIELDS.

As we have seen in the preceding section, for $Z > Z_c$ we are dealing with a Bose condensation phenomenon. In our discussion of the stability of the vacuum based on the coherent state approximation, we have already mixed eigenstates of different charge. For a general quantization we generalize this procedure by replacing H, Eq. (3.10), by

$$H' = H - \mu L\,, \tag{4.1}$$

where

$$L = \int \phi_{\rm op}(\vec{r})\tau_1\phi^{\dagger}_{\rm op}(\vec{r}) \tag{4.2}$$

is proportional [cf. (3.5)] to the total charge operator. We now write

$$\phi_{\rm op}(\vec{r}) = \vec{\phi}(\vec{r}) + \chi_{\rm op}(\vec{r})\,, \tag{4.3}$$

where

$$\vec{\phi}(\vec{r}) = q\,\vec{\phi}_e\,, \tag{4.4}$$

and $\vec{\phi}_e$, q are solutions of the self-consistency problem (3.36) with $p = 0$. We shall ultimately consider the limit $\delta \to 0$. We have argued that the function $\vec{\phi}(r)$ remains well defined in this limit. As long as $\delta \neq 0$, $\mathcal{H}_{\rm eff}$ defines a complete set of single-particle functions in the sense of Eq. (2.11). For $\delta = 0$, we have learned how to adjoin a function to the solutions of the KG equation so as to achieve completeness.

If we substitute (4.3) into (4.1), using (3.36), we find straightforwardly, omitting also the subscript from $\chi_{\rm op}$, that

$$\begin{aligned} H' = {} & -\tfrac{1}{2}e^2\int \frac{\rho(\vec{r}\,')\rho(\vec{r})}{|\vec{r} - \vec{r}\,'|} + q^2\delta + q\delta\left(\int \chi^{\dagger}\tau_1\vec{\phi}_o + \int \vec{\phi}_o^{\dagger}\tau_1\chi\right) \\ & + \int \chi^{\dagger}(\mathcal{H}_{\rm eff} - \mu\tau_1)\chi + \tfrac{1}{2}e^2\int \frac{[\eta(\vec{r}) + \eta^{\dagger}(\vec{r})][\eta(\vec{r}\,') + \eta^{\dagger}(\vec{r}\,')]}{|\vec{r} - \vec{r}\,'|} \\ & + e^2\int \frac{\eta^{\dagger}(\vec{r})[\chi^{\dagger}(\vec{r}\,')\tau_1\chi(\vec{r}\,')]}{|\vec{r} - \vec{r}\,'|} + \text{H.c.} + \tfrac{1}{2}e^2\int \frac{[\chi^{\dagger}(\vec{r})\tau_1\chi(\vec{r})][\chi^{\dagger}(\vec{r}\,')\tau_1\chi(\vec{r}\,')]}{|\vec{r} - \vec{r}\,'|}\,, \end{aligned} \tag{4.5}$$

where [cf. (3.40′)]

$$\rho(\vec{r}) = \vec{\phi}^\dagger(r)\tau_1\vec{\phi}(r)\,, \tag{4.6}$$

$$\eta(\vec{r}) = \vec{\phi}^\dagger(r)\tau_1\chi(r)\,. \tag{4.7}$$

If we ignore terms linear in δ which will ultimately $\to 0$, we see that (4.5) has the structure, in an obvious notation,

$$H' = W_0 + H_2' + H_3' + H_4'\,, \tag{4.8}$$

where

$$W_0 = -\tfrac{1}{2}e^2\int \frac{\rho(\vec{r})\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \tag{4.9}$$

is the condensation energy and

$$H_2' = \int \chi^\dagger(\mathcal{H}_{\text{eff}} - \mu\tau_1)\chi + \tfrac{1}{2}e^2\int\frac{(\eta+\eta^\dagger)(\eta'+\eta^{\dagger\prime})}{|\vec{r}-\vec{r}\,'|} \tag{4.10}$$

is the part quadratic in the field operator χ.

We shall seek a realization of the operator $\chi(\vec{r})$ which brings (4.10) to diagonal form and satisfies the commutation relations

$$[\chi(\vec{r}), \chi^\dagger(\vec{r}\,')] = \tau_1\delta(\vec{r}-\vec{r}\,')\,, \tag{4.11a}$$

$$[\chi(\vec{r}), \chi(\vec{r}\,')] = 0\,. \tag{4.11b}$$

This can be done most expeditiously as follows: Let us define the amplitudes

$$\langle \text{vac}|\chi(\vec{r})|\nu\rangle \equiv \vec{f}_\nu(\vec{r})\,, \tag{4.12a}$$

$$\langle \text{vac}|\chi^\dagger(\vec{r})|\nu\rangle \equiv \vec{g}_\nu^\dagger(\vec{r})\,, \tag{4.12b}$$

where the states $|\nu\rangle$ are quasiparticle states, which we study first in an approximation which neglects H_3' and H_4', Eq. (4.8). In this approximation we obtain by taking matrix elements of the Heisenberg equations of motion for the operators $\chi, \chi^\dagger$,

$$E_\nu\tau_1\vec{f}_\nu(r) = (\mathcal{H}_{\text{eff}} - \mu\tau_1)\vec{f}_\nu(r) + e^2\int|\vec{r}-\vec{r}\,'|^{-1}[\zeta_{f_\nu}(\vec{r}\,') + \zeta_{g_\nu}^*(\vec{r}\,')]\,\tau_1\vec{\phi}(\vec{r})\,, \tag{4.13a}$$

$$-E_\nu\vec{g}_\nu^\dagger(\vec{r})\tau_1 = \vec{g}_\nu^\dagger(\mathcal{H}_{\text{eff}} - \mu\tau_1) + e^2\vec{\phi}^\dagger(\vec{r})\tau_1\int|\vec{r}-\vec{r}\,'|^{-1}[\zeta_{f_\nu}(\vec{r}\,') + \zeta_{g_\nu}^*(\vec{r}\,')]\,, \tag{4.13b}$$

with

$$\zeta_f(\vec{r}) = \bar{\vec{\phi}}(\vec{r})\vec{f}(\vec{r})\,, \tag{4.14}$$

and E_ν measures the energy of the state $|\nu\rangle$ relative to that of the vacuum. For an acceptable quantization, this must be positive for *all* ν.

We demonstrate how (4.13) meets these requirements. By forming the obvious scalar products, we derive from (4.13a) and (4.13b)

$$E_\nu\left(\int\bar{\vec{f}}_\nu\vec{f}_\nu - \int\bar{\vec{g}}_\nu\vec{g}_\nu\right) = \int\vec{f}_\nu^\dagger(\mathcal{H}_{\text{eff}} - \mu\tau_1)\vec{f}_\nu + \int\vec{g}_\nu^\dagger(\mathcal{H}_{\text{eff}} - \mu\tau_1)\vec{g}_\nu + e^2\int\frac{(\zeta_{f_\nu} + \zeta_{g_\nu}^*)'(\zeta_{f_\nu} + \zeta_{g_\nu}^*)^*}{|\vec{r}-\vec{r}\,'|}\,. \tag{4.15}$$

The third term of (4.15) is clearly positive, and by using the expansion theorems (3.58) and (3.59) for $\vec{f}_\nu$ and $\vec{g}_\nu$, respectively, it follows that these terms are positive as well. Since $E_\nu > 0$, we see that two classes of solutions emerge. In the first $\int\bar{\vec{f}}_\nu\vec{f}_\nu = (\vec{f}_\nu, \vec{f}_\nu) > 0$ and $|(\vec{f}_\nu, \vec{f}_\nu)| > |(\vec{g}_\nu, \vec{g}_\nu)|$. These are the appropriate continuations into the supercritical region of the positively normed π^- solutions. For the other class of solutions $(\vec{g}_\nu, \vec{g}_\nu) < 0$ and $|(\vec{g}_\nu, \vec{g}_\nu)| > |(\vec{f}_\nu, \vec{f}_\nu)|$, so that the left-hand side of (4.15) remains positive. In this case $-E_\nu$ corresponds to the solution of the Klein-Gordon equation and we are dealing with a continuation of the π^+ solutions. These assertions become intuitively clear if we allow the explicit e^2 which occurs in (4.13) to approach zero, in which case the two equations would decouple to the limits indicated.

We next notice a symmetry property of the system (4.13). We interchange $\vec{f}_\nu$ with $\vec{g}_\nu$, take the Hermitian conjugate, and replace $E_\nu \to -E_\nu$. This changes (4.13a) into (4.13b) and conversely. The result may be usefully reformulated as follows. We first replace (4.13b) by its transpose so that $\vec{g}_\nu^\dagger\tau_1 \to \tilde{\tau}_1\vec{g}_\nu^* = \tau_1\vec{g}_\nu^*$, etc. A solution to (4.13) is then represented by the double vector

$$F_\nu(r) = \begin{pmatrix} \vec{f}_\nu(r) \\ \vec{g}_\nu^*(r) \end{pmatrix}\,. \tag{4.16}$$

In this new two-component space, the Pauli matrices will be designated as σ_i. Then the symmetry property noted above is that to every solution F_ν with energy $E_\nu > 0$ there corresponds another solution G_ν with equal but opposite E_ν, where

$$G_\nu = \sigma_1 F_\nu^* . \tag{4.17}$$

With this theorem we have accounted for all solutions of the doubled system (4.13). The physical solutions are those with positive energy $E_\nu > 0$. The unphysical solutions are, however, needed for completeness in the doubled space. According to (4.13), the orthonormalization theorem for the F_ν should be

$$F_\nu^\dagger \tau_1 \sigma_3 F_{\nu'} = \delta_{\nu\nu'} . \tag{4.18}$$

The orthogonality follows from (4.13) itself; the normalization follows from the orthogonality plus completeness relations which will be given below. In practice the G_ν are distinguished by the sign of the energy and by the sign of the norm, since from (4.17) and (4.18) follows

$$G_\nu^\dagger \tau_1 \sigma_3 G_{\nu'} = -\delta_{\nu\nu'} . \tag{4.19}$$

The quantization procedure can now be sumarized by the expansion theorems

$$\chi(r) = \sum_\nu [\alpha_\nu \vec{f}_\nu(r) + \alpha_\nu^\dagger \vec{g}_\nu(r)] , \tag{4.20a}$$

$$\chi^\dagger(r) = (\chi(r))^\dagger , \tag{4.20b}$$

$$|\nu\rangle = \alpha_\nu^\dagger |\text{vac}\rangle, \quad \alpha_\nu |\text{vac}\rangle = 0 . \tag{4.21}$$

Insertion of (4.20) into the commutators (4.11) yields the completeness relations

$$\sum_\nu [\vec{f}_\nu(\vec{r}) \vec{f}_\nu^\dagger(\vec{r}\,') - \vec{g}_\nu(\vec{r}) \vec{g}_\nu^\dagger(\vec{r}\,')] = \tau_1 \delta(\vec{r} - \vec{r}\,') , \tag{4.22}$$

$$\sum_\nu [\vec{f}_\nu(\vec{r}) \vec{g}_\nu(\vec{r}\,') - \vec{g}_\nu(\vec{r}) \vec{f}_\nu(\vec{r}\,')] = 0 . \tag{4.23}$$

These, together with their Hermitian conjugates, can be rewritten succinctly in terms of the vectors (4.16) and (4.17) as

$$\sum_\nu [F_\nu(\vec{r}) F_\nu^\dagger(\vec{r}\,') - G_\nu(\vec{r}) G_\nu^\dagger(\vec{r}\,')] = \delta(\vec{r} - \vec{r}\,') \tau_1 \sigma_3 . \tag{4.24}$$

Together with the orthogonality implied by (4.13), the relations lead back to (4.18) and (4.19) and to the further orthogonality relation

$$F_\nu^\dagger \tau_1 \sigma_3 G_{\nu'} = 0 . \tag{4.25}$$

It is finally straightforward to verify that the expansions (4.20) fulfill their aim of diagonalizing H_2'. We obtain

$$H_2' = \sum_\nu E_\nu \alpha_\nu^\dagger \alpha_\nu - \sum_\nu E_\nu (\vec{\bar{g}}_\nu, \vec{g}_\nu) , \tag{4.26}$$

the last term representing a (positive) zero-point energy.

The remaining terms H_3' and H_4', respectively cubic and quartic in the new quasiparticle operators, may be treated by perturbative or other standard methods.

We cannot refrain from carrying the formal development just a little further. It is mainly to observe that the completeness relations (4.23) can be reduced to the form (2.11) appropriate to the KG equation by considering, as after Eq. (4.15), two sets of solutions. For the first we write

$$\vec{f}_\nu = \cosh\theta_{\nu+} \vec{\chi}_{\nu+} , \tag{4.27a}$$

$$\vec{g}_\nu = \sinh\theta_{\nu+} \vec{\chi}_{\nu+} , \tag{4.27b}$$

$$\int \vec{\bar{\chi}}_{\nu+} \vec{\chi}_{\nu+} = 1 \tag{4.28}$$

and for the second

$$\vec{f}_\nu = \sinh\theta_{\nu-} \vec{\chi}_{\nu-} , \tag{4.29a}$$

$$\vec{g}_\nu = \cosh\theta_{\nu-} \vec{\chi}_{\nu-} , \tag{4.29b}$$

$$\int \vec{\bar{\chi}}_{\nu-} \vec{\chi}_{\nu-} = -1 . \tag{4.30}$$

For bound states we choose $\vec{\chi}_\nu$ real. For this case, we state the equations determining this function $\vec{\chi}_\nu$ and the angle θ_ν, obtained by insertion of (4.27) into (4.13) and straightforward algebraic combination, utilizing also (4.28):

$$2 \sinh\theta_\nu \cosh\theta_\nu E_\nu = e^2 \int \zeta_{\chi_\nu}(\vec{r}) \frac{1}{|\vec{r} - \vec{r}\,'|} \zeta_{\chi_\nu}(\vec{r}\,') , \tag{4.31}$$

$$E_\nu \tau_1 \vec{\chi}_\nu(\vec{r}) = \left[\int \mathcal{K}(\vec{r} \,|\, \vec{r}\,')(\cosh^2\theta_\nu + \sinh^2\theta_\nu) + \int \mathcal{L}(\vec{r} \,|\, \vec{r}\,') \, 2 \sinh\theta_\nu \cosh\theta_\nu \right] \vec{\chi}_\nu(\vec{r}\,') , \tag{4.32}$$

where

$$\mathcal{K}(\vec{r} \,|\, \vec{r}\,') = (\mathcal{H}_{\text{eff}} - \mu\tau_1) \delta(\vec{r} - \vec{r}\,') + \mathcal{L}(\vec{r} \,|\, \vec{r}\,') , \tag{4.33}$$

$$\mathcal{L}(\vec{r} \,|\, \vec{r}\,') = \frac{e^2}{|\vec{r} - \vec{r}\,'|} [\tau_1 \vec{\phi}(\vec{r})][\vec{\phi}^\dagger(\vec{r}\,') \tau_1] . \tag{4.34}$$

Equations (4.31) and (4.32) have to be solved by self-consistent methods.

For the set of solutions (4.29) and (4.30), a set analogous to (4.31) and (4.32) can be obtained differing from the latter in form by the replacement $E_\nu \to -E_\nu$.

With the aid of the considerations flowing from (4.27) forward, we can rewrite the expansion (4.20) so as to suggest continuity with the subcritical expansion given in (3.6). For this purpose a change in notation is *de rigeur:*

$$\begin{aligned} \vec{\chi}_{\nu+} &\to \vec{\chi}_\pi , \\ \vec{\chi}_{\nu-} &\to \vec{\chi}_\eta , \\ \alpha_{\nu+} &\to \alpha_\pi , \\ \alpha_{\nu-} &\to \beta_\eta . \end{aligned} \tag{4.35}$$

In place of (4.20) we therefore write

$$\phi_{op}=\vec{\phi}+\sum_{\pi}(\alpha_{\pi}\cosh\theta_{\pi}+\alpha_{\pi}^{\dagger}\sinh\theta_{\pi})\vec{\chi}_{\pi} + \sum_{\eta}(\beta_{\eta}\sinh\theta_{\eta}+\beta_{\eta}^{\dagger}\cosh\theta_{\eta})\vec{\chi}_{\eta}\,. \qquad (4.36)$$

As $(Z-Z_c)\to 0$ the condensate wave function $\vec{\phi}$ and all the phase angles $\theta\to 0$, so that quite formally the two expansions (3.6) and (4.36) look similar in this limit.

In point of fact, however, neither exists mathematically in this limit. The physical counterpart of this is that in the neighborhood of $Z-Z_c=0$ the excitations of unit charge, or more accurately those that can be reached by applying the field operator once to the vacuum, cannot be described simply either as particles or quasiparticles. The description of the eigenstates in terms of either limiting description is more complicated. In the language of thermodynamics this corresponds to an increase in the size of the fluctuations in the neighborhood of a phase transition.

The same point can be emphasized by consideration of Fig. 2. In Fig. 1, the parts of the solid curves marked E_- and E_+ not *too near* $Z=Z_c$ may be considered eigenvalues of the field theory. In Fig. 2 we indicate schematically the continuation of these eigenvalues for increasing Z according to the nonlinear field theory of this paper. The parts of these curves corresponding to $(Z-Z_c)\gg 1$ would emerge from the solution of (4.31) and (4.32) [or (4.13)]. However, for $Z\sim Z_c$ we would have to mix either multiparticle states with the single-particle states valid for $Z<Z_c$ or multi-quasiparticle states with the single-quasiparticle states valid for $Z>Z_c$ in order to reproduce the curves of Fig. 2 in the transition region.

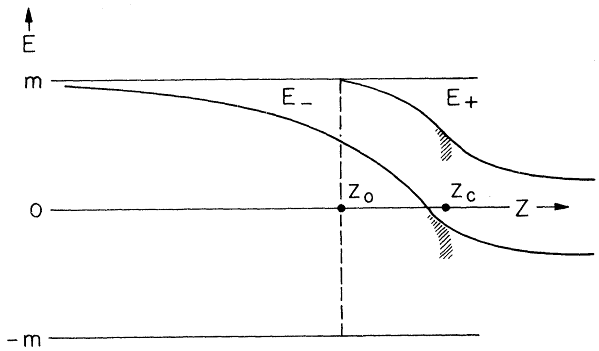

FIG. 2. Schematic representation of those eigenvalues of the field theory, continued into the range $Z>Z_c$, which correspond for $Z<Z_c$ to single π^- and π^+ mesons bound to the nucleus.

V. POSSIBLE APPLICATION AND EXTENSION OF RESULTS.

To provide some balance to the very formal considerations of the preceding portions of this paper, we now attempt to relate our discussion to the structure of nuclei, known and unknown.

The first point which must be emphasized is that the results developed so far are applicable to a situation where the nucleus of charge $Z\sim 10^3$ is held together by a *deus ex machina*. *Under no circumstances* is the condensation energy sufficient to stabilize the nucleus. This is easily seen as follows: The sum of the condensation energy and of the Coulomb energy of the nucleus can, in the semiclassical approximation in which we are working, be written in the form

$$W_{\text{cond}}+W_{\text{Coul}}=\int[(\nabla\psi)^2+m^2\psi^2]+\tfrac{1}{2}\int\frac{\rho_t(\vec{r})\rho_t(\vec{r}')}{|\vec{r}-\vec{r}'|}\,, \qquad (5.1)$$

where ψ is the KG amplitude corresponding to the two-component $\vec{\phi}$, and ρ_t is the sum of the charge densities of the protons and of the pion condensate. As previously noted, the latter arranges itself so as to cancel as much as possible the proton charge inside the nucleus. This must be balanced for overall charge neutrality of the condensate by a positive charge density outside the nucleus. Depending on the actual range of this charge, it is *conceivable* (though we have no reliable numerical results) that the overall Coulomb repulsion, the last term in (5.1), is sufficiently reduced so as to restore stability against fission if the nucleus is otherwise stable.

In this connection, of course, one immediately thinks of the obvious advantages of a negatively charged condensate. We have, however, proved in the last section that *small* charge fluctuations about a neutral condensate increase the total energy of the system. This does not rule against the interest of studying condensates carrying charges of order Z, especially if one simultaneously considers pion-nucleon interactions (see below).

In any event, and this is our elementary point, the best we can do without strong interactions is to reduce the last term of (5.1) to impotence. *We are left with the rest and kinetic energies of the condensate.* These can be compensated, if at all, by the pion-nucleon interaction. It has been claimed[7] that ordinary nuclei contain a neutral pion condensate. This is an extremely interesting suggestion, even if it is moot.[8] If this turns out to be the case, a number of questions come to the fore: (i) Why has this condensate not been observed as yet? (ii) How would we look for it? (iii) What are the implications for the ideas discussed in the body of this paper?

In order to speculate on these questions, let us consider briefly a highly schematic model[9] which predicts such a condensation phenomenon. We describe the condensate by a real (KG) wave function $\psi(r)$ and postulate that the energy functional which determines $\psi(r)$ has the form

$$W[\psi]=\int\{[\nabla\psi(r)]^2[1-Cf(r)]+m^2\psi^2(r)\}+\tfrac{1}{2}D\left[\int\psi^2(r)\right]^2, \tag{5.2}$$

which contains two new constants C and D, both positive, and a function $f(r)$, which we shall tentatively identify with the nucleon matter density. Actually the term proportional to D is not sufficient to stabilize the system, because there is no cutoff on the p-wave interaction. This can be done if we make the replacement

$$\sqrt{C}\,\nabla\psi(r)\to[C(\nabla^2)]^{1/2}\nabla\psi(r), \tag{5.3}$$

where $C(\nabla^2)$ will cut off short-wavelength oscillations of $\psi(r)$. For the time being we neglect the operator character of C.

If we then vary (5.2), we obtain the equation

$$-[1-Cf(r)]\nabla^2\psi(r)+\kappa^2\psi(r)=0, \tag{5.4}$$

$$\kappa^2=m^2+D\int\psi^2(r), \tag{5.5}$$

where for purposes of qualitative discussion we have also dropped a term proportional to the gradient of the function $f(r)$. If we choose $f(r)=$ constant inside the nucleus and zero outside and assume that

$$Cf(0)-1>0, \tag{5.6}$$

then a solution of (5.4) corresponds to finding an s-wave bound state of zero energy for a particle subject to an attractive square-well potential inside the nucleus and a barrier outside. Since both of these can be *adjusted* by adjusting the integral $\int\psi^2(r)$, there are formally an infinite number of solutions for $\psi(r)$. These can be classified by wave-numbers k_m, where it can be shown by standard methods that the values of k_n

$$k_n=\kappa_n/[Cf(0)-1] \tag{5.7}$$

form a monotonically increasing sequence. We shall understand that the main effect of the proper incorporation of the requirement (5.3) is to select the lowest value of k_n. This can still correspond to a solution in which $\psi(r)$ oscillates a number of times inside the nucleus.

Such a condensate, if its amplitude were also small, might be difficult to detect. Such an amplitude must, however, induce corresponding oscillations in the nuclear density which could be detected at sufficiently large momentum-transfer electron and nuclear elastic scattering. It has been asserted that such effects have been seen in electron scattering.[9] Effects on nuclear spectra should also be examined.

For heavier nuclei, we can no longer ignore the Coulomb interaction of the pion cloud with the proton charge distribution. This will result in charge separation in the condensate proper, ultimately to the extreme separation described in the body of of the paper. One may then anticipate additional observable effects in electron scattering and muonic x rays. Firm conclusions (from the model) would, however, require a self-consistent treatment of the nuclear and electromagnetic effects.

It thus is clearly worthwhile to try to understand from first principles if nuclei are condensed in the sense of the present discussion. It may be worthwhile quite independently to investigate phenomenologically the experimental consequences of such a condensate. Finally, one may try to understand if super-superheavy nuclei of the type envisaged in this paper could indeed exist.

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*Work supported in part by the U. S. Atomic Energy Commission.

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[9]This model has so far no known basis in fact, but is introduced only to illustrate the kind of effects to be looked for when a condensate appears. A preliminary account of a possibly more realistic attempt to describe a neutral condensate came to our attention just as this paper was finished: A. B. Migdal, N. A. Kirichenko, and G. A. Sorokin, Phys. Lett. 50B, 411 (1974).