Internal symmetry and the semiclassical method in quantum field theory*

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We apply the methods of Dashen, Hasslacher, and Neveu to quantum field theories with continuous global symmetries. With a U(1) symmetry we show that it is possible to project out a subspace of fixed charge, and to reformulate the theory as one with internal symmetry, but with centrifugal terms arising from rotation in the internal-symmetry space; in the weak-coupling regime, static solutions of this equivalent problem determine the energies of the bound states. Within a particular model in one spatial dimension we demonstrate the existence of such bound states and examine the dependence of their energies upon the charge. The extension of the method to non-Abelian groups is illustrated with SU(2) examples.

I. INTRODUCTION

In a recent series of papers[1,2] Dashen, Hasslacher, and Neveu (DHN) have suggested a method for obtaining bound-state energies in quantum field theory. Working within the framework of the path-integral formulation of quantum mechanics, they show that the stationary-phase approximation to the path integral is the proper generalization of the one-dimensional WKB approximation. In this paper we discuss the application of these methods to theories with global continuous internal symmetries.

The DHN method can take two forms. If all classical periodic solutions of the field equations are known, then the bound-state energies can be obtained by imposing WKB quantization conditions on the action of the classical motion. Alternatively if the coupling is weak and a stable time-independent classical solution is known, there will be a corresponding quantum bound state, as well as nearby states corresponding to excitation of the normal modes of oscillation about the static solution. Their energies will be given by the classical energy of the static solution, plus quantum corrections that can be systematically evaluated. In a somewhat different language, the weak-coupling case has also been discussed by Goldstone and Jackiw.[3]

When there are many coupled fields with some internal symmetry, certain interesting questions arise. The symmetry clearly introduces "symmetry" coordinates[4] which are changed by the symmetry transformation, while the energy of the system is not. When the internal symmetry is global, the number of such coordinates is less than or (more typically) equal to the number of group generators.

Since the potential energy is independent of the transformation, one would expect quantum fluctuations in the symmetry coordinates to be very large. Consequently, one might wonder if "semiclassical" WKB methods could be meaningful in such a context. To put it in other words, one might wonder if the "zero-frequency modes" corresponding to these symmetry coordinates could give rise to infrared problems in the WKB method.

In fact these methods, when used with care, should not present any such problems. Certainly it would be a mistake, in the presence of continuous internal symmetry, even for weak coupling, to perturb about a classical time-independent solution. For every such solution, there will be nearly ones of the same energy, obtained by applying infinitesimal rotations in internal-symmetry space. Consequently, there will be zero-frequency modes which give rise to vanishing denominators in the course of the perturbation calculation. (These are similar to the zero-frequency modes due to translation symmetry which were encountered in Refs. 2 and 3.) One must therefore necessarily include periodic time-dependent solutions which move in internal space, even for weak coupling. (Along the zero-frequency modes, even a small coupling constant is, by comparison to $\omega=0$, infinitely strong.) For strong coupling, the WKB method in any case requires use of all periodic classical solutions, which will include all periodic motions in internal space as well.

This, and many features of the problem, can be illustrated by using a very simple example. Consider the path-integral-WKB treatment of a single nonrelativistic particle on a plane moving under a central potential $V(r)$ which has a minimum at some $r=r_0$. Although a time-independent classical solution exists with $r(t)=r_0$ and $\theta=$ any constant, it is incorrect even for weak anharmonicity to perturb about such a solution. Nontrivial periodic orbits, both circular and noncircular, must be employed. The WKB quantization conditions then pick orbits whose energy and angular momentum approximately equal those of the quantum levels.

Alternatively, and more exactly, one can project out an angular momentum sector and integrate out the θ variables in the path integral, leaving behind a radial problem with an effective potential containing the well-known centrifugal term. WKB methods can then be employed for the radial problem. In particular, for weak coupling, it *is* correct to perturb about the classical stationary solution $r(t)=R$, where R is a minimum of the effective radial potential, since there is no continuous symmetry left in the radial problem. Further, this stationary solution to the effective radial problem and its classical energy correspond in the original r, θ variables, to a circular orbit of appropriate angular momentum.

One might hope that corresponding things happen in field theories with global internal symmetry. In Secs. II and III of this paper we explicitly show this by using a U(1)-symmetric field theory as an example. The "symmetry" coordinate is easily identified here, and is in fact cyclic. It is shown that one can project out a definite charge sector (where the "charge" is of course the internal quantum number) of the propagator of the theory, and write a path integral for it. Then the symmetry coordinate is exactly integrated in the path integral, leaving behind the remaining coordinates and an effective action in these. There are no infrared modes due to internal symmetry in these remaining variables since they are all left invariant by the group.[5] Thus, for weak coupling one can find bound-state energies by perturbing about a *time-independent* solution to the equations of motion of the remaining variables, as obtained from the effective action. We do this, and find that just as in the case of the "kink" solution of Ref. 2, the quantum energy is dominated by the classical energy of this stationary solution for weak coupling. If there is more than one time-independent solution for weak coupling, they correspond to different families of bound states, not accessible to one another in perturbation theory.

Further analogies with the angular momentum problem are also explicitly shown. The stationary solution for the remaining variables is seen to correspond, in terms of the original variables, to time-dependent solutions which rotate globally *in internal space*. The classical energy of this globally rotating solution provides in weak coupling the leading contribution to the bound-state energies. The classical charge of this rotating configuration is also approximately the quantum charge of the sector, with quantum corrections which we evaluate exactly. For large charge the quantum corrections to the classical charge are negligible, in accordance with the correspondence principle.

It is also seen that the equations of motion of the nonsymmetry coordinates contain an added "centrifugal" term dependent on the charge. That rotations in "internal space" can also give rise to centrifugal forces need not cause surprise. After all, such rotations produce nonzero time derivatives of the multiplet fields, and contribute to the kinetic term in the energy, just as ordinary rotations do in nonrelativistic physics. The analogs of "circular orbits" are periodic solutions which rotate globally in internal space, but have no other time dependence. In weak coupling, states of higher values of internal spins come from perturbing about periodic field solutions which rotate globally in internal space with increasing "angular velocity."

In Secs. II and III we illustrate these ideas by using a concrete example of a field theory with U(1) internal symmetry. We consider a two-dimensional Lagrangian with one real scalar field σ and one complex scalar field ϕ:

$$\begin{aligned} L(\sigma,\phi) &= \tfrac{1}{2}(\partial_\mu\sigma)^2 + \tfrac{1}{2}\mu^2\sigma^2 - \tfrac{1}{4}\lambda\sigma^4 \\ &\quad + \tfrac{1}{2}(\partial_\mu\phi^*)(\partial_\mu\phi) - \tfrac{1}{2}m^2|\phi|^2 - \tfrac{1}{4}h(|\phi|^2)^2 \\ &\quad - \tfrac{1}{2}a|\phi|^2(\sigma^2-\mu^2/\lambda)\,. \end{aligned} \tag{1.1}$$

The Lagrangian is clearly U(1)-symmetric with respect to the complex field ϕ. Since the charge operator for the ϕ field should be well defined, we want the vacuum to be symmetric with respect to U(1). Hence, finite-energy configurations of the ϕ field must vanish as $x\to\pm\infty$. However, it is hard to confine a single static scalar multiplet by itself; hence the extra real field σ, which keeps the ϕ field confined. Note the sign of the σ mass term, which leads to a vacuum that breaks the discrete $\sigma\to-\sigma$ symmetry. The resulting σ "kink" solution traps the ϕ field. This is similar to the physically interesting case where quark multiplets are trapped by scalar fields, as in popular confinement models.[2,6] We use a scalar multiplet ϕ instead of quarks, just for simplicity.

Having established in Sec. II the connection between bound states and globally rotating classical solutions, we discuss in Sec. III the existence and qualitative nature of some classical solutions of the system in (1.1). If the complex field ϕ is written in polar form

$$\phi(x,t)=\rho(x,t)e^{i\theta(x,t)}\,, \tag{1.2}$$

then it is shown that for a range of couplings a truly time-independent solution exists of the form

$$\begin{aligned} &\sigma(x,t)=\sigma_0(x)\,, \\ &\rho(x,t)=\rho_0(x), \quad \theta(x,t)=\text{const.} \end{aligned} \tag{1.3}$$

The shapes of σ_0 and ρ_0 are shown qualitatively in

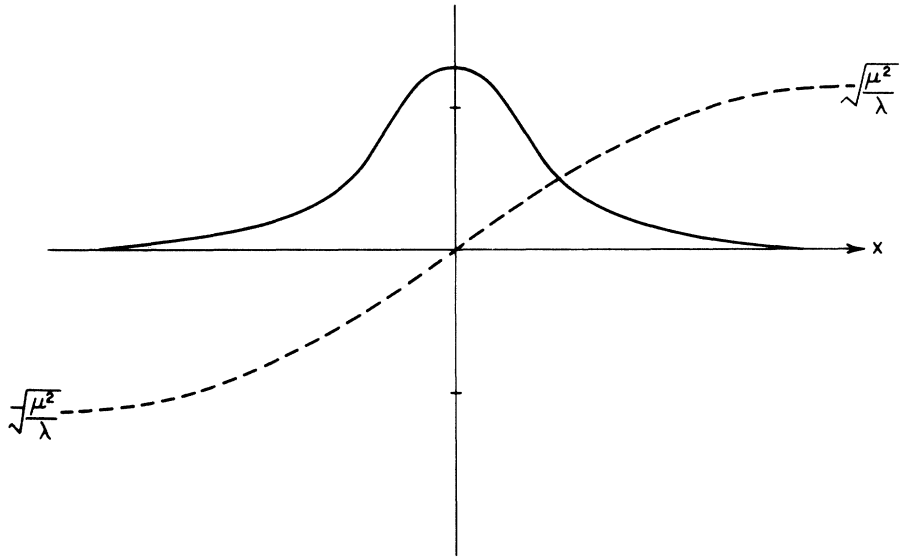

FIG. 1. Qualitative features of a static solution to field equations of the system in (1.1). The dashed line stands for the σ field and the heavy line for the modulus of the complex ϕ field. The phase $\theta(x)$ of the complex field is zero.

Fig. 1. Further, given such a solution, others which rotate globally in the internal U(1) space, viz.,

$$\phi(x,t)=e^{i\omega t}\rho_\omega(x)$$

and (1.4)

$$\sigma(x,t)=\sigma_\omega(x),$$

also are easily obtained for a range of ω. As ω increases the "centrifugal" forces tend to widen the ϕ configuration until, for sufficiently large ω (of the order of the masses), the ϕ field is no longer confined.

This implies in weak coupling [$\lambda, h, a \ll \mu^2, m^2$ in (1.1)] that hadrons become unbound when the internal quantum number is increased too much—a satisfying result in that hadronic particles in nature have low values of internal quantum numbers.

We also find that the bound-state energies vary very weakly with the charge in weak coupling. The charge-dependent part of the energy is of order $(\lambda/\mu^2)^2$ as compared to the charge-independent part. This is similar to what happens with translation symmetry in weak coupling.[2,3] The momentum-dependent part of the energy is two orders of coupling higher than the rest mass of the bound state.

These properties are shown explicitly only for the U(1) group in Secs. II and III. There is every reason to believe that the general approach carries over to higher non-Abelian groups and to adjoint as well as nonadjoint representations. But explicitly identifying and integrating over the symmetry coordinates can be much more difficult. This is partly because the symmetry modes are curvilinear, i.e., they depend upon the field configuration and partly because, for non-Abelian groups, the symmetry coordinates are not all cyclic. Nevertheless, based on the explicit experience with U(1) and on general notions of the correspondence principle it is possible to give weak-coupling approximations to bound-state energies even in the presence of non-Abelian internal-symmetry groups. A discussion of this is given in Sec. IV.

II. U(1) SYMMETRY

Before treating the U(1)-symmetric field theory described in Eq. (1.1), let us consider the corresponding analysis for the prototype U(1) problem, viz., a particle on a plane under the influence of a central potential. Analogy with this simple system was mentioned in the Introduction. It will be useful to sketch its path integral with WKB analysis, in order to point out some crucial features whose analogs exist in the field-theoretic example as well.

Consider a particle on a plane with a Lagrangian

$$L=\tfrac{1}{2}(\dot{x}^2+\dot{y}^2)-V(r), \qquad (2.1)$$

where we will use Cartesian (x, y) or polar (r, θ) coordinates as convenience dictates.

The Lagrangian in (2.1) is clearly U(1)-symmetric, with the angular momentum l as the "charge" of that symmetry. The spectrum of this system for any given l can be obtained from the poles of

$$G_l(E)=\mathrm{Tr}_l\left[\frac{1}{H-E}\right]$$

$$=\frac{1}{2\pi}\int_0^{2\pi}d\alpha\, e^{-il\alpha}\,\mathrm{Tr}\left[\frac{e^{i\hat{L}\alpha}}{H-E}\right] \tag{2.2}$$

$$=\frac{i}{2\pi}\int^{2\pi}d\alpha\, e^{-il\alpha}\int^{\infty}dT\, e^{iET}\int dx\,dy\langle x,y|e^{-iHT+i\hat{L}\alpha}|x,y\rangle \tag{2.3}$$

$$=\frac{i}{2\pi}\int d\alpha\, e^{-il\alpha}\int dT\, e^{iET}\int r_0 dr_0 d\theta_0\langle r_0,\theta_0|e^{-iHT+i\hat{L}\alpha}|r_0,\theta_0\rangle\,, \tag{2.4}$$

where it has been assumed that the angular momentum operator $\hat{L}$ has only integral eigenvalues. Noting that $\hat{L}$ is the generator of rotations, we have

$$\langle r_0,\theta_0|e^{-iHT+i\hat{L}\alpha}|r_0,\theta_0\rangle=\langle r_0,\theta_0+\alpha|e^{-iHT}|r_0,\theta_0\rangle\,. \tag{2.5}$$

This is just the amplitude to go from one point to a rotated point in time T, and a path integral can be written for it. It is tempting to write the path integral in polar coordinates, using naively the polar form of the Lagrangian in the infinitesimal action,

$$\Delta S=\mathfrak{L}\Delta t=\left[\tfrac{1}{2}\dot{r}^2+\tfrac{1}{2}r^2\dot{\theta}^2-V(r)\right]\Delta t\,. \tag{2.6}$$

That would be incorrect, and amounts to assuming $\Delta\theta\sim O(\Delta t)$, which is not the correct order of infinitesimals in a path integral. This has been pointed out by Edwards and Gulyaev,[7] who also describe the correct way to transform to polar coordinates, starting from the known Cartesian form of the path integral. The transition amplitude in (2.5) is given by the path integral

$$\int D[x,y]\exp\left[i\int_0^T\left(\frac{\dot{x}^2}{2}+\frac{\dot{y}^2}{2}-V(r)\right)dt\right], \tag{2.7}$$

where all paths begin at (r_0,θ_0) and end at $(r_0,\theta_0+\alpha)$ in time T. Dividing the time T into a lattice $t_0=0$, $t_1,\ldots,t_{n-1},t_n=T$, with

$$t_i-t_{i-1}\equiv\epsilon,\quad x(t_i)\equiv x_i,\quad y(t_i)\equiv y_i\,,$$

the path integral (2.7) is

$$\lim_{n\to\infty}\frac{1}{\Delta^n}\int\prod_{i=1}^{n-1}dx_i\,dy_i\exp\left\{i\sum_i\left[\frac{(x_i-x_{i-1})^2}{2\epsilon}+\frac{(y_i-y_{i-1})^2}{2\epsilon}-V(x_i\,y_i)\epsilon\right]\right\}\,, \tag{2.8}$$

where $\Delta=(2\pi i\epsilon)$. Now, going to polar coordinates at every time index i,

$$dx_i\,dy_i=r_i\,dr_i\,d\theta_i\,,$$

and

$$\frac{1}{2\epsilon}[(x_i-x_{i-1})^2+(y_i-y_{i-1})^2]=\frac{1}{2\epsilon}[r_i^2+r_{i-1}^2-2r_ir_{i-1}\cos(\theta_i-\theta_{i-1})]\,. \tag{2.9}$$

If $\theta_i-\theta_{i-1}\equiv\Delta\theta_i$ were of order ϵ, one could expand the cosine, retain the first two terms, and get a kinetic term $(1/2\epsilon)[(r_i-r_{i-1})^2+r_ir_{i-1}(\Delta\theta_i)^2]$, which is what one would have obtained if the infinitesimal action in the path integral had been $L\epsilon$, with L given by its polar form in (2.6). But $\Delta\theta$ is *not* of order ϵ in a path integral, where, just as in Brownian motion, $(\Delta\theta)^2$ is of order ϵ, so that higher terms in the cosine expansion must be retained. Edwards and Gulyaev point out that the cosine can be retained in its entirety and the θ path integral evaluated exactly.

Inserting (2.9) into (2.8), (2.7), (2.5), and (2.4), $G_l(E)\equiv i\int_0^{\infty}dT\,e^{iET}G_l(T)$, with

$$G_l(T)=\frac{1}{2\pi}\int r_0dr_0d\theta_0\int d\alpha\,e^{-il\alpha}\int\prod_{i=1}^{n-1}\frac{dr_ir_id\theta_i}{\Delta^n}\exp\left\{i\sum_{i=1}^{n}\left[\frac{r_i^2+r_{i-1}^2}{2\epsilon}-V(r_i)\epsilon-\frac{r_ir_{i-1}}{\epsilon}\cos(\theta_i-\theta_{i-1})\right]\right\}\,. \tag{2.10}$$

All the angular integrals are exactly evaluated by using

$$d\alpha=d\theta_n,\ \text{where}\ \theta_n\equiv\theta(T)=\theta_0+\alpha\,,$$

$$l\cdot\alpha=l(\theta_n-\theta_0)=\sum_{i=1}^{n}l(\Delta\theta_i)\,,$$

and

$$\prod_{i=1}^{n-1} d\theta_i \, d\alpha = \prod_{i=1}^{n} d\theta_n = \prod_{i=1}^{n} d(\Delta\theta_i) . \tag{2.11}$$

The angular integrals in (2.10) are

$$\frac{1}{2\pi}\int d\theta_0 \prod_{i=1}^{n} \int d(\Delta\theta_i) \exp\left[- il(\Delta\theta_i) + \frac{r_i r_{i-1}}{i\epsilon}\cos(\Delta\theta_i)\right] = \prod_{i=1}^{n} 2\pi I_l\left(\frac{r_i r_{i-1}}{i}\right) \quad \text{(where } I_l \text{ is a Bessel function[7])}$$

$$\underset{\epsilon\to 0}{\sim} \prod_{i=1}^{n}\left(\frac{2\pi i\epsilon}{r_i r_{i-1}}\right)^{1/2} \exp\left[\frac{r_i r_{i-1}}{i\epsilon} - \frac{(l^2-\frac{1}{4})i}{2r_i r_{i-1}}\right]. \tag{2.12}$$

Inserting this into (2.10) we get

$$G_l(T) = \frac{1}{\Delta^{n/2}}\int \prod_{i}^{n} dr_i \exp\left\{ i\sum_{i=1}^{n}\left[\frac{(r_i - r_{i-1})^2}{2\epsilon} - V(r_i)\epsilon - \frac{(l^2-\frac{1}{4})\epsilon}{2r_i r_{i-1}}\right]\right\}$$

$$= \int dr_0 \int D[r(t)] \exp\left[i\int\left(\frac{\dot r^2}{2} - V(r) - \frac{(l^2-\frac{1}{4})}{2r^2}\right)dt\right]. \tag{2.13}$$

Thus the problem reduces to an effective radial problem with an effective potential

$$U(r) = V(r) + \frac{(l^2-\frac{1}{4})}{2r^2} . \tag{2.14}$$

This is only too well known, and is most easily obtained by writing the Schrödinger differential equation in polar coordinates. However, it is the path-integral method which is easier to generalize to field theory. Note also that had one used the naive polar path integral with the polar form of L in (2.6), or equivalently dropped the quartic and higher powers of $\Delta\theta_i$ in the cosine in (2.9), one would have obtained a very similar result, with l^2 replacing the correct factor $(l^2-\frac{1}{4})$ in the effective potential $U(r)$.

The radial problem can now be treated by the WKB methods of Ref. 1. The stationary-phase points of the integrand in (2.13) are the classical periodic solutions with period T of

$$\ddot r + \frac{dU(r)}{dr} = 0 , \tag{2.15}$$

The WKB quantization condition may then be imposed on these solutions. Alternatively, for weak coupling one can perturb the path integral about the stationary solution

$$r(t) = R ,$$

where R is a minimum of $U(r)$. Expanding the path integral in powers of $x(t) \equiv r(t) - R$, we get

$$G_l(T) = e^{-iU(R)T}\int dx_0 \int D[x] \exp\left\{ i\int\left[\frac{\dot x^2}{2} - \frac{x^2}{2}U''(R) - \frac{x^3}{3!}U'''(R)\cdots\right]dt\right\}$$

$$\equiv e^{-iU(R)T} D(T) . \tag{2.16}$$

If $U'''(R)$ and higher derivatives are small ("weak coupling") and if R is large enough so that the corrections due to the semi-infinite range of r (from 0 to ∞) (see comment No. 6 below) can be ignored, then $D(T)$ is just the trace and path integral of a harmonic oscillator. It is a product of Gaussian integrals and has the well-known result

$$D(T) = \frac{1}{2i\sin\{[U''(R)]^{1/2}T/2\}}$$

$$= \sum_p \exp[-i(p+\tfrac{1}{2})\sqrt{U''}\,T] , \tag{2.17}$$

$$G_l(E) = i\int_0^\infty dT\, e^{iET} G_l(T)$$

$$= \sum_p i\int dT \exp\{i[E - U(R) - (p+\tfrac{1}{2})\sqrt{U''}\,]\,T\}$$

$$= \sum_p \frac{1}{E - U(R) - (p+\tfrac{1}{2})[U''(R)]^{1/2}} . \tag{2.18}$$

Thus, in this approximation, for any given l, the system has bound states at

$$E_{l,p} = U(R) + (p+\tfrac{1}{2})[U''(R)]^{1/2} + O(U''') , \tag{2.19}$$

where the l dependence lies in the potential $U(R)$

and the location of its minimum R.

Now let us make several observations about this well-known result and its simple derivation, which must be borne in mind in analyzing the field-theoretic problem:

1. The energy levels have a contribution $U(R)$ from the classical energy of a stationary solution at the bottom of the effective potential $U(r) = V(r) + \tilde{l}^2/2r^2$, where $\tilde{l}^2 = l^2 - \frac{1}{4}$. In terms of the (r, θ) coordinates, there is clearly a circular orbit at $r = R$ with $\theta(t) = \tilde{l}t/R^2$, *whose energy is also* equal to $U(R)$.

2. Thus, in weak coupling one is essentially perturbing the path integral about a circular orbit of appropriate angular momentum $\tilde{l}$ (which is slightly different from the desired quantum value l).

3. The quantum corrections and the level splitting come from the term $(p + \frac{1}{2})\sqrt{U''}$, which is the result of fluctuations about the circular orbit. These fluctuations are restricted to paths which keep l fixed in the result (2.19).

4. Varying the quantum value of l amounts to varying the classical circular orbit about which the perturbation is done.

5. The neglect of $U'''(R)$ and higher derivatives (weak coupling) does not necessarily follow from the corresponding statement about the original potential $V(r)$, because of the added centrifugal term. Even if $V(r)$ is purely harmonic,

$$\left| \frac{\partial^n U(r)}{\partial r^n} \right|_{r=R;n>2} = \frac{(l^2 - \frac{1}{4})}{2R^{n+2}} (n+1)! \,.$$

This is not small, unless R is very large. We will find that in the field-theory example, the corresponding weak-coupling condition is generally satisfied.

6. The range of the radial variable r extends only from 0 to $+\infty$, and not from $-\infty$ to $+\infty$. Thus, even if U''', etc. in (2.16) are completely neglected so that the path integral for the remaining harmonic oscillator is a product of Gaussian integrals, the limits of integration can be finite, leading to corrections to the simple answer in (2.17) for $D(T)$. If R is very large, it is clear that these corrections are negligible. This problem, also evident in the WKB method starting from the wave equation, leads to the so-called Langer modification.[8] If $x = \ln r$ is used as a variable, then since x ranges from $-\infty$ to $+\infty$, the end-point problem does not exist. In the one-particle problem, this manipulation effectively leads to a modified centrifugal term where $l^2 - \frac{1}{4}$ is replaced by $(l^2 - \frac{1}{4}) + \frac{1}{4} = l^2$.

7. Note, however, that these difficulties exist in the Gaussian path integral after the classical energy $U(R)$ has been factored out [Eq. (2.16)]. Hence, they affect only the quantum correction. It will be seen that in the weak-coupling field theory, the classical energy dominates and is unaffected by this problem.

8. One could attempt to "derive" the quantization of l from the WKB method. This would involve using all periodic orbits, circular and noncircular, on the plane. In our approach, we instead assume the well-known result that l must be an integer, and project out a given l sector by Eq. (2.2). We will similarly assume, in the field-theoretic case, that the internal charges are quantized, since this is a very familiar result independent of the details of dynamics. Our main interest is in the energies of the bound states with some given charge.

Let us now turn to the field theory described by the Lagrangian in (1.1). The complex ϕ field can be written as

$$\phi(x,t) = \rho(x,t) e^{i\theta(x,t)} = \phi_1(x,t) + i\phi_2(x,t) \,. \tag{2.20}$$

The Lagrangian takes the form

$$L = \tfrac{1}{2}(\partial_\mu \sigma)^2 + \tfrac{1}{2}(\partial_\mu \phi_1)^2 + \tfrac{1}{2}(\partial_\mu \phi_2)^2 - V(\sigma, \phi_1 \phi_2) \tag{2.21a}$$

$$= \tfrac{1}{2}(\partial_\mu \sigma)^2 + \tfrac{1}{2}(\partial_\mu \rho)^2 + \tfrac{1}{2}\rho^2 (\partial_\mu \theta)^2 - V(\sigma, \rho)$$

$$\equiv L_0(\sigma, \rho) + \tfrac{1}{2}\rho^2 (\partial_\mu \theta)^2 \,, \tag{2.21b}$$

where $L_0(\sigma, \rho)$ does not depend on the θ field or its derivatives. The Lagrangian is clearly invariant under the U(1) internal-symmetry transformation

$$\sigma(x,t) \to \sigma(x,t) \,,$$

$$\rho(x,t) \to \rho(x,t) \,,$$

and (2.22)

$$\theta(x,t) \to \theta(x,t) + \alpha \,,$$

where α is any global (space-time–independent) constant. The corresponding conserved internal quantum number is the charge given by

$$Q = \frac{i}{2} \int \phi^* \frac{\partial}{\partial t} \phi + \text{Hermitian conjugate.} \tag{2.23}$$

The propagator, for any given eigenvalue q of the charge operator Q, is given by

$$G_q(E) = \frac{i}{2\pi} \int_0^\infty dT\, e^{+iET} \int_0^{2\pi} d\alpha\, e^{-iq\alpha} \,\mathrm{Tr}[e^{-iHT} e^{iQ\alpha}]$$

$$\equiv i \int_0^\infty dT\, e^{iET} G_q(T) \tag{2.24}$$

As is well known, given the canonical commutation relation for the ϕ field, the operator Q gen-

erates precisely the global U(1) transformation in (2.22). Thus, in terms of eigenstates of the field operators σ and ϕ

$$\langle\sigma(x),\rho(x)e^{i\theta(x)}|\,e^{-iHT}e^{iQ\alpha}|\sigma(x),\rho(x)e^{i\theta(x)}\rangle=\langle\sigma(x),\rho(x)e^{i[\theta(x)+\alpha]}|\,e^{-iHT}|\sigma(x)\rho(x)e^{i\theta(x)}\rangle\,. \tag{2.25}$$

Hence

$$G_q(T)=\frac{1}{2\pi}\int d\alpha\, e^{+iq\alpha}\int d\sigma_0(x)\int d\rho_0(x)\rho_0(x)d\theta_0(x)\int D[\tfrac{1}{2}\rho^2]\,D[\sigma]\,D[\theta]\,e^{i\int dt\int L(\sigma,\rho,\theta)dx} \tag{2.26}$$

where in the path integral all paths start from $\rho_0(x),\sigma_0(x),\theta_0(x)$ at $t=0$ and return to $\rho_0(x),\sigma_0(x),\theta_0(x)+\alpha$ after time T.

For the moment, let us naively replace the Lagrangian in the path integral by its polar form (2.21b). As with the one-particle problem, a more careful evaluation starting with the "Cartesian" form (2.21a) leads to a correction which, for the sake of simplicity of presentation, we will discuss at the end.

Working in a box of finite volume (the infinite-volume limit can be taken at the end) the field $\theta(x,t)$ can be decomposed into a Fourier sum,

$$\theta(x,t)=b(t)+\sum_{k_n\neq 0}b_n(t)e^{ik_nx}\equiv b(t)+\tilde{\theta}(x,t)\,, \tag{2.27}$$

where the zero-momentum mode has been given a separate symbol "b", and the remaining space-dependent modes have been summed to $\tilde{\theta}(x,t)$. Using coordinates $\rho(x)$, $\sigma(x)$, $\tilde{\theta}(x)$, and b, it is clear that (i) the coordinate b is cyclic and (ii) the symmetry transformation (2.22) translates the coordinate b by a constant α, while leaving $\rho(x)$, $\sigma(x)$, and $\tilde{\theta}(x)$ unaffected. Further,

$$\begin{aligned}\tfrac{1}{2}\int\rho^2(\partial_\mu\theta)^2dx&=\tfrac{1}{2}\int\rho^2[(\dot{b}+\dot{\tilde{\theta}})^2-(\tilde{\theta}')^2]\,dx\\&=\tfrac{1}{2}\int\rho^2(x,t)[(\dot{\tilde{\theta}}(x,t))^2-(\tilde{\theta}'(x,t))^2]+\tfrac{1}{2}(\dot{b})^2\int\rho^2(x,t)dx+\dot{b}\int\rho^2(x,t)\dot{\tilde{\theta}}(x,t)dx\,,\end{aligned}$$

where dots and primes refer to time and space differentiation, respectively. Hence, the Lagrangian in (2.21) can be written as

$$\int L\,dx=\int L'(\rho,\sigma,\tilde{\theta})dx+\tfrac{1}{2}(\dot{b})^2A(t)+\dot{b}\,B(t)\,, \tag{2.28}$$

where

$$L'=L_0(\rho,\sigma)+\tfrac{1}{2}\rho^2[\dot{\tilde{\theta}}^2-\tilde{\theta}'^2]\,,$$

$$A(t)=\int\rho^2(x,t)dx\,,\ \text{and}\ B(t)=\int\rho^2(x,t)\dot{\tilde{\theta}}(x,t)dx\,, \tag{2.29}$$

Therefore,

$$G_q(T)=\int d[\xi]\exp\left(i\int L'dx\,dt\right)\frac{1}{2\pi}\int d\alpha\,db_0\boldsymbol{D}[b(t)]\exp\left[i\int dt(\tfrac{1}{2}A\dot{b}^2+B\dot{b})\right]e^{iq\alpha}\,, \tag{2.30}$$

where $d[\xi]$ stands for the path integrals *and* the initial-value (trace) integrals for the variables $\rho(x)$, $\tilde{\theta}(x)$, and $\sigma(x)$. That is,

$$d[\xi]\equiv\rho_0(x)\,d\rho_0(x)\,d\tilde{\theta}_0(x)\,d\sigma_0(x)\,\boldsymbol{D}[\tfrac{1}{2}\rho^2]\,D[\sigma]\,D[\tilde{\theta}]\,,$$

and $L'=L'(\sigma,\tilde{\theta},\rho)$. Thus Eq. (2.30) shows the explicit dependence of the integral on the symmetry variable b, which can now be integrated out. The path integral $\boldsymbol{D}[b(t)]$ and the end-point integrals $d\alpha db_0$ can all be evaluated together easily by familiar methods. Divide the time T into $n+1$ lattice points $t_0=0,t_1,t_2,\ldots,t_{n-1}$, $t_n=T$, and let $\epsilon=t_i-t_{i-1}$, $b_i\equiv b(t_i)$, and $b_n=b(T)=b_0+\alpha$.

Then

$$d\alpha db_0\boldsymbol{D}[b(t)]\to(1/\Delta^{n/2})\prod_{i=0}^{n}db_i\,,$$

where Δ is the usual path-integral measure $2\pi i\epsilon$. Further,

$$q\alpha = q(b_n - b_0) = \sum_{i=1}^{n} q(b_i - b_{i-1}) \equiv q\sum_{i=1}^{n} \Delta b_i \,. \tag{2.31}$$

Then

$$\frac{1}{2\pi}\int d\alpha \int db_0 \int Db(t) \exp\left[i\int dt(\tfrac{1}{2}A\dot{b}^2 + B\dot{b}) + iq\alpha \right] = \frac{1}{2\pi}\int \prod_{i=0}^{n} \frac{d(\Delta b_i)}{\Delta^{n/2}} \exp\left\{ i\sum_{i=1}^{n}\left[(\Delta b_i)^2 \frac{A_i}{2\epsilon} + (\Delta b_i)B_i + q(\Delta b_i)\right]\right\}$$

$$= \prod_{i=1}^{n} \frac{\exp[-i(q+B_i)^2\epsilon/2A_i]}{A_i^{1/2}} \text{ by Gaussian integration} \tag{2.32}$$

Inserting this into (2.30) we get

$$G_q(T) = \int d[\xi] \exp\left(i\int L_{\text{eff}}\, dt \right) , \tag{2.33}$$

where

$$L_{\text{eff}} = L_0(\rho,\sigma) + \tfrac{1}{2}\rho^2[(\dot{\tilde{\theta}})^2 - (\nabla\tilde{\theta})^2] - \frac{(q+\int\rho^2\dot{\tilde{\theta}}dx)^2}{2\int\rho^2 dx} \,. \tag{2.34}$$

Equation (2.33) is the analog here of the radial integral (2.13) in the one-particle problem. The role of the single radius variable r there is now played by the infinite collection of variables $\rho(x)$, $\sigma(x)$, and $\tilde{\theta}(x)$, collectively symbolized here by ξ. The last term in L_{eff} in (2.34) is the analog here of the centrifugal potential $(l^2 - \frac{1}{4})/2r^2$ in (2.13).

Thus far, the treatment does not require coupling constants to be small. All that has happened is that the symmetry variable corresponding to the global internal symmetry has been explicitly integrated. In evaluating the remaining highly nontrivial path integrals in (2.33) over the ξ variables, one resorts to WKB approximations. If one's ability to solve the classical field equations corresponding to the Lagrangian L_{eff} were good enough to obtain all periodic solutions, one could use the WKB method even for strong coupling with reasonable hope of a good approximation.

Alternatively, in a weak-coupling case one can perturb the action in (2.33) about a time-independent classical solution, as was done with the "kink" solution for the $\lambda\phi^4$ theory in Ref. 2.

Our method of integrating out the symmetry variable is useful especially for weak coupling because (i) it reduces the problem to one where the remaining variables do not have a continuous internal symmetry; (ii) the potentially troublesome "infrared" modes due to the symmetry have been integrated out,[5] and (iii) since L_{eff} in the remaining variables is a function of q, a correspondence is set up between the WKB energy levels and the value of the internal quantum number for the levels. These ideas can be generalized to more complicated internal groups and representations (see Sec. IV).

Let us proceed with our U(1) example for the weak-coupling case, i.e., $\lambda, a, h \ll \mu^2, m^2$ in (1.1). In that case, one perturbs L_{eff} in (2.33) about a time-independent solution of the corresponding Euler-Lagrange equations.

These equations are obtained by varying L_{eff} with respect to $\rho(x,t)$, $\sigma(x,t)$, and $\tilde{\theta}(x,t)$ and their derivatives [variations with respect to $\tilde{\theta}(x,t)$ implies those with respect to the coefficients b_n in (2.27)]. Upon inserting $L_0(\sigma,\rho)$ from (2.21) and (1.1) into L_{eff} in (2.34), it is easy to see that

$$\tilde{\theta}(x,t) = 0$$

is a solution of the Euler-Lagrange equations corresponding to L_{eff}. The corresponding equations obeyed by σ and ρ are

$$\Box\sigma + \frac{\partial V}{\partial\sigma} = 0$$

and (2.35)

$$\Box\rho + \frac{\partial V}{\partial\rho} - \frac{q^2\rho}{[\int\rho^2(x,t)dx]^2} = 0 \,.$$

It is argued in Sec. III that these equations have a time-independent solution $\{\rho_0(x), \sigma_0(x)\}$ for a range of the couplings $\{\lambda, a, h\}$ and the charge q. It is further argued that in the weak-coupling case, i.e., $\lambda, a, h \ll \mu^2, m^2$ in (1.1), the classical energy $U_q(\rho_0(x), \sigma_0(x))$ of this solution is of order μ^2/λ. This inverse relationship between the energy of the classical stationary solution and the coupling is also found for the "kink" solution of Ref. 2.

Let us for the moment assume this and perturb the action in the path integral for $G_q(T)$ about this

solution $\{\rho_0, \sigma_0, \tilde{\theta} = 0\} \equiv \xi_0$,

$$G_q(T) = e^{-iU_q(\rho_0, \sigma_0)T} \Delta_q(T), \tag{2.36}$$

where

$$\Delta_q(T) = \int d[\xi] \exp\left\{ i \int dT [L_{\text{eff}}(\xi) + U_q(\xi_0)] \right\}. \tag{2.37}$$

The reader is reminded that ξ stands for the collection of variables $\rho(x, t)$, $\sigma(x, t)$, and $\tilde{\theta}(x, t)$ and $d[\xi]$ is the trace and path integration as defined in (2.31). Recall that $\tilde{\theta}(x)$ is constrained to satisfy $\int \tilde{\theta}(x) dx = 0$. Therefore,

$$G_q(E) = i \int dT \exp\left[i\left(E - U_q - \frac{1}{T} \ln \Delta_q(T)\right) T \right]. \tag{2.38}$$

Clearly, the last few equations are analogous to Eqs. (2.16) to (2.18) in the example of one particle on a plane. There in a given angular momentum sector l, $G_l(E)$ had a series of poles corresponding to bound-state energies. The lowest such energy had a classical piece $U(R)$ and a quantum (zero-point energy) correction of $\frac{1}{2}\hbar[U''(R)]^{1/2}$. This quantum correction arose from the factor $D(T)$ in Eq. (2.16) which was the value of the path integral for $G_l(T)$ perturbed about the classical time-independent solution. Similarly, $G_q(E)$ here gives bound-state energies which have a classical piece $U_q(\rho_0, \sigma_0)$ and quantum corrections arising from $\Delta(T)$. Unlike the single-particle problem, however, the field-theoretic example gives a classical energy $U_q(\rho_0, \sigma_0)$ of order μ^2/λ, while the quantum corrections from $\ln\Delta_q(T)$ can be seen to be of order unity. This is similar to the contributions to the energy of the "kink" in Ref. 2. It seems to be a general feature of time-dependent nonvanishing solutions of weak-coupling field theories, where, by rescaling fields, an inverse power of the coupling constant can be factored out of the entire Lagrangian.

Thus, to leading order in μ^2/λ, the energy of the lowest bound state with charge q is the classical value $U_q(\sigma_0, \rho_0)$. Both the quantum correction to this energy and the splitting between excited levels with the same charge are of order unity. Finally, as pointed out in Sec. III, the time-independent solution $\{\sigma_0(x), \rho_0(x), \tilde{\theta}(x) = 0\}$ of the Euler-Lagrange equations (2.35) arising from L_{eff} is related to a time-dependent classical solution of the original Lagrangian in (1.1). This latter solution rotates globally in the U(1) space and has a classical charge q and energy $U_q(\sigma_0, \rho_0)$.

We do not evaluate the quantum correction to the energy in this paper for several reasons. Firstly, while we can obtain the existence and qualitative features of the solution (ρ_0, σ_0) in Sec. III, we do not know its precise analytic form, other than within a variational class. If the classical solution were precisely known, the procedure for evaluating $\Delta_q(T)$ would be similar in principle to what was done in Ref. 2. One perturbs the path integral in $[\xi]$ about the solution ξ_0, and the leading quantum correction is obtained by solving for the eigenfrequencies of the corresponding linearized problem. In practice, because of the presence of polar coordinates, additional difficulties exist here. One is the field-theoretic generalization of the Langer problem mentioned in the one-particle example. The range of the "radial field" $\rho(x)$ extends only from 0 to ∞ at each x. Hence the Gaussian integrals over the normal modes of the linearized problem will have truncated limits, leading to corrections. Secondly, the derivation above used an action $ds = L(\rho, \sigma, \theta)dt$ in the path integral, with L given by the polar form (2.21b). As we saw earlier, this is incorrect even for the one-particle problem. A more careful treatment, briefly outlined below, leads to a much more complicated effective action in the place of $\int L_{\text{eff}} dt$ in the path integral in (2.33). The resulting equations of motion are very similar to (2.35), but the quantum corrections will be very hard to evaluate in practice. These corrections are nevertheless of order unity, and we also expect them to be finite once mass counterterms are introduced as in Ref. 2, since the underlying theory is renormalizable.

Finally let us return, as promised, to the propriety of using the polar form of the Lagrangian in the path integral in Eq. (2.26). We saw in the one-particle problem that replacing $(\dot{x}^2 + \dot{y}^2)$ by $(\dot{r}^2 + r^2\dot{\theta}^2)$ in the path integral is incorrect, but a more careful treatment of infinitesimals merely led to a simple modification, viz., it replaced l^2 by $l^2 - \frac{1}{4}$ in the effective radial potential. A corresponding thing happens in the field-theoretic example as well. The problem concerns the time-derivative part of the Lagrangian which has the form $\frac{1}{2}(\dot{\phi}_1^2 + \dot{\phi}_2^2)$ in terms of the Cartesian fields. In the path integral this would give rise to a kinetic term

$$K \equiv \frac{[\phi_{1,i}(x) - \phi_{1,i-1}(x)]^2}{2\epsilon^2} + \frac{[\phi_{2,i}(x) - \phi_{2,i-1}(x)]^2}{2\epsilon^2}, \tag{2.39}$$

where time is divided into a lattice of spacing ϵ and $\{\phi_{1,i}(x), \phi_{2,i}(x)\}$ and $\{\phi_{1,i-1}(x), \phi_{2,i-1}(x)\}$ describe fields at two neighboring times. Using polar fields and the cosine law,

$$K = \frac{1}{2\epsilon^2}\{\rho_i^2(x)+\rho_{i-1}^2(x) - 2\rho_i(x)\rho_{i-1}(x) \times \cos[\theta_i(x) - \theta_{i-1}(x)]\}. \quad (2.40)$$

If the cosine of $\theta_i(x) - \theta_{i-1}(x) \equiv \Delta\theta_i(x)$ were expanded in powers of $\Delta\theta_i(x)$, and only the first two terms retained, this would be equivalent to the j polar form $\frac{1}{2}(\dot{\rho}^2 + \rho^2\dot{\theta}^2)$ used in Eq. (2.21). This is incorrect, for it assumes that $\Delta\theta(x) \sim \epsilon$, which is not the correct order of infintesimals in the path integral.

It is, however, possible to retain the cosine in its entirety and integrate the cyclinal coordinate b, where $\theta_i(x)$ is written as before as

$$\theta_i(x) = b_i + \tilde{\theta}_i(x).$$

The relevant part of the path integral involving b [in place of (2.32a)] then has the form

$$\int \prod_{i=1}^{n} d(\Delta b_i) \exp\left[i \sum_i \left(q(\Delta b_i) + \frac{1}{\epsilon}\int \rho_i(x)\rho_{i-1}(x)\cos[\Delta b_i + \Delta\tilde{\theta}_i(x)]\,dx \right)\right]. \quad (2.41)$$

After a suitable change of variables, the exponent in (2.41) obviously has the form

$$i \sum_i [q(\Delta b_i) + A_i \cos(\Delta b_i + \delta_i)], \quad (2.42)$$

where A_i and δ_i are functions of space integrals involving $\rho_i(x)$, $\rho_{i-1}(x)$, and $\tilde{\theta}_i(x)$. Thus (2.41) when integrated is again a product of Bessel functions, as was the corresponding (2.12). The Bessel functions can again be expanded and leading terms in ϵ retained. These steps are straightforward but the resulting expression, a path integral in the remaining variables, is too lengthy to be presented here. We will merely mention the following features. The path integral involves an effective action in the exponent, whose stationary-phase paths satisfy equations of motion. A time-independent solution to these exists, with $\tilde{\theta}(x) = 0$, and $\sigma(x)$ and $\rho(x)$ satisfying (2.35), with the sole modification that $q^2 \to (q^2 - \frac{1}{4})$ in the "centrifugal" term of the ρ equation in (2.35). Quantum corrections to the classical energy of this time-independent solution, while more complicated in their evaluation, can still be seen to be of order unity, one order of λ/μ^2 higher than the classical energy. We will discuss the classical solution and its energy in Sec. III using the correct factor $(q^2 - \frac{1}{4})$ instead of q^2 in (2.35).

III. BOUND-STATE SOLUTIONS AND THEIR ENERGIES

As was shown in Ref. 1 and in the previous section, the WKB treatment of quantum field theory leads to the consideration of the corresponding classical field theory. In this section we investigate the classical field theory with the Lagrangian (1.1), with particular emphasis on the case of weak coupling, where the WKB method simplifies considerably. To simplify our equations we rescale as follows:

$$x \to x/\mu, \quad t \to t/\mu,$$
$$\sigma(x) \to \left(\frac{\mu^2}{\lambda}\right)^{1/2}\sigma(x), \quad (3.1)$$
$$\phi(x) \to \left(\frac{a\mu^2}{\lambda h}\right)^{1/2}\phi(x).$$

Also, we define

$$d = \frac{a^2}{\lambda h}, \qquad f = \frac{\lambda m^2}{a\mu^2}. \quad (3.2)$$

If we write the complex field $\phi(x)$ in polar form, as in (2.20), the Lagrangian takes the form

$$L = \mu\left(\frac{\mu^2}{\lambda}\right)\int dx \left\{ \tfrac{1}{2}\dot{\sigma}^2 - \tfrac{1}{2}\sigma'^2 + \tfrac{1}{2}\sigma^2 - \tfrac{1}{4}\sigma^4 + d\left[\frac{1}{2}\,\frac{\lambda}{a}(\dot{\rho}^2 - \rho'^2 + \rho^2\dot{\theta}^2 - \rho^2\theta'^2) - \tfrac{1}{2}f\rho^2 - \tfrac{1}{2}(\sigma^2 - 1)\rho^2 - \tfrac{1}{4}\rho^4\right]\right\}, \quad (3.3a)$$

where dots and primes refer to time and space derivatives, respectively. The corresponding equations of motion are

$$[\Box - 1 + \sigma^2 + d\rho^2]\sigma = 0,$$
$$\left[\frac{\lambda}{a}(\Box - (\partial_\mu\theta)^2) + (f + \sigma^2 - 1) + \rho^2\right]\rho = 0, \quad (3.3b)$$
$$\partial_\mu[\rho^2\partial_\mu\theta] = 0.$$

Since we will be interested only in solutions of the field equations whose energy is finite when measured relative to the ground state, we must first determine the ground state and its energy. We expect the fields in this state to be uniform in both space and time. In this case the Euler-La-

grange equations become

$$0=\sigma(\sigma^2-1+d\rho^2),$$
$$0=\rho(f+\sigma^2-1+\rho^2). \tag{3.4}$$

There are four solutions to these equations:

$$\text{I}\quad \sigma^2=1,\quad \rho=0;$$
$$\text{II}\quad \sigma^2=\frac{1-d(1-f)}{1-d},\quad \rho^2=\frac{f}{d-1}; \tag{3.5}$$
$$\text{III}\quad \sigma=0,\quad \rho^2=1-f;$$
$$\text{IV}\quad \sigma=\rho=0.$$

(The value of θ is arbitrary, as long as it is constant in space and time.) Depending upon the values of f and d, either I, II, or III will be the lowest in energy. The others will have finitely greater energy densities, hence infinitely higher energies. If we wish to have solutions of the general form shown in Fig. 1, solution I must be lowest in energy. The necessary and sufficient conditions for this are

$$f>0,$$
$$d(1-f)<1. \tag{3.6}$$

Henceforth, we assume that these conditions are satisfied.

We can now add an (infinite) constant to the Lagrangian so that solution I will correspond to zero energy:

$$L\to L-\mu\left(\frac{\mu^2}{\lambda}\right)\int dx\{\tfrac{1}{4}\}. \tag{3.7}$$

Note that this has absolutely nothing to do with the infinite shift in the ground-state energy resulting from the quantum zero-point oscillations, which will be discussed later.

As was shown in Sec. II, evaluating the path integral over the mode corresponding to global shifts of θ leads to consideration of solutions of effective Euler-Lagrange equations involving a centrifugal-barrier term. (An analogous result can be obtained in the classical field theory from the conservation of the classical charge.) In particular, for the case of weak coupling we need only find solutions in which ρ, $\tilde{\theta}$, and σ are time-independent. [$\tilde{\theta}$ is defined in (2.27).] Such solutions must satisfy

$$0=-\sigma''-\sigma+\sigma^3+d\rho^2\sigma,$$
$$0=-\frac{\lambda}{a}\rho''+\frac{\lambda}{a}\tilde{\theta}'^2\rho-\frac{\lambda}{a}\left(\frac{\lambda h}{a\mu^2}\right)^2\frac{(q^2-\frac{1}{4})}{[\int\rho^2(x)dx]^2}\rho$$
$$+(f+\sigma^2-1)\rho+\rho^3, \tag{3.8}$$
$$0=\frac{d}{dx}(\rho^2\tilde{\theta}'),$$

where we have rescaled according to (3.1).

Before proceeding to the consideration of Eqs. (3.8), it is useful to study the corresponding equations in the absence of the centrifugal-barrier term, namely

$$0=-\sigma''-\sigma+\sigma^3+d\rho^2\sigma, \tag{3.9a}$$
$$0=-\frac{\lambda}{a}\rho''-\frac{\lambda}{a}\tilde{\theta}'^2\rho+(f+\sigma^2-1)\rho+\rho^3, \tag{3.9b}$$
$$0=\frac{d}{dx}(\rho^2\tilde{\theta}'). \tag{3.9c}$$

If the energy is to be finite, σ and ρ must approach their ground-state values asymptotically as x approaches $\pm\infty$. In particular, we will be interested in solutions which satisfy

$$\rho(\pm\infty)=0,$$
$$\sigma(\pm\infty)=\pm 1. \tag{3.10}$$

These boundary conditions, together with (3.9c) and the definition of $\tilde{\theta}$, imply that

$$\tilde{\theta}(x)=\text{constant}=0. \tag{3.11}$$

We must show that (3.9) does in fact have a stable solution which satisfies (3.10). We begin by noting that in the case of static fields the potential energy is given by

$$V=\mu\left(\frac{\mu^2}{\lambda}\right)\int dx\left\{\tfrac{1}{2}\sigma'^2-\tfrac{1}{2}\sigma^2+\tfrac{1}{4}\sigma^4+\tfrac{1}{4}\right.$$
$$\left.+d\left[\frac{1}{2}\frac{\lambda}{a}\rho'^2+\tfrac{1}{2}(f+\sigma^2-1)\rho^2+\tfrac{1}{4}\rho^4\right]\right\}, \tag{3.12}$$

where, because of (3.11), we have omitted terms involving $\tilde{\theta}(x)$. Stable solutions of (3.9) correspond to local minima of the potential energy in the infinite-dimensional space of field configurations. The signs of the quartic terms in the Lagrangian ensure that the potential energy is bounded from below; the absolute minimum corresponds to solution I of (3.5). Further, no field configuration satisfying the boundary conditions (3.10) can be deformed into a field configuration satisfying different boundary conditions (e.g., solution I) without crossing an infinite potential barrier, arising from the infinite length of space. Thus, among field configurations satisfying (3.10), there must be at least one which is a local minimum of the potential energy. (In fact, there will be an infinity of minima since substituting $x'=x+a$ for x will generate a new configuration with the same potential energy.) We are particularly interested in finding solutions for which $\rho(x)$ is nonvanishing.

If $\rho(x)$ were identically zero, we could solve (3.9) analytically, obtaining

$$\sigma(x,t)=\tanh\left(\frac{x-x_0}{\sqrt{2}}\right),$$
$$\rho(x,t)=0. \tag{3.13}$$

To determine whether or not this is a stable solution, we must see whether the matrix of second derivatives of the potential energy has any negative eigenvalues. In this system with an infinite number of degrees of freedom, the problem becomes that of finding the eigenvalues of the following equations:

$$-\psi''+\left(-1+3\tanh^2\frac{x-x_0}{\sqrt{2}}\right)\psi=k\psi, \tag{3.14a}$$

$$-\frac{\lambda}{a}\chi''+\left(-1+f+\tanh^2\frac{x-x_0}{\sqrt{2}}\right)\chi=\bar{k}\chi. \tag{3.14b}$$

Both of these are exactly soluble Schrödinger equations.[9] The lowest eigenvalue of (3.14a) is zero, as shown in Ref. 2, while the lowest eigenvalue of (3.14b) is

$$\bar{k}_0=f-1+\frac{\lambda}{4a}\left[\left(1+\frac{8a}{\lambda}\right)^{1/2}-1\right]. \tag{3.15}$$

Thus, (3.13) will be unstable if

$$f<1-\frac{\lambda}{4a}\left[\left(1+\frac{8a}{\lambda}\right)^{1/2}-1\right], \tag{3.16}$$

and stable otherwise. If (3.16) is satisfied, the minimum of the potential energy does not occur at (3.13), but rather at some other field configuration, in which $\rho(x)$ is not identically zero and a solution of the form we are seeking does exist. [Note that the conditions (3.16) and (3.6) are consistent with weak coupling, viz., $a,\lambda,h\ll m^2,\mu^2$.] It is difficult to find this solution analytically, but an approximation can be obtained by variational methods, as we shall see presently.

[If (3.16) is not satisfied, then (3.13) is a local minimum of the potential energy. This in itself does not preclude the possibility of other minima; however, it is easy to show that if $f>1$, there are no minima in which $\rho(x)$ is not zero everywhere. If $f>1$, all terms in the potential energy which contain $\rho(x)$ are positive. Therefore, if we begin with any field configuration and replace $\rho(x)$ by $b\rho(x)$, the potential energy will decrease monotonically as we let b approach zero.

We also note that if f were allowed to be negative, no field configuration satisfying (3.10) could be even a local minimum of V, since such configurations asymptotically approach solution I of (3.5), which is no longer a stable solution.]

We are now ready to consider Eqs. (3.8). Again, the constraint of finite energy imposes the condition that the fields approach their ground-state values asymptotically for large $|x|$, and again we restrict ourselves to solutions satisfying (3.10) and (3.11). These boundary conditions will also ensure the finiteness of $\int\rho^2(x)dx$, so the effective Lagrangian will be well defined. Since we are dealing with static solutions, we can define a constant, ω^2, by

$$\left(\frac{\lambda h}{a\mu^2}\right)^2(q^2-\tfrac{1}{4})=\frac{\omega^2}{\mu^2}\left[\int\rho^2(x)dx\right]^2. \tag{3.17}$$

(If it were not for the $-\frac{1}{4}$ arising from quantum effects, ω would have a simple interpretation as the frequency of rotation in internal space.) We can now write (3.8) as

$$0=-\frac{\lambda}{a}\rho''+\left(f-\frac{\lambda\omega^2}{a\mu^2}+\sigma^2-1\right)\rho+\rho^3. \tag{3.18}$$

This is of just the same form as (3.9) with $\tilde{\theta}(x,t)=0$, and f replaced by

$$g=f-\frac{\lambda\omega^2}{a\mu^2}.$$

Everything that we have said about the solutions of (3.9) can be immediately applied. In particular, there will be solutions only when g is both greater than 0 and less than a critical value which is no greater than 1. The physical interpretation of this is that for sufficiently large q the "centrifugal force" arising from the rotation in internal space is so great that the potential of the σ field can no longer bind the ρ field. This result, which clearly carries over in weak-coupling WKB approximation to the quantum system as well. is physically satisfying. It says that bound hadrons no longer exist for sufficiently large values of internal quantum numbers. Rewriting (3.8) in the form (3.18) has several advantages, since the latter has the same form as the original field equation (3.9). If a static solution to the original equations is known precisely, or within a variational class, the static solutions of (3.8) are immediately available by replacing the constant f by the constant g. The extra centrifugal term in (3.8) acts just like a mass term for a static ρ field. Further, consider a globally rotating solution to the original equations (3.3b), viz.,

$$\sigma(x,t)=\sigma_0(x),$$
$$\rho(x,t)=\rho_0(x), \tag{3.19}$$
$$\theta(x,t)=\frac{\omega}{\mu}t.$$

Then $\rho_0(x)$ clearly satisfies (3.18). Thus static solutions of the effective equations of motion containing the centrifugal term are essentially globally rotating solutions of the parent equations of

motion, just as in the single-particle example in Sec. II. Finally, as can be checked easily, the rotating solution (3.19) has an energy equal to $U_q(\rho_0, \sigma_0) \equiv -L_{\text{eff}}(\rho_0, \sigma_0)$ [see Eq. (2.34)].

So far, we have dealt primarily with the existence of solutions to (3.8) and (3.9); we now turn our attention to the form of these solutions. Because of the complexity of (3.8) and (3.9), it is difficult to obtain analytic expressions for the solutions; instead, we use a variational approach to finding the minima of the potential energy (3.12) (with g replacing f). We use the trial field configuration (see Fig. 2)

$$\sigma(x) = \begin{cases} -1, & x < -L \\ x/L, & -L < x < L \\ +1, & x > L \end{cases}$$

$$\rho(x) = \begin{cases} 0, & |x| > R \\ \frac{c}{R}(R - |x|), & |x| < R \end{cases} \tag{3.20}$$

varying c^2, R, and L so as to minimize V. The motivation for this form should be clear: Even in the presence of a nonzero ρ field, the σ field should behave qualitatively just as it does in (3.20); i.e., it should be very close to +1 or −1 through most of space, crossing from one to the other in a relatively narrow region, which we can arbitrarily take to be centered at $x = 0$. The ρ field should be nonvanishing only near the region where σ^2 is sufficiently small.

Using (3.20), we obtain

$$V = \begin{cases} \frac{1}{\mu}\frac{\mu^4}{\lambda}\left\{\frac{1}{L} + \frac{4}{15}L + d\left[\frac{\lambda}{a}\frac{c^2}{R} + \frac{1}{3}(g-1)c^2R + \frac{c^2R^3}{30L^2} + \frac{c^4R}{10}\right]\right\}, & R < L \\ \frac{1}{\mu}\frac{\mu^4}{\lambda}\left\{\frac{1}{L} + \frac{4}{15}L + d\left[\frac{\lambda}{a}\frac{c^2}{R} + \frac{1}{3}gc^2R - \frac{2}{15}\frac{c^2L^3}{R^2} - \frac{2c^2L}{3} + \frac{1}{2}c^2\frac{L^2}{R} + \frac{c^4R}{10}\right]\right\}, & R > L. \end{cases} \tag{3.21}$$

If we vary c^2 and R so as to minimize V, we obtain the conditions

$$R^2 = L^2\left\{(1-g) + \left[(1-g)^2 + \frac{18\lambda}{aL^2}\right]^{1/2}\right\}, \quad R < L$$
$$c^2 = \tfrac{4}{3}(1-g) - \frac{8\lambda}{aR^2}, \tag{3.22}$$

$$0 = g\left(\frac{R}{L}\right)^3 + 2\left(\frac{R}{L}\right)^2 - \frac{9}{2}\left(1 + \frac{2\lambda}{aL^2}\right)\left(\frac{R}{L}\right) + 2, \quad R > L.$$
$$c^2 = \frac{8}{3}\frac{L}{R} - \left(1 + \frac{2\lambda}{aL^2}\right)\left(\frac{L}{R}\right)^2 - 2g$$

(If these yield a negative value for c^2, then $c^2 = 0$ is the solution.) We may also try to vary L, but

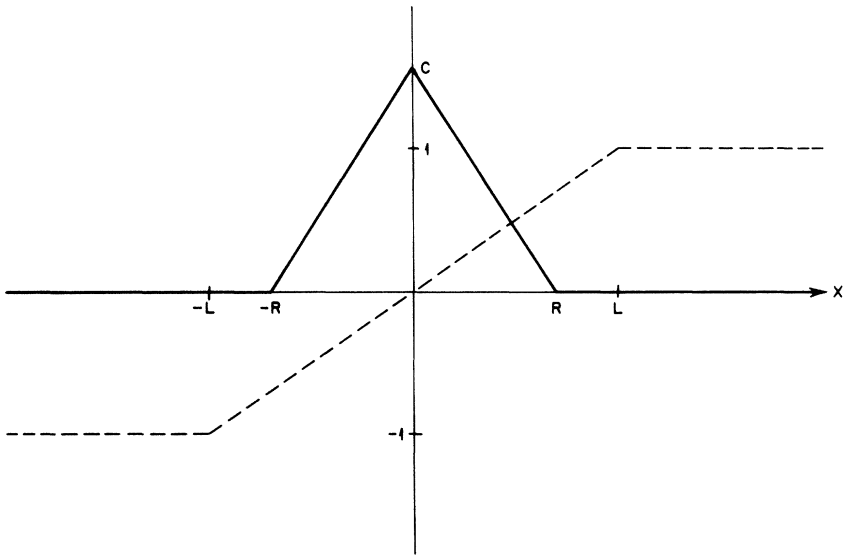

FIG. 2. Field configurations used in the variational calculation. The dashed line stands for the σ field and the solid line for the ρ field.

the equations become prohibitively complicated. Instead, we restrict our consideration to the case of small d, where we may write

$$L = L_0 + dL_1 + O(d^2). \tag{3.23}$$

Choosing $L_0{}^2 = \frac{15}{4}$ will minimize that part of V which is zeroth order in d; L_1 then appears only in terms of order d^2, and will therefore be neglected.

If we consider variations of λ/a, we find that R increases and c^2 decreases as λ/a is increased. This is to be expected, since gradient terms in the potential energy should make the variation in ρ more gradual, both by decreasing its magnitude and by spreading it over a broader region. More important is the behavior of R and c^2 as ω, and thus g, are varied; a short calculation shows that both R and c^2 increase with increasing ω. This is perfectly reasonable, since increasing the "centrifugal force" should drive $\int \rho^2(x)dx$ upward.

However, the quantities of primary interest for us are the energy and the charge, given by

$$E = \mu\left(\frac{\mu^2}{\lambda}\right)\left\{\frac{19}{2\sqrt{15}} + d\left[(f-g)\left(\tfrac{2}{3}c^2R\right) - \tfrac{1}{10}c^4R\right]\right\} \tag{3.24a}$$

and

$$q^2 - \tfrac{1}{4} = \left(\frac{\mu^2}{\lambda}\right)d\left(\frac{a\mu^2}{\lambda h}\right)(f-g)\left(\tfrac{1}{3}c^2R\right)^2. \tag{3.24b}$$

These also increase if g is decreased while f is held fixed. If we eliminate $(f-g)$ from (3.24), we obtain

$$E = \mu\left(\frac{\mu^2}{\lambda}\right)\left\{\frac{19}{2\sqrt{15}} + d\left[6\,\frac{\lambda h^2}{a^3}\left(\frac{\lambda}{\mu^2}\right)^2\frac{(q^2-\frac{1}{4})}{c^2R} - \tfrac{1}{10}c^4R\right]\right\}. \tag{3.25}$$

It is important to keep in mind that this expression does not explicitly display the entire dependence of the energy on the charge, since R and c^2 are both charge-dependent. In Fig. 3 we plot the dependence of E on q for typical values of the parameters; other values give similar results.

The WKB method reduces to the consideration of the static solutions of (3.8) only for the case of weak coupling, i.e., when a, h, and $\lambda \ll \mu^2$ and m^2. In this case we can make the following remarks:

(1) For small q ($q \ll \mu^2/\lambda$), the variation of R and c^2 with charge is small.

(2) For q of order unity the charge-dependent part of the energy will be of order $(\lambda/\mu^2)^2$ compared to the charge-independent part. This is smaller than the quantum corrections. However, when q is of order μ^2/λ, the variation of the energy with increasing charge will be significant. At the same time the weak-coupling approximation will still be valid.

(3) For any given set of parameters, there will be a maximum charge, determined by the condition that g be positive. This maximum charge will be of order μ^2/λ.

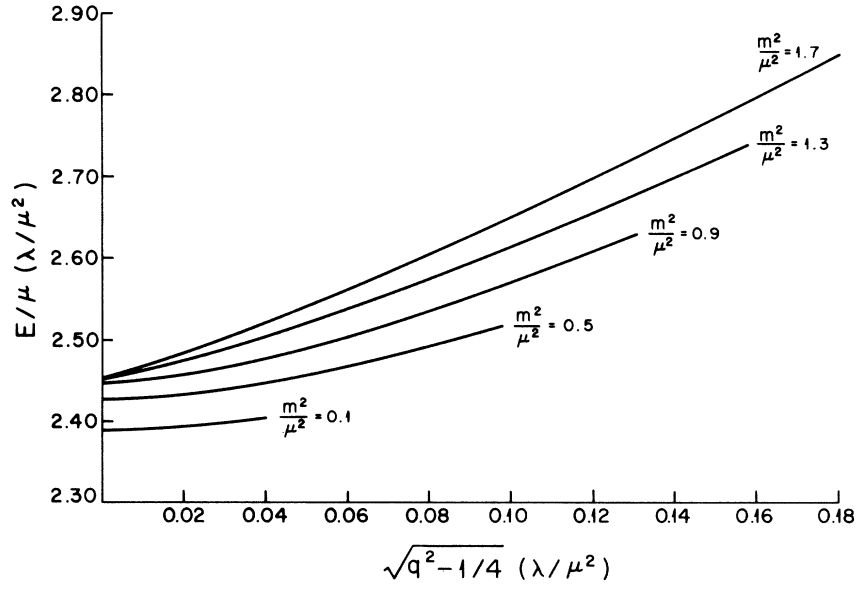

FIG. 3. Dependence of energy upon charge for solutions of the variational problem, with $\lambda/a = 0.5$ and $a/h = 0.1$. The curves terminate at the charge where binding ceases to be possible.

IV. HIGHER SYMMETRY GROUPS

The methods outlined explicitly in the previous two sections for U(1) symmetry should hold, subject to the same limitations, for higher groups, and for fields transforming as adjoint as well as nonadjoint representations of these groups.

However, certain complications arise when we consider the extension of these methods to theories with non-Abelian internal-symmetry groups. To illustrate this, we use a theory with an SU(2) internal symmetry, namely the theory of (1.1) with $\phi(x,t)$ as a triplet of real fields transforming under the $I=1$ representation of SU(2) and with $|\phi|^2 \equiv \sum_{i=1}^{3}\phi_i^2$. We can define one "radial" and two "angular" fields in terms of the original Cartesian fields by

$$\begin{aligned}\phi_1(x,t) &= \rho(x,t)\sin\theta(x,t)\cos\psi(x,t),\\ \phi_2(x,t) &= \rho(x,t)\sin\theta(x,t)\sin\psi(x,t),\\ \phi_3(x,t) &= \rho(x,t)\cos\theta(x,t).\end{aligned} \tag{4.1}$$

Under the infinitesimal global isospin rotation generated by $\epsilon_1 I_1 + \epsilon_2 I_2 + \epsilon_3 I_3$, these fields transform as

$$\begin{aligned}\delta\rho(x,t) &= 0,\\ \delta\theta(x,t) &= \epsilon_1\cos\psi(x,t) + \epsilon_2\sin\psi(x,t),\\ \delta\psi(x,t) &= -\epsilon_1\cot\theta(x,t)\sin\psi(x,t)\\ &\quad + \epsilon_2\cot\theta(x,t)\cos\psi(x,t) + \epsilon_3.\end{aligned} \tag{4.2}$$

In the U(1) case, the symmetry coordinate was seen to be $b(t)$, defined by (2.27). However, it is easy to see that if we were to define b_θ, b_ψ, $\bar{\theta}$, and $\bar{\psi}$ in analogy with the U(1) example, $\bar{\theta}$ and $\bar{\psi}$ would not be left invariant by the transformation (4.2), so b_θ and b_ψ cannot be the symmetry coordinates. The coordinate system containing the symmetry coordinates is in fact a much more complicated curvilinear one, i.e., it depends upon the point in function space.

Rather than immediately proceeding to construct this coordinate system, let us outline the steps which our method would prescribe if we did know the coordinate system. First, we would project onto a subspace of definite I by inserting the appropriate projection operator, obtaining

$$\begin{aligned}G_I(E) &= \int_0^\infty dT\, e^{iET} G_I(T),\\ G_I(T) &= \frac{2I+1}{2\pi}\int d\alpha(1-\cos\alpha)\sum_{m=-I}^{I} e^{i\alpha m}\,\mathrm{Tr}\, e^{i(HT+\alpha I_3)},\\ \mathrm{Tr}\, e^{i(HT+\alpha I_3)} &= \int \rho_0^2 d\rho_0\, d\cos\theta_0\, d\psi_0 \langle \rho_0(x), \theta_0(x), \psi_0(x)+\alpha | e^{iHT} | \rho_0(x), \theta_0(x), \psi_0(x)\rangle .\end{aligned} \tag{4.3}$$

[In writing the projection in this manner we are assuming, on the basis of the usual quantum results, that I will be restricted to integral values. An analogous assumption was made in the U(1) example.] Next we would transform to the coordinate system containing the symmetry coordinates and write a path integral for the final term in (4.3). Integrating over α and the symmetry coordinates, we would have been left with a path integral over the remaining coordinates with an integrand involving an effective action, depending on I, which no longer contained the global internal symmetry. [The corresponding path integrals over $\theta(t)$ and $\phi(t)$ for a single particle in three dimensions have been evaluated exactly in Ref. 7.]

The manner of treating this resultant path integral depends upon whether the coupling is strong or weak. The strong-coupling case is quite difficult to handle; at the very least, we have to know all of the periodic solutions of the effective action. For weak coupling, on the other hand, it is sufficient to expand about the stationary-phase point determined by a static solution of the effective action. Furthermore, the bound-state energies in the weak-coupling limit will be dominated by the value of the effective action at this stationary point, with corrections being smaller by a factor of λ/μ^2.

So far we have not made any assumptions about the relation of the effective action in the path integral to any classical quantities. However, in the weak-coupling situation, the results of the U(1) example, as well as the correspondence principle, lead us to expect a connection with classical field theory, at least when the internal quantum number is sufficiently large. In particular, we may expect in this limit that (1) the equations which determine the static stationary point of the path integral are the same as the classical effective field equations (i.e., those equations obtained from the Euler-Lagrange equations by eliminating the symmetry coordinates in favor of the conserved charges), and (2) that the value of the effective action appearing in the path integral is, when eval-

uated at this point, equal to the energy of the corresponding solution of the classical effective equations (i.e., the energy of a classical solution whose only time dependence is in the symmetry coordinates).

If the internal quantum number is small, we should certainly expect the quantum-mechanical equations to differ from the classical ones. For the U(1) example, the effect of quantum mechanics was to modify the charge dependence by such factors as the replacement of q^2 by $q^2-\frac{1}{4}$ in (3.8). Recall, however, that in the weak-coupling regime the charge dependence of the energy was negligible for small quantum numbers, so such factors could become important for calculating energies only for large changes, in which case the difference between q^2 and $q^2-\frac{1}{4}$ is negligible, as required by the correspondence principle. If this property remains true in theories with non-Abelian internal-symmetry groups, we may expect the relation between the quantum and classical field theories outlined in the previous paragraph to hold for all values of the internal quantum numbers, provided only that the coupling is weak.

Let us assume that these properties hold, and see how far we can proceed without actually obtaining an explicit expression for the symmetry coordinates. Even at the classical level, we cannot write the effective equations of motion without obtaining the symmetry coordinates. Despite this, we can recognize the static solutions of these equations. Such solutions must be solutions of the original Euler-Lagrange equations with the additional property that their only time dependence is in the symmetry coordinates. In other words, the field configuration at time $t+\delta t$ must be obtained from that at time t by a global isospin rotation; using (4.2), we see that $\sigma(x,t)$ and $\rho(x,t)$ must be time-independent, while $\dot\theta(x,t)$ and $\dot\psi(x,t)$ must be of the form

$$\begin{aligned}\dot\theta(x,t)&=\omega_1(t)\cos\psi(x,t)+\omega_2(t)\sin\psi(x,t),\\ \dot\psi(x,t)&=-\omega_1(t)\cot\theta(x,t)\sin\psi(x,t)\\ &\quad+\omega_2(t)\cot\theta(x,t)\cos\psi(x,t)+\omega_3(t),\end{aligned}\tag{4.4}$$

where ω_1, ω_2, and ω_3 are functions of time, but not of space. The field equations obtained from the Lagrangian are

$$\begin{aligned}&[\Box-1+\sigma^2+d\rho^2]\sigma=0,\\ &\left[\frac{\lambda}{a}(\Box-\dot\theta^2+\theta'^2-\sin^2\theta\,\dot\phi^2+\sin^2\theta\,\phi'^2)\right.\\ &\qquad\left.+(f+\sigma^2-1)+\rho^2\right]\rho=0,\\ &\frac{d}{dt}[\rho^2\sin^2\theta\,\dot\phi]-\frac{d}{dx}[\rho^2\sin^2\theta\,\phi']=0,\\ &\frac{d}{dt}[\rho^2\dot\theta]-\frac{d}{dx}[\rho^2\theta']-\rho^2\sin\theta\cos\theta(\dot\phi^2-\phi'^2)=0.\end{aligned}\tag{4.5}$$

We can now use (4.4) to write all of the time derivatives of θ and ψ in (4.5) in terms of $\vec\omega$, $\dot{\vec\omega}$, and the values of the fields (but not their derivatives). We next write expressions for the components of the classical isospin in terms of these variables:

$$\begin{aligned}I_1&=\omega_1\int dx\,\rho^2\sin\psi\cos\psi\sin^2\theta+\omega_2\int dx\,\rho^2(\sin^2\psi+\cos^2\psi\cos^2\theta)+\omega_3\int\rho^2\sin\theta\cos\theta\cos\psi,\\ I_2&=\omega_1\int dx\,\rho^2(\cos^2\psi+\sin^2\psi\cos^2\theta)+\omega_2\int dx\,\rho^2\sin\psi\cos\psi\sin^2\theta-\omega_3\int\rho^2\sin\theta\cos\theta\sin\psi,\\ I_3&=-\omega_1\int\rho^2\sin\theta\cos\theta\sin\psi+\omega_2\int\rho^2\sin\theta\cos\theta\cos\psi+\omega_3\int\rho^2\sin^2\theta.\end{aligned}\tag{4.6}$$

Using (4.6), we can obtain $\vec\omega$ in terms of $\vec I$. If we write the equations which follow from the fact that $d\vec I/dt=0$, we can also obtain an expression for $\dot{\vec\omega}$ in terms of $\vec I$ and the values of the fields. Substituting into (4.5), we can now obtain a set of differential equations involving only spatial derivatives. If we solve these and then let the fields develop in time according to (4.4), we will have obtained the desired classical solutions.

Rather than exhibit the most general form of such equations, we note that any field configuration in which θ and ψ are independent of space, though not necessarily of time, satisfies (4.4). In such a case, it is easy to show that the static solutions of the classical effective equations must satisfy

$$\begin{aligned}0&=-\sigma''-\sigma+\sigma^3+d\rho^2\sigma,\\ 0&=-\frac{\lambda}{a}\rho''+(f+\sigma^2-1)\rho+\rho^3\\ &\quad-\frac{\lambda}{a}\left(\frac{\lambda\hbar^2}{a\mu^2}\right)\frac{I^2}{[\int\rho^2dx]^2}\rho,\end{aligned}\tag{4.7}$$

where we have rescaled as in (3.1). These equations are of just the same form as the ones we obtained in the U(1) example (3.8). Therefore, we

can immediately apply all the results of Sec. III to the present example, merely substituting isospin for charge.

It is important not to confuse the representation under which the fundamental fields transform with the representation under which the bound states transform. The latter may be any representation which can be obtained from the decomposition of direct products of the representation of the fundamental fields. Thus, in the above example the isospin of the bound states may take on any integral value, within the limits imposed by the dynamics.

We need not, however, have taken the fundamental fields to be in the adjoint ($I=1$) representation. As an example of an alternative representation, we consider the theory with the same Lagrangian, but with ϕ now being an isospin doublet and with $|\phi|^2 \equiv \phi^\dagger\phi$. Again, it is most convenient to work in terms of "radial" and "angular" fields defined by

$$\phi(x,t) = \begin{pmatrix} a(x,t)+ib(x,t) \\ c(x,t)+id(x,t) \end{pmatrix},$$

$$a(x,t) = \rho(x,t)\cos\tfrac{1}{2}\beta(x,t)\cos\left[\frac{\alpha(x,t)+\gamma(x,t)}{2}\right],$$

$$b(x,t) = -\rho(x,t)\cos\tfrac{1}{2}\beta(x,t)\sin\left[\frac{\alpha(x,t)+\gamma(x,t)}{2}\right], \tag{4.8}$$

$$c(x,t) = \rho(x,t)\sin\tfrac{1}{2}\beta(x,t)\cos\left[\frac{\alpha(x,t)-\gamma(x,t)}{2}\right],$$

$$d(x,t) = \rho(x,t)\sin\tfrac{1}{2}\beta(x,t)\sin\left[\frac{\alpha(x,t)-\gamma(x,t)}{2}\right].$$

Once again, there is the complication that the symmetry coordinates cannot be simply defined. However, for weak coupling we can make the same assumptions that we did in the triplet example and proceed as we did there. In particular, if we restrict ourselves to cases where the angular fields are space-independent, we will once again find that the classical solutions we seek must once again satisfy (4.7). There is, however, one difference from the previous example. Since the fundamental fields now belong to a half-integral representation, I can now take on both integral and half-integral values.

No new complications in principle occur when these methods are applied to other groups or representations; however, it may become more difficult to obtain the classical field solutions. In the examples we have considered, it was possible to obtain globally rotating classical solutions from truly static ones simply by a change in the mass parameter. This is unlikely to be true for larger representations in which there are more than one "radial" field. [For example, the $I=2$ representation of SU(2) will have two "radial" and three "angular" fields.] Nevertheless, we expect that in the weak-coupling limit there will be bound states of increasing internal quantum numbers whose energies can be obtained from corresponding classical globally rotating solutions.

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[4]We use the following terminology. It is assumed that fields can be expressed in terms of orthogonal coordinates such that the global internal symmetry acts only on a minimal subset (the "symmetry coordinates"), leaving the others (the "remaining" or "nonsymmetry" coordinates) unchanged. For the case of U(1), such a decomposition is explicitly shown. We will use the term cyclic coordinates in the conventional sense. The Lagrangian will not depend on the cyclic coordinates at all, only on their time derivatives. For a non-Abelian group, not all symmetry coordinates are cyclic.

[5]There will be of course be additional zero-frequency modes due to space-time symmetries. We do not discuss them here. They may be treated by methods similar to those employed for translation symmetry. (See Refs. 2 and 3.)

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