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Reggeon calculus for the production amplitude. I

Jochen Bartels*
Fermi National Accelerator Laboratory, Batavia, Illinois 60510†
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We derive and discuss a representation of $2 \to n$ production amplitude in the multi-Regge limit which exhibits signature factors and singularity structure allowed by the Steinmann relations. We demonstrate how each term of the representation can be understood in terms of the more familiar $2 \to 2$ amplitudes, and prove that the representation can be cast into a factorizing form.

I. INTRODUCTION

Within the last years, substantial progress has been made in understanding the nature of Regge cuts in $2 \to 2$ processes. Almost ten years ago, Gribov, Pomeranchuk, and Ter-Martirosyan[1] showed that multiparticle intermediate states in the t-channel partial-wave unitarity relations generate branch points in the angular momentum plane. More recently, White[2] rederived and extended these results on a more rigorous basis. Rather independent of this approach, Gribov derived a Reggeon diagram technique[3] by considering the high-energy behavior of a certain class of Feynman diagrams within the simple ϕ^3 model. The structure of the rules, however, turned out to be of much more general validity, and they satisfy completely the t-channel unitarity relations. They have the character of a three-dimensional nonrelativistic field theory (one time, two space dimensions), where the coupling between the fields as well as the form of the bare propagator are not specified. Very recently,[4,5] the application of renormalization-group techniques to this Reggeon field theory has led to very interesting results which, in particular, strongly emphasize the importance of cuts in addition to Regge poles.

In comparison with this, the situation in inelastic processes to which Regge ideas have been applied (particle production, inclusive processes) is less satisfactory. Migdal, Polyakov, and Ter-Martirosyan[4] extended the idea of Reggeon field theory to such inelastic processes, but their rules do not take care of signature or singularity structure, which turned out to be rather crucial in at least some of these processes, and therefore their results must be correct only as an approximation. A derivation of a Reggeon calculus which can compare with Gribov's work on the $2 \to 2$ process as well as an investigation of crossed-channel unitarity contributions are still outstanding. On the other hand, it is now generally believed that Regge cuts must play an important role in inelastic processes: The assumption that pure Regge-pole exchange dominates the high-energy behavior leads to serious theoretical inconsistencies by requiring the decoupling of the Pomeron from a large number of processes,[6] most likely even from elastic $2 \to 2$ scattering.[7] However, these decoupling arguments would become invalid if cut contributions would turn out to be equally important as poles. This, in fact, is strongly suggested by the results of Reggeon field theory in $2 \to 2$ processes.

This makes it very desirable to extend Gribov's Reggeon calculus to these inelastic processes. The experience with $2 \to 2$ processes suggests two ways which might be promising. The one is a set of discontinuity formulas, derived from partial-wave unitarity relations in the crossed channels, the other is a Reggeon diagram technique which solves these discontinuity formulas. The success of Gribov's work suggests deriving such a diagram technique again from a careful study of hybrid Feynman diagrams. This is what we are doing in this and the following paper, restricting ourselves to the multi-Regge limit of production amplitudes. The triple-Regge limit and other inelastic processes will, hopefully, be the object of future investigations.

Multi-Regge behavior in $2 \to n$ production amplitudes with pure pole exchange has been suggested already some time ago.[8] Attempts to include Regge cuts and to extend Gribov's Reggeon calculus to this kind of process have been started by several authors,[9-12] but they remained incomplete and have not yet reached the level where underlying general rules become visible. The reason for this failure is essentially the presence of signature factors. To illustrate this in more detail, we remember that when the $2 \to 2$ amplitude is written as

$$T(s, t) = \frac{1}{2\pi i} \int dj\, s^j \xi_j f_j(t)\,, \tag{1.1}$$

$$\xi_j = \frac{e^{-i\pi j} + \tau}{\sin \pi j} \tag{1.2}$$

the partial wave $f_j(t)$ is a real analytic function. This is the reason why in Gribov's Reggeon diagram technique signature factors of internal lines can always be extracted in such a way that the amplitude is written as signature factor times real analytic function. In Reggeon diagrams of production amplitudes, however, it has not yet been possible to write the amplitude as a phase factor times a real function, and no general scheme has been found for combining the signature factors of internal lines in any suitable way.

Knowing that it was the signature which caused the trouble in applying Gribov's technique of hybrid Feynman diagrams to production amplitudes, it is natural to go back to the simplest multi-Regge amplitudes with pure pole exchange (Fig. 1) and examine the signature structure. Studying the diagram of Fig. 1 in the multi-Regge limit,

$$s = s_{abc} \to \infty, \quad s_{ab} \to \infty, \quad s_{bc} \to \infty,$$
$$\eta = \frac{s_{ab}s_{bc}}{s_{abc}}, \quad t_1, t_2 \text{ fixed}, \tag{1.3}$$

Drummond *et al.*[13] have pointed out that the factorizing form

$$T_{2\to 3} = g(t_1) s_{ab}{}^{\alpha_1} \xi_{\alpha_1} f_{\alpha_1\alpha_2}(t_1, t_2, \eta) s_{bc}{}^{\alpha_2} \xi_{\alpha_2} g(t_2),$$
$$\xi_\alpha = \frac{e^{-i\pi\alpha} + \tau}{\sin\pi\alpha} \tag{1.4}$$

is not appropriate, because the Reggeon-particle-Reggeon coupling function $f_{\alpha_1\alpha_2}$ still contains phase factors and has a nontrivial singularity structure in η as well.[14] Instead of (1.4), a more suitable representation has been found:

$$T_{2\to 3} = g(t_1) g(t_2) s^{\alpha_1} s_{bc}{}^{\alpha_2 - \alpha_1} \xi_{\alpha_1} \xi_{\alpha_2\alpha_1} V_L + s^{\alpha_2} s_{ab}{}^{\alpha_1 - \alpha_2} \xi_{\alpha_2} \xi_{\alpha_1\alpha_2} V_R,$$
$$\xi_{\alpha_2\alpha_1} = \frac{e^{-i\pi(\alpha_2 - \alpha_1)} + \tau_1\tau_2}{\sin\pi(\alpha_2 - \alpha_1)} \tag{1.5}$$

where V_L, V_R are now real functions of t_1, t_2, η, and are supposed to have no singularities in η around $\eta = 0$.

A decomposition similar to (1.5) has also been suggested[5,6] for the $2 \to 4$ amplitude, and it has been shown that (1.5) as well as the analogous representation for the $2 \to 4$ amplitude are identical to the factorizing forms (1.4) and a similar one for the $2 \to 4$ case, respectively. The representation (1.5), however, has the advantage over (1.4) by exhibiting explicitly all phase factors; the remaining coefficient functions V_R and V_L are real. In view of the difficulties with signature factors mentioned above, this representation seems to be a strong candidate for a form in which Regge cuts might easily be included.

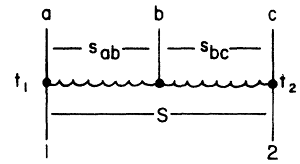

FIG. 1. Double-Regge limit of the $2 \to 3$ amplitude with Regge-pole exchange.

Our subsequent investigation will show that this is, indeed, correct. More precisely, we shall find that the representation (1.5), written as a double Mellin transform;

$$T_{2\to 3} = \left(\frac{-1}{4i}\right)^2 \int dj_1 dj_2 [s^{j_1} s_{bc}{}^{j_2 - j_1} \xi_{j_1} \xi_{j_2 j_1} F_L(j_1 j_2 t_1 t_2 \eta) + s^{j_2} s_{ab}{}^{j_1 - j_2} \xi_{j_2} \xi_{j_1 j_2} F_R(j_1 j_2 t_1 t_2 \eta)], \tag{1.6}$$

remains valid for all Reggeon diagrams (one example in Fig. 2), the functions F_L, F_R being real analytic. (1.6) is thus the generalization of (1.1). Moreover, the way in which a given Reggeon diagram contributes to F_L and F_R, is a straightforward extension of Gribov's Reggeon diagram technique for the $2 \to 2$ amplitude: The underlying structure is, again, a nonrelativistic field theory with one time and two space dimensions, and what is new is the vertex where the produced particle couples to Reggeons. This result is also generalized to more than 3 outgoing particles: We shall describe the generalization of (1.6), and the way in which Regge cuts contribute to the (always real analytic) coefficient functions is a simple generalization of the $2 \to 3$ rules. In a special case, when all Reggeons are Pomerons, and only the region $t_i \sim 0$, $j_i \sim 1$ is considered, our diagram technique reproduces the field-theoretical rules of Migdal, Polyakov, and Ter-Martirosyan.[4] But our rules are also applicable to other Reggeons than the Pomeron and are of the same general validity as Gribov's calculus for the $2 \to 2$ amplitude.

Comparing (1.6) with (1.1), we are obviously confronted with new features which are not present in the familiar $2 \to 2$ scattering amplitude: two terms instead of one, and a new signature factor. We therefore feel that it is necessary to discuss this representation in a little more detail. In case

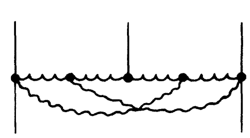

FIG. 2. A cut contribution to the $2 \to 3$ amplitude.

of the $2 \to 3$ amplitude, we will mainly review the arguments of Drummond *et al.*[13] and Weis[16] which lead to (1.5), but in order to derive the analogous representation for the general $2 \to n$ case we need some more insight. We also will show that our representation is compatible with the factorizing form [like (1.4) for the $2 \to 3$ case].

Thus our study naturally breaks into two parts. In the first part (this paper) we are concerned with a discussion of the representation (1.5) and its analog for the $2 \to n$ amplitude. Section II contains a review of the $2 \to 3$ amplitude which enables us to find in Sec. III the general scheme of this decomposition. All this will be based on multi-Regge amplitudes with pure pole exchange. In the following paper we then turn to an extensive study of hybrid Feynman diagrams and will find (a) that the representation found in part I remains valid in the presence of cuts and (b) that the underlying structure of how the cut contributions are computed is that of a Reggeon field theory.

II. SIGNATURE AND SINGULARITY STRUCTURE IN THE 2→3 AMPLITUDE

We said already in the Introduction that the correct treatment of signature plays an important role in the study of the production amplitude. We therefore start with a reconsideration of signature and singularity structure in the $2 \to 3$ amplitude. This will mainly be a review of arguments given by Drummond *et al.*[13] and Weis.[15,16] In the next section we shall extend this consideration to the $2 \to n$ amplitude and find the representation which will be of importance later on. For a general review on Regge behavior in inelastic processes we refer to the article of Brower, DeTar, and Weis,[17] which also contains a complete list of references.

Let us now start with the $2 \to 3$ process of Fig. 1 in the double-Regge limit:

$$s,\, s_{ab},\, s_{bc} \to \infty\,, \qquad \eta = \frac{s_{ab}s_{bc}}{s}\,,\quad t_1,\, t_2 \text{ fixed}\,. \tag{2.1}$$

Following the argument of Drummond *et al.*[13] we write the amplitude as a sum of four terms with only right-hand or left-hand cuts in the energy variables (Fig. 3). The first part has only right-hand cuts in s, s_{ab}, s_{bc}, and in the double-Regge limit behaves as

$$g(t_2)\,\frac{(-s_{ab})^{\alpha_1}}{\sin\pi\alpha_1}\,\Gamma_{\alpha_1\alpha_2}(t_1, t_2, \eta)\,\frac{(-s_{bc})^{\alpha_2}}{\sin\pi\alpha_2}\,g(t_2)\,. \tag{2.2}$$

Here $g(t)$ denotes the Reggeon-particle-particle vertex function. The function $\Gamma_{\alpha_1\alpha_2}$ is real analytic, and as a function of η has the form[13]

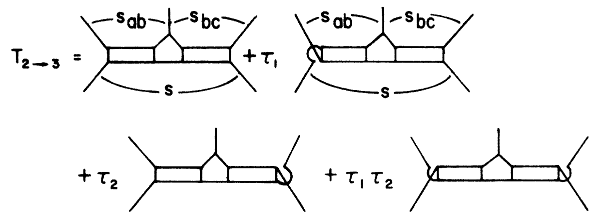

FIG. 3. Signature decomposition of the $2 \to 3$ amplitude.

$$\Gamma_{\alpha_1\alpha_2} = (-\eta)^{-\alpha_1}\tilde{V}_L(t_1, t_2, \eta) + (-\eta)^{-\alpha_2}\tilde{V}_R(t_1, t_2, \eta)\,, \tag{2.3}$$

where $\tilde{V}_L$ and $\tilde{V}_R$ have no further singularities around $\eta = 0$ (the meaning of the subscripts will become clear later). Now (2.2) can be rewritten as

$$\frac{g(t_1)g(t_2)}{\sin\pi\alpha_1\sin\pi\alpha_2}\,[\,(-s)^{\alpha_1}(-s_{bc})^{\alpha_2-\alpha_1}\tilde{V}_L + (-s)^{\alpha_2}(-s_{ab})^{\alpha_1-\alpha_2}\tilde{V}_R]\,. \tag{2.4}$$

At this point it is necessary to remember an important analyticity property of multiparticle amplitudes which is closely related to the Steinmann relations.[18] In the physical region the amplitude is not allowed to have simultaneous discontinuities in energy variables of overlapping channels [two channels are defined to be overlapping when they have particles in common but are not subchannels of each other. For our $2 \to 3$ amplitude (ab) and (bc) are overlapping channels, while (ab) is a subchannel of (abc)]. Since Regge behavior in a given channel can be thought of as resulting from the superposition of energy thresholds in this channel, one concludes that Regge behavior, such as given in (2.2), is not in agreement with this required analyticity property. In other words, $\Gamma_{\alpha_1\alpha_2}$ must necessarily contain η factors if the amplitude is to behave like $s^{\text{power}} \times s_{ab}^{\text{power}}$ or $s^{\text{power}} \times s_{bc}^{\text{power}}$ rather than $s_{ab}^{\text{power}} \times s_{bc}^{\text{power}}$. This explains (2.3) and (2.4): The first term has simultaneous singularities in s and s_{bc}, the second one has them in s and s_{ab}. Graphically, their content of discontinuities is shown in Fig. 4. The form (2.3), which is necessary to satisfy the analyticity requirements, has

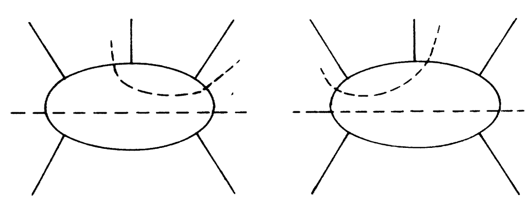

FIG. 4. The two allowed sets of simultaneous energy discontinuities.

been derived from the dual model[15] as well as from hybrid Feynman diagrams, the ladder model, and the Van Hove model.[14] A partial-wave analysis based on S-matrix principles leads to the same form.[19,20]

We now turn to the question of phase factors in Fig. 3. The first term has only right-hand cuts in all three energy variables, and in the physical region one has to approach these cuts from above. Starting with, say, s_{ab} on the negative real axis, which is free from singularities, we move to the positive real axis:

$$s_{ab}+i\epsilon=(-s_{ab})e^{-i\pi}, \quad s_{ab}>0\ . \tag{2.5}$$

Similarly,

$$s_{bc}+i\epsilon=(-s_{bc})e^{-i\pi}, \quad s_{bc}>0 \tag{2.6}$$

$$s+i\epsilon=(-s)e^{-i\pi}, \quad s>0 \tag{2.7}$$

and (2.4) becomes

$$\frac{g(t_1)g(t_2)}{\sin\pi\alpha_1\sin\pi\alpha_2}\left[s^{\alpha_1}s_{bc}{}^{\alpha_2-\alpha_1}e^{-i\pi\alpha_1}e^{-i\pi(\alpha_2-\alpha_1)}\tilde{V}_L + s^{\alpha_2}s_{ab}{}^{\alpha_1-\alpha_2}e^{-i\pi\alpha_2}e^{-i\pi(\alpha_1-\alpha_2)}\tilde{V}_R\right]. \tag{2.8}$$

In the second term of Fig. 3, $s_{ab}<0$, $s_{bc}>0$, and $s<0$, and only s_{bc} is multiplied by a phase factor; in the third term $s_{ab}>0$, $s_{bc}<0$, and $s<0$, and so forth. The whole amplitude then becomes

$$\frac{g(t_1)g(t_2)}{\sin\pi\alpha_1\sin\pi\alpha_2}\left[s^{\alpha_1}s_{bc}{}^{\alpha_2-\alpha_1}(e^{-i\pi\alpha_1}+\tau_1)(e^{-i\pi(\alpha_2-\alpha_1)}+\tau_1\tau_2)\tilde{V}_L+s^{\alpha_2}s_{ab}{}^{\alpha_1-\alpha_2}(e^{-i\pi\alpha_2}+\tau_2)(e^{-i\pi(\alpha_1-\alpha_2)}+\tau_1\tau_2)\tilde{V}_R\right]\ . \tag{2.9}$$

We can still put this in a more symmetric form by making use of sine factors contained in $\tilde{V}_R$ and $\tilde{V}_L$:

$$\tilde{V}_L=\frac{\sin\pi\alpha_2}{\sin\pi(\alpha_2-\alpha_1)}V_L\ ,\qquad \tilde{V}_R=\frac{\sin\pi\alpha_1}{\sin\pi(\alpha_1-\alpha_2)}V_R\ . \tag{2.10}$$

Formula (2.9) can then be written as

$$T_{2\to3}=g(t_1)g(t_2)(s^{\alpha_1}s_{bc}{}^{\alpha_2-\alpha_1}\xi_{\alpha_1}\xi_{\alpha_2\alpha_1}V_L + s^{\alpha_2}s_{ab}{}^{\alpha_1-\alpha_2}\xi_{\alpha_2}\xi_{\alpha_1\alpha_2}V_R)\ , \tag{2.11}$$

with

$$\xi_\alpha=\frac{e^{-i\pi\alpha}+\tau}{\sin\pi\alpha}\ ,$$

$$\xi_{\alpha_1\alpha_2}=\frac{e^{-i\pi(\alpha_1-\alpha_2)}+\tau_1\tau_2}{\sin\pi(\alpha_1-\alpha_2)}\ .$$

We still mention that (2.11) is equivalent to the factorized form (1.4). Obviously, one can write (2.9) as

$$g(t_1)g(t_2)s_{ab}{}^{\alpha_1}s_{bc}{}^{\alpha_2}\xi_{\alpha_1}\xi_{\alpha_2}[\eta^{-\alpha_1}\phi^{\alpha_1}_{\alpha_1\alpha_2}\tilde{V}_L+\eta^{-\alpha_2}\phi^{\alpha_2}_{\alpha_1\alpha_2}\tilde{V}_R]\ , \tag{2.12}$$

with

$$\phi^{\alpha_i}_{\alpha_1\alpha_2}=e^{i\pi\alpha_i}-\frac{\tau_1\tau_2}{(e^{-i\pi\alpha_1}+\tau_1)(e^{-i\pi\alpha_2}+\tau_2)}\times(e^{-i\pi\alpha_i}-e^{+i\pi\alpha_i})\ .$$

The phase factors $e^{\pm i\pi}$ are connected with the $\pm i\epsilon$ prescription,

$$\eta^{-\alpha_i}\phi^{\alpha_i}_{\alpha_1\alpha_2}=(\eta+i\epsilon)^{-\alpha_i}-\frac{\tau_1\tau_2}{(e^{-i\pi\alpha_1}+\tau_1)(e^{-i\pi\alpha_2}+\tau_2)}\times[(\eta+i\epsilon)^{-\alpha_i}-(\eta-i\epsilon)^{-\alpha_i}]\ , \tag{2.13}$$

and (2.12) is the same as

$$T_{2\to3}=g(t_1)g(t_2)s_{ab}{}^{\alpha_1}s_{bc}{}^{\alpha_2}\xi_{\alpha_1}\xi_{\alpha_2}\times\left[\Gamma_{\alpha_1\alpha_2}(\eta+i\epsilon)-\frac{\tau_1\tau_2}{(e^{-i\pi\alpha_1}+\tau_1)(e^{-i\pi\alpha_2}+\tau_2)}\operatorname{disc}_\eta\Gamma_{\alpha_1\alpha_2}\right]. \tag{2.14}$$

Here the term in brackets has to be compared with $f_{\alpha_1\alpha_2}$ in (1.4). Because of the complicated structure of $f_{\alpha_1\alpha_2}$ we consider the form (2.11) to be more suitable than (2.14). It is, however, sometimes useful to remember that (2.11) can be written in the form (2.14). The validity of (2.11) has been proved by White's partial-wave analysis of the five-point function,[20] whereas the form (2.14) appears in dual models,[15] ladder graphs, and hybrid Feynmann diagrams.[14]

Before we turn to the $2\to4$ amplitude and, more generally, to the $2\to n$ amplitude, we want to say few more words about (2.11). The first term can be written as

$$g(t_1)\left(\frac{s}{s_{bc}}\right)^{\alpha_1}\xi_{\alpha_1}[V_L(\eta)s_{bc}{}^{\alpha_2}\xi_{\alpha_2\alpha_1}g(t_2)] \tag{2.15}$$

and has the same form as a $2\to2$ amplitude with the exchange of Reggeon α_1. The bracketed term in (2.15), however, is not simply a Reggeon–two-particle vertex, but a $2\to 2$ amplitude by itself. It describes the scattering process particle 2 + Reggeon $\alpha_1 \to b+c$ (Fig. 5), and $\xi_{\alpha_2\alpha_1}$ is the generalization of the signature factor ξ_{α_2} when one external particle has noninteger spin α_1.[21] As Reggeon α_1 becomes a physical particle—α_1 even (odd) for $\tau_1 = +(-)$—the signature factor ξ_{α_1} develops a pole (particle pole in the t_1 channel), and its residue is

$$\frac{2}{\pi}\, g(t_1) s_{ab}{}^{\alpha_1}[\eta^{-\alpha_1} V_L(\eta) s_{bc}{}^{\alpha_2} \xi_{\alpha_2} g(t_2)] \; . \tag{2.16}$$

In (2.16), ξ_{α_2} is what $\xi_{\alpha_2\alpha_1}$ has reduced to, and the vertex $\eta^{-\alpha_1}V_L(\eta)$ has a simple form, too. For general α_1, α_2, $\tilde{V}_L$ and $\tilde{V}_R$ are of the form (at least around $\eta = 0$)[14]

$$\tilde{V}_L(\alpha_1\alpha_2 t_1 t_2 \eta) = \frac{1}{\Gamma(-\alpha_1)\Gamma(-\alpha_2)} \times \sum_{\kappa=0}^{\infty} \frac{\Gamma(-\alpha_1+\kappa)\Gamma(-\alpha_2+\alpha_1-\kappa)}{\kappa!} \times \eta^{\kappa}\beta(\alpha_1-\kappa;\, t_1, t_2) \,, \tag{2.17}$$

$$\tilde{V}_R(\alpha_1\alpha_2 t_1 t_2 \eta) = \frac{1}{\Gamma(-\alpha_1)\Gamma(-\alpha_2)} \times \sum_{\kappa=0}^{\infty} \frac{\Gamma(-\alpha_2+\kappa)\Gamma(-\alpha_1+\alpha_2-\kappa)}{\kappa!} \times \eta^{\kappa}\beta(\alpha_2-\kappa;\, t_1, t_2) \,,$$

but when α_1 is an integer, V_L reduces to a polynomial in η of degree α_1 [the same happens with V_R when α_2 becomes an integer; this explains our use of the labels L and R: V_L (V_R) reduces to a polynomial when the Reggeon on the left- (right-) hand side of the vertex becomes a physical state]. Thence, $\eta^{-\alpha_1}V_L(\eta)$ is a polynomial in η^{-1} of degree α_1. But since η^{-1} is linearly related to the cosine of the Toller angle ω,

$$\eta^{-1} = \frac{m_b{}^2 - t_1 - t_2 + 2(t_1 t_2)^{1/2}\cos\omega}{\lambda(t_1, t_2, m_b{}^2)} \quad , \tag{2.18}$$

$\eta^{-\alpha_1}V_L(\eta)$ is a polynomial in $\cos\omega$, too. This means that for integer α_1 nonsense helicity states are decoupled, and $\eta^{-\alpha_1}V_L$ is a superposition of physical helicity states of Reggeon α_1. Putting this all together, we have found that, for integer α_1, the bracket term in (2.16) is really a $2\to 2$ scattering amplitude with physical particles.

For the second term in (2.11), the same holds for α_1 and α_2, t_1 and t_2, s_{ab} and s_{bc} interchanged. Finally, we note that (2.11) has no pole when the denominator $\sin\pi(\alpha_1-\alpha_2)$ vanishes in $\xi_{\alpha_1\alpha_2}$ and $\xi_{\alpha_2\alpha_1}$. This is most easily seen when we rewrite (2.17) into the form[17]

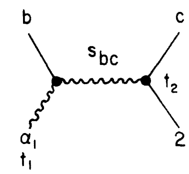

FIG. 5. The Reggeon-particle → particle-particle scattering amplitude described by the bracketed term of (2.15).

$$\eta^{-\alpha_1}\tilde{V}_1 + \eta^{-\alpha_2}\tilde{V}_2 = \frac{1}{\Gamma(-\alpha_1)\Gamma(-\alpha_2)}\,\frac{1}{2\pi i} \times \int dn\, \Gamma(-n)\Gamma(n-\alpha_1) \times \Gamma(n-\alpha_2)\eta^{-n}\beta(n, t_1, t_2) \,, \tag{2.19}$$

where the contour of the n integration lies to the left of the poles of $\Gamma(-n)$ and to the right of those of $\Gamma(n-\alpha_1)\Gamma(n-\alpha_2)$. For $\alpha_1 - \alpha_2$ integer, (2.19) remains finite and, because of (2.3) and (2.14), $T_{2\to 3}$ has no pole either.

This completes our discussion of the $2\to 3$ amplitude. The main result is the representation (2.11), and we have illustrated in some detail its properties: how it reflects the s-channel discontinuity structure, and how each term can be understood in terms of the more familiar $2\to 2$ amplitudes. In the next section, this discussion will help us to construct the analog to (2.11) of the $2\to n$ amplitude.

III. THE $2\to n$ AMPLITUDE

As a preparation, we take the $2\to 4$ amplitude (Fig. 6). η variables are defined by

$$\eta_b = \frac{s_{ab}s_{bc}}{s_{abc}} \;, \quad \eta_c = \frac{s_{bc}s_{cd}}{s_{bcd}} \;, \tag{3.1}$$

and in the multi-Regge limit

$$s,\, s_{ab},\, s_{bc},\, s_{cd} \to \infty \,, \qquad \eta_b, \eta_c, t_1, t_2, t_3 \text{ fixed} \,. \tag{3.2}$$

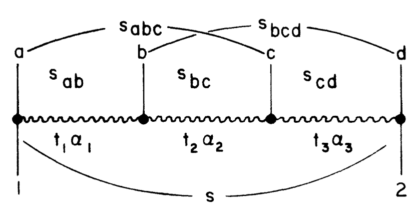

FIG. 6. The $2\to 4$ amplitude with Regge-pole exchange.

we have the further relation

$$\eta_b \eta_c = \frac{s_{ab}s_{bc}s_{cd}}{s} . \tag{3.3}$$

One realizes that not all energy variables are independent. A minimal set of variables would be s_{ab}, s_{bc}, s_{cd}, η_b, η_c, t_1, t_2, t_3, but it is convenient to use the other energy variables, too. By application of the same arguments as we have used for the $2 \to 3$ amplitude, Weis[16] has shown that in the limit (3.2) $T_{2\to 4}$ can be written as

$$\begin{aligned} T_{2\to 4} = g(t_1)g(t_3)[& s^{\alpha_1} s_{bcd}{}^{\alpha_2-\alpha_1} s_{cd}{}^{\alpha_3-\alpha_2} \xi_{\alpha_1}\xi_{\alpha_2\alpha_1}\xi_{\alpha_3\alpha_2} V_L(\eta_b)V_L(\eta_c) + s^{\alpha_2} s_{ab}{}^{\alpha_1-\alpha_2} s_{cd}{}^{\alpha_3-\alpha_2} \xi_{\alpha_2}\xi_{\alpha_1\alpha_2}\xi_{\alpha_3\alpha_2} V_R(\eta_b)V_L(\eta_c) \\ & + s^{\alpha_3} s_{abc}{}^{\alpha_2-\alpha_3} s_{ab}{}^{\alpha_1-\alpha_2} \xi_{\alpha_3}\xi_{\alpha_2\alpha_3}\xi_{\alpha_1\alpha_2} V_R(\eta_b)V_R(\eta_c) + s^{\alpha_3} s_{abc}{}^{\alpha_1-\alpha_3} s_{bc}{}^{\alpha_2-\alpha_1} \xi_{\alpha_3}\xi_{\alpha_1\alpha_3}\xi_{\alpha_2\alpha_1} V_L(\eta_b)V_R(\eta_c) \\ & + s^{\alpha_1} s_{bcd}{}^{\alpha_3-\alpha_1} s_{bc}{}^{\alpha_2-\alpha_3} \xi_{\alpha_1}\xi_{\alpha_3\alpha_1}\xi_{\alpha_2\alpha_3} V_L(\eta_b)V_R(\eta_c)]. \end{aligned} \tag{3.4}$$

It is now quite tempting to apply the arguments of the previous section to this representation. With the understanding that the appearance of $(s_{ik\ldots}^{\text{power}})$ signifies a nonvanishing discontinuity in $s_{ik\ldots}$ one associates with each term in (3.4) a certain set of simultaneous discontinuities (Fig. 7). Apparently, these are all allowed sets of s cuts, because any other set would contain either intersecting cut lines which correspond to discontinuities in overlapping channels or a smaller number of simultaneous discontinuities.

Let us now pick out one term of (3.4), say, the first one. We write it as

$$g(t_1)\left(\frac{s}{s_{bcd}}\right)^{\alpha_1} \xi_{\alpha_1} \left\{ V_L(\eta_b)\left(\frac{s_{bcd}}{s_{cd}}\right)^{\alpha_2} \times \xi_{\alpha_2\alpha_1}[V_L(\eta_c) s_{cd}{}^{\alpha_3} \xi_{\alpha_3\alpha_2} g(t_3)] \right\} . \tag{3.5}$$

Paying no attention to the content of the curly brackets we find that (3.5) has the form of a $2\to 2$ scattering amplitude [Fig. 8(a)]: $1+2 \to a +$ cluster(bcd) with a large rapidity gap between a and cluster (bcd). Turning now to the curly brackets, but still neglecting the content of the square brackets, one again finds the form of a $2\to 2$ scattering amplitude [Fig. 8(b)]: Reggeon α_1 + particle $2 \to b +$ cluster(cd) with a rapidity gap between b and cluster (cd). We note the appearance of $\xi_{\alpha_2\alpha_1}$ instead of ξ_{α_2}: This is due to the incoming Reggeon α_1 with nonintegral spin. When α_1 takes a physical integer value, then $\xi_{\alpha_2\alpha_1} = \xi_{\alpha_2}$ and $\eta_b{}^{-\alpha_1} V_L(\eta_b)$ reduces to a polynomial in helicity of Reggeon α_1. Finally, the same argument applies to the square brackets in (3.5): It is the amplitude for Reggeon α_2 + particle $2 \to c + d$ [Fig. 8(c)].

The way in which we have reduced this term of the $2\to 4$ amplitude to a sequence of $2\to 2$ amplitude can nicely be illustrated in Fig. 7(a). First we considered the amplitude $1+2 \to a +$ cluster(bcd): The lowest cut line belongs to the energy s, which

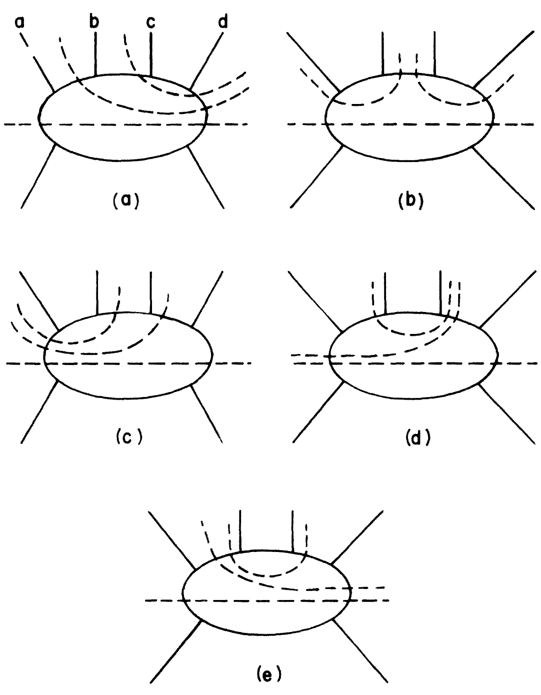

FIG. 7. The five allowed sets of simultaneous energy discontinuities of the $2\to 4$ amplitude.

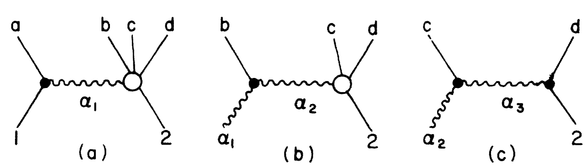

FIG. 8. Reduction of Fig. 7(a) to $2\to 2$ amplitudes: (a) $1+2 \to a$ + cluster (bcd); (b) Reggeon $\alpha_1 + 2 \to b$ + cluster (cd); (c) Reggeon $\alpha_2 + 2 \to c + d$.

has to be large for this process, the next lower line belongs to the way in which the outgoing particles are clustered. In the next step we considered the cluster (bcd) alone and applied to it the same argument again: The lower line denotes the overall cluster energy s_{bcd}, the next line denotes the way in which the outgoing particles have to be clustered. In each of these steps, the power of the large energy is the Reggeon between the outgoing cluster, and the signature factor refers to the exchanged Reggeon as well as the incoming Reggeon. As to the question whether to use V_L or V_R for the Reggeon-particle-Reggeon vertex, it is obvious that it is answered by the requirement of the nonsense helicity decoupling. Comparing this with our Fig. 7(a), we see that for any vertex the label L or R denotes the side of the vertex where one (or more) cut lines leaves the diagram.

The other terms in (3.4) can be treated in a quite similar way. We demonstrate this still for the last term. The analog to (3.5) is

$$g(t_1)\left(\frac{s}{s_{bcd}}\right)^{\alpha_1}\xi_{\alpha_1}\left\{\left[V_L(\eta_b)s_{bc}{}^{\alpha_2}\xi_{\alpha_2\alpha_3}V_R(\eta_c)\right]\times\left(\frac{s_{bcd}}{s_{bc}}\right)^{\alpha_3}\xi_{\alpha_3\alpha_1}g(t_3)\right\}. \quad (3.6)$$

The first step is the same as in the previous case [Fig. 9(a)]. The curly brackets in (3.6) now describe Reggeon α_1 + particle $2 \to$ cluster$(bc)+d$ with a rapidity gap between cluster (bc) and d [Fig. 9(b)]. When α_1 becomes a physical integer, the residue of the pole in ξ_{α_1} becomes

$$\frac{2}{\pi}g(t_1)s_{ab}{}^{\alpha_1}\left\{\eta_b{}^{-\alpha_1}[V_L(\eta_b)s_{bc}{}^{\alpha_2}\xi_{\alpha_2\alpha_3}V_R(\eta_c)]\times\left(\frac{s_{bcd}}{s_{bc}}\right)^{\alpha_3}\xi_{\alpha_3}g(t_3)\right\}. \quad (3.7)$$

As to the decoupling of nonsense helicity states, we have to specify the reference frame the helicity of Reggeon α_1 refers to. One possible frame is the c.m. system of cluster (bc) and Reggeon α_3 with the Toller angle ω_{bc}, the other is the c.m. system of particle b and Reggeon α_2 with the Toller angle ω_b. In the first case, one can show that $\eta_b{}^{-1}$ can be expressed in terms of $\eta_c, \omega_b, \omega_{bc}$,

$$\eta_b{}^{-1}=\eta_c{}^{-1}P(\cos\omega_b, \sin\omega_b, \cos\omega_{bc}, \sin\omega_{bc}), \quad (3.8)$$

where the function P is linear in all variables, and since $\eta_b{}^{-\alpha_1}V_L(\eta_b)$ is a polynomial of degree α_1 in $\eta_b{}^{-1}$, we have polynomial in $\sin\omega_{bc}$ and $\cos\omega_{bc}$ as well, and the nonsense helicity states are absent. Within the second reference frame, the cosine of the Toller angle ω_b is linearly related to $\eta_b{}^{-1}$ by (2.18), and the decoupling of nonsense helicity works, too.

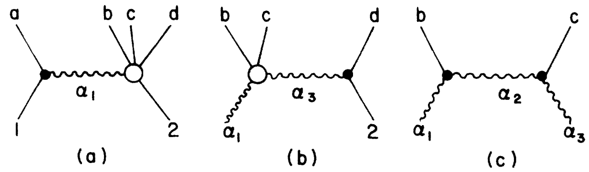

FIG. 9. Reduction of Fig. 7(e): (a) $1+2 \to a$ +cluster (bcd); (b) Reggeon $\alpha_1+2 \to$ cluster $(bc)+d$; (c) Reggeon α_1 + Reggeon $\alpha_3 \to b+c$

In the last step, we consider the term in square brackets in (3.6) which describes the process Reggeon α_1 +Reggeon $\alpha_3 \to b+c$ [Fig. 9(c)]. It is important to note that the signature factor $\xi_{\alpha_2\alpha_3}$ of this amplitude refers only to Reggeon α_2 and to Reggeon α_3 but not Reggeon α_1.[22] When α_1 and α_3 are physical integers, then $\xi_{\alpha_2\alpha_3}=\xi_{\alpha_2}$, and $\eta_b{}^{-\alpha_1}V_L(\eta_b)$ and $\eta_c{}^{-\alpha_3}V_R(\eta_c)$ reduce to polynomials in $\cos\omega_b$ and $\cos\omega_c$, respectively.

It is easy to trace this reduction procedure in Fig. 7(e). What is new in comparison with Fig. 7(a) is the appearance of a Reggeon+Reggeon $\to$ particle+particle amplitude. In writing down its high-energy form, we have to remember that the signature factor refers only to one of the incoming Reggeons: to that Reggeon which in the next large cluster (or in the preceding step of our analysis) has been exchanged between the outgoing particles (or clusters).

This completes our analysis of (3.4). It has taught us that the arguments of the previous section are completely sufficient for the understanding of the more complicated $2\to 4$ amplitude. However, what we have gained in addition is a simple set of heuristic rules which allow us to construct, for a given set of discontinuity lines (Fig. 7), the corresponding term in the decomposition (3.5). We shall formulate these rules in a moment.

So far, we have been concerned only with the two simplest cases $2\to 3$ and $2\to 4$. In the same way as we derived in Sec. II the representation (2.11) [and Weis has derived (3.4)], we could also proceed with $2\to 5$, etc. However, the algebra of phase factors becomes more and more tedious. On the other hand, we have derived a scheme in which both $2\to 3$ and $2\to 4$ fit naturally, and we might expect that it applies to the general case, too. As a proof that this is indeed the case, we show in the Appendix that our representation [(2.11), (3.4), and the analog for $2\to n$ which we will describe in a moment] can be written in the factorized form (Fig. 10)

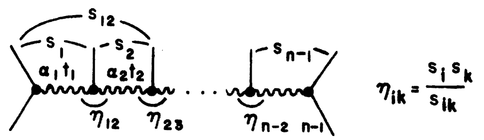

FIG. 10. Multi-Regge limit of the $2 \to n$ amplitude.

$$g(t_1)s_1{}^{\alpha_1}\xi_{\alpha_1} f_{\alpha_1\alpha_2} s_2{}^{\alpha_2}\xi_{\alpha_2} f_{\alpha_2\alpha_3}\cdots \times f_{\alpha_{n-2}\alpha_{n-1}} s_{n-1}{}^{\alpha_{n-1}}\xi_{\alpha_{n-1}} g(t_{n-1}) , \quad (3.9)$$

where $f_{\alpha_1\alpha_2}$ is given by (2.12):

$$f_{\alpha_1\alpha_2} = \eta_{12}{}^{-\alpha_1}\phi^{\alpha_1}_{\alpha_1\alpha_2}\tilde{V}_L(\eta_{12}) + \eta_{12}{}^{-\alpha_2}\phi^{\alpha_2}_{\alpha_1\alpha_2}\tilde{V}_R(\eta_{12}) . \quad (3.10)$$

This form (3.9) emerges in dual models[15] as well as in hybrid Feynman diagrams,[11] and this is, for the moment, all we have on the $2 \to n$ amplitude. Showing the identity of (3.9) and our representation has in addition the other virtue that it demonstrates the absence of poles due to the signature factors $\xi_{\alpha_1\alpha_2}$, etc.; we have shown above that $f_{\alpha_1\alpha_2}$ does not have them, and therefore they are absent in (3.9) as well.

We have now to specify the analog of (2.11) and (3.4) for the general $2 \to n$ case. To do this, we simply summarize the rules which allowed us to construct the $2 \to 3$ and $2 \to 4$ amplitude and, for illustration, apply them to one more complicated diagram. These construction rules will then provide a general definition of our representation.

The rules are the following.

(a) First draw all allowed sets of discontinuity sets. A set is defined to be allowed if (i) lines do not intersect and (ii) no further lines can be drawn without violating (i). (Intersecting lines correspond to discontinuities in overlapping channels.)

For any set [for illustration we choose one of the $2 \to 5$ process (Fig. 11)]:

(b) Consider it as a $2 \to 2$ scattering amplitude $1 + 2 \to \text{cluster}(abc) + \text{cluster}(de)$. The clusters are given by the two cut lines above the overall s cut. For large s this amplitude is dominated by Reggeon exchange between these two clusters:

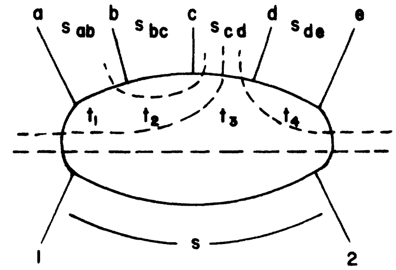

FIG. 11. One allowed set of discontinuities for the $2 \to 5$ amplitude.

$$V_{abc}\left(\frac{s}{s_{abc}s_{de}}\right)^{\alpha_3}\xi_{\alpha_3}V_{de} . \quad (3.11)$$

For any of the clusters, say (abc):

(c) Consider it as a $2 \to 2$ scattering amplitude $1 + \text{Reggeon}\ \alpha_3 \to a + \text{cluster}(bc)$. The clustering of the outgoing particles is again given by the cut lines above the overall cut of this amplitude. For large s_{abc} the Reggeon exchange between the outgoing clusters is given by

$$V_{abc} = V_a\left(\frac{s_{abc}}{s_{bc}}\right)^{\alpha_1}\xi_{\alpha_1\alpha_3}V_{bc} . \quad (3.12)$$

Here $\xi_{\alpha_1\alpha_3}$ indicates that our incoming particle has noninteger spin.

For cluster V_{bc}:

(d) There are only two outgoing particles. If we had more, we would have to repeat step (c) until we end up with only two single outgoing particles. Considering V_{bc} as the amplitude Reggeon α_1 + Reggeon $\alpha_3 \to b + c$, we obtain

$$V_{bc} = V_b s_{bc}{}^{\alpha_2}\xi_{\alpha_2\alpha_1}V_c , \quad (3.13)$$

where the signature factor refers only to α_1, which in the previous step has been the exchanged Reggeon.

(e) Repeating this procedure for the other cluster in (3.11),

$$V_{de} = V_d s_{de}\xi_{\alpha_4\alpha_3}V_e , \quad (3.14)$$

we put all parts (3.11)–(3.14) together and obtain

$$V_a\left(\frac{s_{abc}}{s_{bc}}\right)^{\alpha_1}\xi_{\alpha_1\alpha_3}V_b s_{bc}{}^{\alpha_2}\xi_{\alpha_2\alpha_1}V_c\left(\frac{s}{s_{abc}s_{de}}\right)^{\alpha_3}\xi_{\alpha_3}V_d s_{de}{}^{\alpha_4}\xi_{\alpha_4\alpha_3}V_e = s^{\alpha_3}s_{abc}{}^{\alpha_1-\alpha_3}s_{de}{}^{\alpha_4-\alpha_3}s_{bc}{}^{\alpha_2-\alpha_1} \times \xi_{\alpha_3}\xi_{\alpha_1\alpha_3}\xi_{\alpha_2\alpha_1}\xi_{\alpha_4\alpha_3}V_a V_b V_c V_d V_e . \quad (3.15)$$

(f) Finally, we set $V_a = g(t_1)$, $V_e = g(t_4)$ and label the other vertex functions by L or R, according to whether to the left- or right-hand sides of the produced particle cut lines are leaving the amplitude: $V_b \to V_L(\eta_b, t_1, t_2, \alpha_1\alpha_2)$, etc. Then (3.15) becomes

$$s^{\alpha_3}s_{abc}{}^{\alpha_1-\alpha_3}s_{de}{}^{\alpha_4-\alpha_3}s_{bc}{}^{\alpha_2-\alpha_1}\xi_{\alpha_3}\xi_{\alpha_1\alpha_3}\xi_{\alpha_2\alpha_1}\xi_{\alpha_4\alpha_3} \times g(t_1)V_L(\eta_b)V_R(\eta_c)V_L(\eta_d)g(t_4) . \quad (3.16)$$

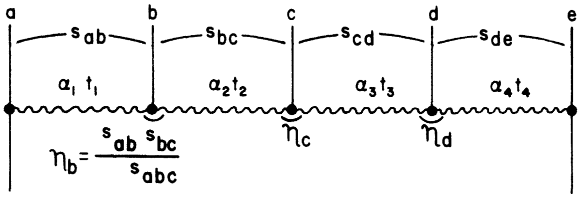

FIG. 12. Notations for the $2\to 5$ amplitude.

IV. SUMMARY

In this paper, we have found a representation of the production amplitude in the multi-Regge limit. It is a sum of terms, each of which represents a certain set of allowed simultaneous discontinuities and allows a simple interpretation in terms of $2\to 2$ amplitudes. Furthermore, the phase factors are extracted, and the remaining functions are real. It is this representation which remains valid when Regge cuts are included. This will be shown in the following part of our study.

ACKNOWLEDGMENT

I am grateful to H. D. I. Abarbanel for very helpful discussions.

APPENDIX

In this appendix we prove that our decomposition of the $2\to n$ amplitude which we have described in Sec. III can always be written in the factorizing form (Fig. 10)

$$g(t_1)\,s_1^{\ \alpha_1}\xi_{\alpha_1}f_{\alpha_1\alpha_2}s_2^{\ \alpha_2}\xi_{\alpha_2}\cdots \times f_{\alpha_{n-2}\alpha_{n-1}}s_{n-1}^{\ \alpha_{n-1}}\xi_{\alpha_{n-1}}g(t_{n-1}) \tag{A1}$$

when $f_{\alpha_1\alpha_2}$ is given in (2.12):

$$f_{\alpha_1\alpha_2}=\eta_{12}^{\ -\alpha_1}\phi^{\alpha_1}_{\alpha_1\alpha_2}\tilde{V}_L(\eta_{12})+\eta_{12}^{\ -\alpha_2}\phi^{\alpha_2}_{\alpha_1\alpha_2}\tilde{V}_R(\eta_{12})\,,$$
$$\phi^{\alpha_{1,2}}_{\alpha_1\alpha_2}=e^{i\pi\alpha_{1,2}}-\frac{\tau_1\tau_2}{(e^{-i\pi\alpha_1}+\tau_1)(e^{-i\pi\alpha_2}+\tau_2)}\times(e^{-i\pi\alpha_{1,2}}-e^{i\pi\alpha_{1,2}})\,. \tag{A2}$$

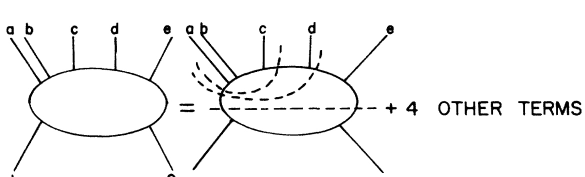

FIG. 13. Decomposition of the $2\to 4$ amplitude in (A9), $1+2\to$ cluster $(ab)+c+d+e$, into the allowed sets of simultaneous singularities.

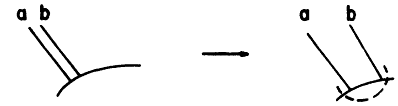

FIG. 14. Replacement to be made in Fig. 13 in order to obtain Fig. 15.

For convenience, we shall write (A2) as

$$f_{\alpha_1\alpha_2}=\eta_{12}^{\ -\alpha_1}\Omega^{\alpha_1}_{\alpha_1\alpha_2}V_L(\eta_{12})+\eta_{12}^{\ -\alpha_2}\Omega^{\alpha_2}_{\alpha_2\alpha_1}V_R(\eta_{12})\,, \tag{A3}$$

with

$$\Omega^{\alpha_1}_{\alpha_1\alpha_2}=\phi^{\alpha_1}_{\alpha_1\alpha_2}\frac{\sin\pi\alpha_2}{\sin\pi(\alpha_2-\alpha_1)}\,,\qquad \Omega^{\alpha_2}_{\alpha_2\alpha_1}=\phi^{\alpha_2}_{\alpha_1\alpha_2}\frac{\sin\pi\alpha_1}{\sin\pi(\alpha_1-\alpha_2)}\,. \tag{A4}$$

By some algebra one can check that

$$\xi_{\alpha_1}\Omega^{\alpha_2}_{\alpha_2\alpha_1}=\xi_{\alpha_1\alpha_2},\quad \xi_{\alpha_2}\Omega^{\alpha_1}_{\alpha_1\alpha_2}=\xi_{\alpha_2\alpha_1}\,, \tag{A5}$$

and

$$\Omega^{\alpha_1}_{\alpha_1\alpha_3}\Omega^{\alpha_3}_{\alpha_3\alpha_2}+\Omega^{\alpha_1}_{\alpha_1\alpha_2}\Omega^{\alpha_3}_{\alpha_3\alpha_1}=\Omega^{\alpha_1}_{\alpha_1\alpha_2}\Omega^{\alpha_3}_{\alpha_3\alpha_2}\,. \tag{A6}$$

For our proof we proceed as follows. In Sec. II we have demonstrated how our decomposition for the $2\to 3$ amplitude can be written in the factorized form. Assuming that we have proven this already for the $2\to(n-1)$ amplitude, we shall then show the validity for the $2\to n$ amplitude. For sake of simplicity, however, we illustrate in the following how one proceeds from the $2\to 4$ to the $2\to 5$ amplitude, and it will then be clear that with the same procedure the general step from $2\to(n-1)$ to $2\to n$ can be performed.

We now turn to the $2\to 5$ amplitude (Fig. 12) and demonstrate that the factorized form

$$g(t_1)s_{ab}^{\ \alpha_1}\xi_{\alpha_1}f_{\alpha_1\alpha_2}s_{bc}^{\ \alpha_2}\xi_{\alpha_2}f_{\alpha_2\alpha_3}\,s_{cd}^{\ \alpha_3}\xi_{\alpha_3}f_{\alpha_3\alpha_4}s_{de}^{\ \alpha_4}\xi_{\alpha_4}g(t_4) \tag{A7}$$

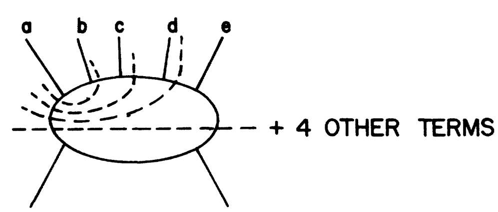

FIG. 15. Five possible sets of discontinuities in the $2\to 5$ amplitude, as obtained from Figs. 13 and 14. They represent the first term of (A8).

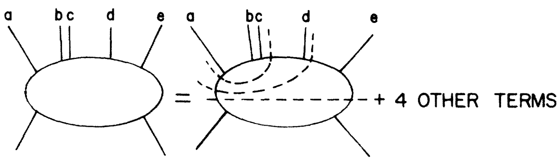

FIG. 16. Decomposition of the $2 \to 4$ amplitude (A12): $1+2 \to a$ + cluster $(bc)+d+e$.

can be cast into our decomposition (which now has already 14 terms). In (A7), we insert for $f_{\alpha_1\alpha_2}$ formula (A3), combine $\xi_{\alpha_1}\Omega^{\alpha_2}_{\alpha_2\alpha_1}=\xi_{\alpha_1\alpha_2}$ with (A4), and set $s_{bc}/\eta_b = s_{abc}/s_{ab}$:

$$[g(t_1)\xi_{\alpha_1\alpha_2} s_{ab}{}^{\alpha_1-\alpha_2} V_R(\eta_b)]\xi_{\alpha_2} s_{abc}{}^{\alpha_2} f_{\alpha_2\alpha_3}\cdots + g(t_1)\xi_{\alpha_1} s_{ab}{}^{\alpha_1} V_L(\eta_b)\eta_b{}^{-\alpha_1}\Omega^{\alpha_1}_{\alpha_1\alpha_2}\xi_{\alpha_2} s_{bc}{}^{\alpha_2} f_{\alpha_2\alpha_3}\cdots . \tag{A8}$$

In the brackets of the first term, we recognize the amplitude 1 + Reggeon $\alpha_2 \to a+b$. Treating it as a vertex particle 1 – Reggeon α_2 – cluster(ab), we write the first term of (A8) as

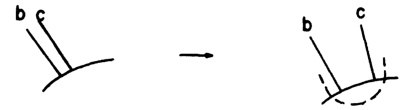

FIG. 17. Replacement in Fig. 16 which leads to Fig. 18.

$$V_{ab}\xi_{\alpha_2} s_{abc}{}^{\alpha_2} f_{\alpha_2\alpha_3}\cdots g(t_4) . \tag{A9}$$

It has precisely the form of a $2 \to 4$ amplitude, and hence, can be decomposed according to our scheme (Fig. 13). To obtain the complete singularity structure, however, we have to take care of the internal structure of V_{ab}. According to what we said in Sec. III, Regge behavior within V_{ab} corresponds to another discontinuity line in our diagrams. As depictured in Fig. 14, we have to insert a new cut wherever we see the cluster (ab). The right-hand side of Fig. 13 is thus transformed to Fig. 15.

In the second term of (A8), we use (A3) for $f_{\alpha_2\alpha_3}$:

$$g(t_1)s_{ab}{}^{\alpha_1}\xi_{\alpha_1}[V_L(\eta_b)\eta_b{}^{-\alpha_1}\Omega^{\alpha_1}_{\alpha_1\alpha_2}]s_{bc}{}^{\alpha_2}\xi_{\alpha_2}[V_L(\eta_c)\eta_c{}^{-\alpha_2}\Omega^{\alpha_2}_{\alpha_2\alpha_3} + V_R(\eta_c)\eta_c{}^{-\alpha_3}\Omega^{\alpha_3}_{\alpha_3\alpha_2}]s_{cd}{}^{\alpha_3}\xi_{\alpha_3} f_{\alpha_3\alpha_4} s_{de}{}^{\alpha_4}\xi_{\alpha_4} g(t_4) . \tag{A10}$$

Here we take the second term in the second square brackets and combine:

$$\eta_b{}^{-\alpha_1}\eta_c{}^{-\alpha_3} s_{ab}{}^{\alpha_1} s_{bc}{}^{\alpha_2} s_{cd}{}^{\alpha_3} = s_{abc}{}^{\alpha_1} s_{bcd}{}^{\alpha_3} s_{bc}{}^{\alpha_2-\alpha_1-\alpha_3}, \quad \Omega^{\alpha_1}_{\alpha_1\alpha_2}\xi_{\alpha_2}\Omega^{\alpha_3}_{\alpha_3\alpha_2} = \Omega^{\alpha_1}_{\alpha_1\alpha_3}\xi_{\alpha_2\alpha_3} + \Omega^{\alpha_3}_{\alpha_3\alpha_1}\xi_{\alpha_2\alpha_1} . \tag{A11}$$

This leads to

$$g(t_1)s_{abc}{}^{\alpha_1}\xi_{\alpha_1}[s_{bc}{}^{-\alpha_1}\Omega^{\alpha_1}_{\alpha_1\alpha_3} V_L(\eta_b) s_{bc}{}^{\alpha_2-\alpha_3}\xi_{\alpha_2\alpha_3} V_R(\eta_c) + s_{bc}{}^{-\alpha_3}\Omega^{\alpha_3}_{\alpha_1\alpha_3} V_L(\eta_b) s_{bc}{}^{\alpha_2-\alpha_1}\xi_{\alpha_2\alpha_1} V_R(\eta_c)] \times s_{bcd}{}^{\alpha_3}\xi_{\alpha_3} f_{\alpha_3\alpha_4} s_{de}{}^{\alpha_4}\xi_{\alpha_4} g(t_4) . \tag{A12}$$

One notices that in (A12) the term in brackets has the same structure as $f_{\alpha_1\alpha_2}$ in (A3) and can be written as

$$F^{bc}_{\alpha_1\alpha_3} = s_{bc}{}^{-\alpha_1}\Omega^{\alpha_1}_{\alpha_1\alpha_3} V^{bc}_L + s_{bc}{}^{-\alpha_3}\Omega^{\alpha_3}_{\alpha_3\alpha_1} V^{bc}_R . \tag{A13}$$

Thus (A12) is like a $2 \to 4$ amplitude and can be decomposed into 5 terms (Fig. 16). But again we have to remember that V^{bc}_L and V^{bc}_R have internal Regge exchange themselves. Wherever we meet the cluster (bc), we make the replacement of Fig. 17, and Fig. 16 is transformed into Fig. 18.

Next we turn to the first term in (A10). For $f_{\alpha_3\alpha_4}$ we use (A3):

$$g(t_1)\xi_{\alpha_1} s_{ab}{}^{\alpha_1}[V_L(\eta_b)\eta_b{}^{-\alpha_1}\Omega^{\alpha_1}_{\alpha_1\alpha_2}]\xi_{\alpha_2} s_{bc}{}^{\alpha_2}[\eta_c{}^{-\alpha_2}\Omega^{\alpha_2}_{\alpha_2\alpha_3} V_L(\eta_c)]\xi_{\alpha_3} s_{cd}{}^{\alpha_3}[\eta_d{}^{-\alpha_3}\Omega^{\alpha_3}_{\alpha_3\alpha_4} V_L(\eta_c) + \eta_d{}^{-\alpha_4}\Omega^{\alpha_4}_{\alpha_4\alpha_3} V_R(\eta_c)] \times \xi_{\alpha_4} s_{dc}{}^{\alpha_4} g(t_4) . \tag{A14}$$

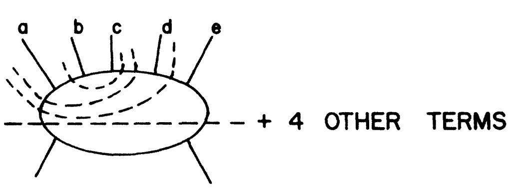

FIG. 18. Five other possible sets of discontinuities in the $2 \to 5$ amplitude, as obtained from Figs. 16 and 17. They represent (A12).

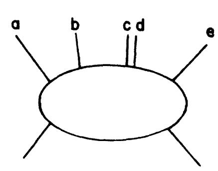

FIG. 19. $2 \to 4$ amplitude $1+2 \to a+b$ + cluster (cd) $+e$ (A15).

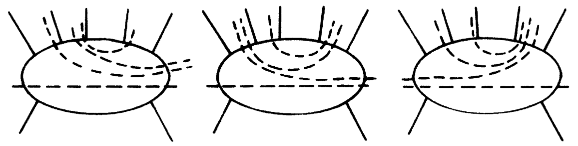

FIG. 20. Three other possible sets of discontinuities in the $2 \to 5$ amplitude, as obtained from Fig. 19 when clustering among particles a and b is omitted. They represent (A15).

For the term with $V_R(\eta_c)$ we repeat the previous analysis and arrive at

$$g(t_1)s_{ab}{}^{\alpha_1}\xi_{\alpha_1}[V_L(\eta_b)\eta_b{}^{-\alpha_1}\Omega^{\alpha_1}_{\alpha_1\alpha_2}] \times s_{bcd}{}^{\alpha_2}\xi_{\alpha_2}F^{cd}_{\alpha_2\alpha_4}s_{cde}{}^{\alpha_4}\xi_{\alpha_4}g(t_4)\,, \quad \text{(A15)}$$

where $F^{cd}_{\alpha_2\alpha_4}$ is the analog of (A13). (A15) is like a $2 \to 4$ amplitude (Fig. 19), except for the vertex of particle b: Instead of $f_{\alpha_1\alpha_2}$ there is only the first term of (A3). Therefore, when we apply our decomposition to (A15) and draw the five diagrams with cut lines, all discontinuity sets involving $V_R(\eta_b)$ have to be left out, and we end up with only three diagrams (Fig. 20).

In the final step we examine the first term in (A14):

$$g(t_1)\xi_{\alpha_1}s_{ab}{}^{\alpha_1}[V_L(\eta_b)\eta_b{}^{-\alpha_1}\Omega^{\alpha_1}_{\alpha_1\alpha_2}] \times \xi_{\alpha_2}s_{bc}{}^{\alpha_2}[V_L(\eta_c)\eta_c{}^{-\alpha_2}\Omega^{\alpha_2}_{\alpha_2\alpha_3}]\xi_{\alpha_3}s_{cd}{}^{\alpha_3} \times [V_L(\eta_d)\eta_d{}^{-\alpha_3}\Omega^{\alpha_3}_{\alpha_3\alpha_4}]\xi_{\alpha_4}s_{de}{}^{\alpha_4}g(t_4)\,, \quad \text{(A16)}$$

which by use of (A5) can be written

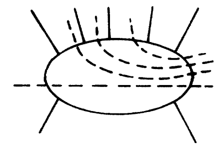

FIG. 21. The last set of discontinuities from (A17).

$$g(t_1)s^{\alpha_1}s_{bcde}{}^{\alpha_2-\alpha_1}s_{cde}{}^{\alpha_3-\alpha_2}s_{de}{}^{\alpha_4-\alpha_3}\xi_{\alpha_1}\xi_{\alpha_2\alpha_1}\xi_{\alpha_3\alpha_2}\xi_{\alpha_4\alpha_3} \times g(t_1)V_L(\eta_b)V_L(\eta_c)V_L(\eta_d)g(t_4)\,, \quad \text{(A17)}$$

which obviously is Fig. 21 and completes the set of allowed discontinuity sets (Figs. 15, 18, 20, and 21).

It is not difficult to see how our argument is generalized to more than 3 produced particles. Proceeding from the left to the right end of the diagram, we first decompose $f_{\alpha_1\alpha_2}$ and treat the V_R term. Being left with the V_L, we decompose $f_{\alpha_2\alpha_3}$ and treat its V_R. Repeating this for each vertex, we are finally left with only V_L's from all vertices and we handle this like (A16). Translating this into our diagrams with sets of discontinuity lines, we create in the first step all configurations where particles a and b are combined into a cluster, then all diagrams with cluster (bc) and so forth. In the final step, particles c and d are clustered, but no clustering of a and b appears (otherwise we would have double counting, for clustering of a and b has been covered in the first step). Obviously, in this way all possible configurations are created without any double counting. This completes our proof.

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[21] I am indebted to H. D. I. Abarbanel for a helpful discussion on this point.
[22] When making the analytic continuation of a $2 \to 2$ amplitude in the spins of both incoming particles with helicities λ_1 and λ_2, one has to distinguish between the two cases $|\lambda_1| \geq |\lambda_2|$ and $|\lambda_1| < |\lambda_2|$. Each of them has to be treated separately. Having made the analytic continuation, setting $\lambda_1 = \alpha_1$, $\lambda_2 = \alpha_2$, and taking the high-energy limit of the amplitude, one finds the signature factor $\xi_{\alpha\alpha_i}$ where α is the exchanged Reggeon and $\alpha_i = \alpha_1$ or α_2 according to whether the continuation has been made from $\lambda_1 \geq \lambda_2$ or $\lambda_1 < \lambda_2$.