Axial-vector anomaly in asymptotically free theories*

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We show the absence of radiative corrections to the axial-vector anomaly in non-Abelian gauge theories of strong interactions.

I. INTRODUCTION

The quark constituent picture has been useful in providing a simple and concrete model for the description of the low-lying hadronic states and their properties. One of the criteria in hadron model building is the correct prediction of the low-energy theorem for the decay $\pi^0\to 2\gamma$. It has been shown that the low-energy theorem is related to the anomalous partial conservation law of the axial-vector current (PCAC anomaly).[1] Adler and Bardeen[2] have shown that the PCAC anomaly is given by the basic fermion triangle graphs in quantum electrodynamics (QED) and the σ model. This important result tells us that the $\pi^0\to 2\gamma$ amplitude can be used to distinguish between various quark charge schemes. The theorem was also proved in QED by Zee[3] using the Callan-Symanzik equation[4]; this technique was later extended to the σ model by Shei and Zee.[5]

A very popular model of strong interaction is the quark model with non-Abelian gauge fields. This is the only renormalizable theory which exhibits the property of being asymptotically free.[6] Therefore, it provides an appealing explanation of Bjorken scaling in the SLAC-MIT deep-inelastic scattering experiments. In constructing models along this line, the correct prediction for the low-energy theorem of $\pi^0\to 2\gamma$ is often required. It is natural to ask whether the PCAC anomaly is independent of the strong interactions. To the best of our knowledge, this general problem has not been studied. The purpose of this paper is to show that, in non-Abelian gauge theories of strong interactions, the PCAC anomaly is still given by the basic fermion triangle graphs.

The plan of this paper is as follows: We describe the non-Abelian gauge model of strong interactions in Sec. II. In Sec. III, the problem of the radiative corrections to the PCAC anomaly is discussed. We show that the argument by Zee[3] can be easily adapted to non-Abelian gauge theories of strong interactions. In the last section, we first comment on our results and the Crewther relation.[7] We then comment on the difference between our model and the model considered by Chanowitz.[8] It is also pointed out why the argument we use here is not applicable to his model.

II. MODEL OF STRONG INTERACTIONS

For definiteness, let us consider a non-Abelian gauge model of three quark triplets with a gauge symmetry SU(3)′ [color SU(3)]. The Lagrangian is of the form

$$\mathcal{L} = -\tfrac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + i\bar\psi \not{D}\psi - m_0\bar\psi\psi, \tag{1}$$

where

$$D_\mu = \partial_\mu + i g_0 B^a_\mu T^a, \tag{2}$$

$$G^a_{\mu\nu} = \partial_\mu B^a_\nu - \partial_\nu B^a_\mu - g_0 f^{abc} B^b_\mu B^c_\nu; \tag{3}$$

f^{abc} are the structure constants and T^a are matrices acting on the color indices of quarks. In this model the color gauge symmetry is exact. The hadronic vector and axial-vector currents are all singlets in the color space SU(3)′. They satisfy conservation laws and partial conservation laws, respectively:

$$\partial_\mu V^\mu_k = 0, \tag{4}$$

$$\partial_\mu A^\mu_k = m_0\bar\psi i\gamma_5\lambda_k\psi \equiv D_k, \tag{5}$$

where

$$V^\mu_k \equiv \bar\psi\gamma^\mu\frac{\lambda_k}{2}\psi, \tag{6}$$

$$A^\mu_k \equiv \bar\psi\gamma^\mu\gamma_5\frac{\lambda_k}{2}\psi, \tag{7}$$

and the λ_k are the usual Gell-Mann matrices acting in SU(3) space. We remark here that there is no anomaly in Eq. (5), because the gauge fields are SU(3) singlets.

III. ABSENCE OF RADIATIVE CORRECTIONS TO THE PCAC ANOMALY

To study the low-energy theorem for the decay $\pi^0\to 2\gamma$, we consider the amplitude[9]

$$R^0_{D\mu\nu}(k,q) = i\int d^4x\, d^4y\, e^{i(kx+qy)}\langle 0|T\partial A(0)J_\mu(x)J_\nu(y)|0\rangle = \epsilon_{\mu\nu\lambda\sigma}k^\lambda q^\sigma f(k^2,q^2,(k+q)^2). \quad (8)$$

For clarity, we have dropped subscripts on D and A in Eq. (8). It is understood that D and A have subscript 3 and J_μ is the electromagnetic current. In what follows we treat the strong interactions to all orders but the weak and electromagnetic interactions to the lowest order. The Callan-Symanzik equation is obtained by varying $R^0_{D\mu\nu}$ with respect to m_0 with the bare coupling constant g_0 and the cutoff Λ held fixed. One finds that

$$m_0\frac{\partial}{\partial m_0}R^0_{D\mu\nu}(k,q) = R^0_{D\mu\nu}(k,q) + iR^0_{D\mu\nu s}(k,q,0), \quad (9)$$

where

$$R^0_{D\mu\nu s}(k,q,0) = i\int d^4x\, d^4y\, d^4z\, e^{i(kx+qy)}\langle 0|T\partial A(0)J_\mu(x)J_\nu(y)s(z)|0\rangle, \quad (10)$$

$$s \equiv -m_0\bar\psi\psi. \quad (11)$$

It is easy to see that the following Ward identity is true:

$$i(k+q)^\lambda iR^0_{\lambda\mu\nu s}(k,q,0) = R^0_{D\mu\nu}(k,q) + iR^0_{D\mu\nu s}(k,q,0), \quad (12)$$

where

$$R^0_{\lambda\mu\nu s}(k,q,0) \equiv i\int d^4x\, d^4y\, d^4z \times e^{i(kx+qy)}\langle 0|TA_\lambda(0)J_\mu(x)J_\nu(y)s(z)|0\rangle. \quad (13)$$

It is important to realize that there is no anomalous term in the Ward identity (12). This can be proved by appealing to Weinberg's theorem[10] as in Adler's review in Ref. 1. We only have to remind ourselves that in the renormalizable gauge the power counting for the superficial degree of divergence is exactly the same as in QED.

Substituting Eq. (12) into Eq. (10) we find that

$$m_0\frac{\partial}{\partial m_0}R^0_{D\mu\nu}(k,q) = -(k+q)^\lambda R^0_{\lambda\mu\nu s}(k,q,0). \quad (14)$$

The operators D, s, J_μ, and A_λ are all cutoff independent owing to the CVC (conservation of vector current) and PCAC conditions.[11] This implies that the amplitudes $R^0_{D\mu\nu}$ and $R^0_{\lambda\mu\nu s}$ are all cutoff independent. From now on we drop the superscript in $R^0_{D\mu\nu}$ and $R^0_{\lambda\mu\nu s}$ to emphasize that they are renormalized cutoff-independent quantities. Now we can express Eq. (14) in terms of the renormalized mass m and the renormalized coupling constant g.[12] We obtain

$$\eta\left[m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g}\right]R_{D\mu\nu}(k,q) = -(k+q)^\lambda R_{\lambda\mu\nu s}(k,q,0), \quad (15)$$

where

$$\eta m = m_0\frac{\partial m}{\partial m_0}\bigg|_{g_0,\Lambda}, \quad \eta\beta(g) = m_0\frac{\partial g}{\partial m_0}\bigg|_{g_0,\Lambda}. \quad (16)$$

Let us turn to the behavior of $R_{\lambda\mu\nu s}$ in the limit $k, q \to 0$.[13] From Bose statistics and parity one finds that

$$R_{\lambda\mu\nu s}(k,q,0) = a\epsilon_{\lambda\mu\nu\sigma}(k-q)^\sigma + \text{terms of higher order in } k, q. \quad (17)$$

On the other hand, the gauge invariance of $R_{\lambda\mu\nu s}$ requires that

$$ik^\mu R_{\lambda\mu\nu s}(k,q,0) = -ia\epsilon_{\lambda\mu\nu\sigma}k^\mu q^\sigma + \cdots = 0$$

or

$$a = 0. \quad (18)$$

This together with Eq. (15) implies that

$$\eta\left[m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g}\right]f(0,0,0) = 0. \quad (19)$$

Now we recall that our model has only one mass so that $f(0,0,0)$ is independent of the mass m. Thus we are led to the result

$$\eta\beta(g)\frac{\partial}{\partial g}f(0,0,0) = 0. \quad (20)$$

It is known that $\eta\beta(g) \neq 0$. Therefore, $f(0,0,0)$ is independent of the coupling constant g. In other words, the PCAC anomaly is given by the basic fermion triangle graphs in the non-Abelian gauge theories of strong interactions. This is our essential conclusion. For simplicity, we have considered the case of exact SU(3). Our result remains the same in the case of broken SU(3) caused by the mass splitting in quarks.[14]

IV. REMARKS

Before we conclude this paper, we would like to make some pertinent remarks. Our first comment concerns the validity of the Crewther relation $S = KR$.[6] It has been shown by Bég[15] that in asymptotically free theories the coefficient function of A_λ in the short-distance expansion[16] of $V_\mu^k(x)V_\nu^l(0)$ is given by the free field result in the short-distance limit, i.e., K is independent of the coupling constant g. The constant R, which is related to the c-number term in the short-distance expansion of $A_\mu^k(y)A_\nu^l(0)$, can be obtained by a technique similar to that of Appelquist and Georgi[17] and Zee.[18] One finds that R is also given by the free-field value. In this note we have shown that the PCAC anomaly is given by the free-field value. Therefore, we conclude that the Crewther relation is valid in non-Abelian gauge theories of strong interactions.

Our last comment is on the comparison of our result with that of Chanowitz.[8] He considered the weak and electromagnetic corrections to the low-energy theorem of $\pi^0 \to 2\gamma$ in a spontaneously broken non-Abelian gauge theory. He has found the absence of corrections to the fourth order in the coupling constant. Nothing is known about the higher-order corrections in his case. One might be tempted to apply our technique to the spontaneously broken gauge theories of weak and electromagnetic interactions. Unfortunately this is not a fruitful pursuit. This can be understood in the following way. One essential difference is that in the model we consider here there is only one mass, while in the spontaneously broken gauge theories there is more than one mass. Therefore, Eq. (19) and the statement that $f(0,0,0)$ is independent of mass are no longer true. To carry out the generalization to the cases involving several masses, such as the σ model, it is essential that[5]

$$c_i \frac{\partial \mathcal{L}_i}{\partial c_i} = \mathcal{L}_i \qquad (21)$$

for each constant c_i in the interaction Lagrangian. Unfortunately, in the non-Abelian gauge theories $g_0(\partial/\partial g_0)\mathcal{L}_i \neq \mathcal{L}_i$. Therefore, our method is not applicable to spontaneously broken non-Abelian gauge theories of weak and electromagnetic interactions.

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[12]We define the wave-function renormalization constants for both quarks and non-Abelian gauge particles at the off-mass-shell point $p^2 = -m^2$. Similarly, the renormalized coupling constant g is defined at the off-mass-shell point $p_1^2 = p_2^2 = p_3^2 = -m^2$. Therefore the coupling constant g is free from infrared divergence.

[13]The amplitudes $R_{D\mu\nu}$ and $R_{\lambda\mu\nu s}$ are analytic at $k^2 = q^2 = (k+q)^2 = 0$. In particular, $f(0,0,0)$ exists. This can be understood as follows. The singularities of Feynman graphs are given by Landau rules. The operators D, J_μ, and A_λ are all octets in the usual SU(3), while the non-Abelian gauge particles are singlets in the usual SU(3). We can see readily that, when we cut the graphs, the intermediate states contain at least one quark and one antiquark which have nonzero masses. Therefore the amplitudes $R_{D\mu\nu}$ and $R_{\lambda\mu\nu s}$ are analytic at $k^2 = q^2 = (k+q)^2 = 0$.

[14]The axial-vector current A_5^μ is constructed out of $\mathcal{P}$ and $\mathfrak{N}$ quarks. In the broken-SU(3) model, we have

$m_{\varphi} = m_{\mathfrak{N}} = m_N \neq m_\lambda$. Therefore, instead of Eq. (9) we have equations for $m_N^0(\partial/\partial m_N^0)R_{D\mu\nu}^0$ and $m_\lambda^0(\partial/\partial m_\lambda^0)R_{D\mu\nu}^0$. We also have two Ward identities similar to Eq. (12). With these changes our argument goes through.

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