Extended particles in quantum field theories*

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The method of collective coordinates developed in the study of strong-coupling theory is used for the quantization of the kink solution of a two-dimensional nonlinear field theory. The position of the kink is treated as a collective coordinate, which represents the position of a particle. It is separated from the rest of the coordinates, which represent the internal degrees of freedom of an extended particle. Two similar but different methods are presented; the one is nonrelativistic and suited for the weak-coupling limit, while the other is relativistic.

The present discussion is in the same spirit as our recent papers on strong-coupling theories.[1,2] The method of collective coordinates used in these papers is fairly general and especially suited for the description of extended particles in quantum field theory. As an example we consider in this paper the two-dimensional field theory described by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi\,\partial_\mu\phi + \tfrac{1}{2}\phi^2 - \frac{1}{4\lambda^2}\phi^4 . \tag{1}$$

ϕ is a real field and a convenient unity of mass has been chosen. This field theory is well known to possess the so-called classical kink solutions[3,4]

$$\phi^0(x - X) \equiv \lambda \tanh\left(\frac{x-X}{\sqrt{2}}\right). \tag{2}$$

The parameter X indicates the kink position.

In quantum theory, we start from the transition probability between initial and final states described by the wave functions Ψ_i and Ψ_f,

$$S_{fi} = \int \mathfrak{D}\phi \exp\left[i\int dx\,dt\,\mathcal{L}(\phi)\right]\Psi_f^*[\phi(x,+\infty)] \times \Psi_i[\phi(x,-\infty)], \tag{3}$$

where one integrates over ϕ with the boundary conditions $\phi \to \pm\lambda$ for $x\to\pm\infty$, and consider quantum fluctuations around classical solutions with kink parameters determined as functionals of the field ϕ. Two different methods will be discussed.

First we can write

$$1 = \int \prod_t \left[\mathfrak{D}X(t)\,\delta\left(\int dx\,\phi(x,t)\frac{\partial\phi^0(x-X(t))}{\partial x}\right)\right] J ,$$
$$J = \prod_t \left(\int dx\,\phi(x,t)\frac{\partial^2\phi^0(x-X(t))}{\partial x^2}\right) \tag{4}$$

if we restrict (3) to the case where the argument of the δ function vanishes only once at every time, namely if we consider the case where there is only one kink. Note that the argument of the δ function vanishes at least once because of the boundary conditions. Inserting (4) into (3) we exchange orders of integrations over ϕ and X and let $\phi(x,t) = \phi^0(x - X(t)) + \eta(x,t)$. The δ condition which has been inserted by means of formula (4) now determines how the kink-position degree of freedom is specified as functional of ϕ. This condition has been chosen such that the term involving $\partial\phi/\partial t \equiv \dot\phi$ can be rewritten as

$$\int dx\,\dot\phi^2 = \int dx\,\dot\eta^2 + \dot X^2(M_0 + 2\xi), \tag{5}$$

where

$$M_0 = \frac{2\sqrt{2}}{3}\lambda^2 ,$$
$$\xi = \int dx\,\frac{\partial\eta(x,t)}{\partial x}\,\frac{\partial\phi^0(x-X(t))}{\partial x}.$$

The Hamiltonian formalism is established in a similar way as described in Ref. 1 through the functional integration in phase space. We insert the identities

$$J = \int \mathfrak{D}\tilde p \exp\left(-\frac{i}{2}\int dt\,\frac{\tilde p^2(t)}{M_0(1+\xi/M_0)^2}\right),$$
$$1 = \int \mathfrak{D}\tilde\pi\,\delta\left(\int dx\,\tilde\pi\,\frac{\partial\phi^0(x-X(t))}{\partial x}\right) \times \exp\left(-\frac{i}{2}\int dx\,dt\,\tilde\pi^2(x,t)\right) \tag{6}$$

into the functional expression of S_{fi} and perform the following change of variables to eliminate the explicit $\dot X$, $\dot\eta$ dependence from the Hamiltonian:

$$\tilde{p} = \hat{p} - \dot{X} M_0 \left(1 + \frac{\xi}{M_0}\right)^2 ,$$
$$\tilde{\pi} = \pi - \dot{\eta} - \frac{\xi}{M_0} \dot{X} \frac{\partial \phi^0(x - X(t))}{\partial x} . \tag{7}$$

In order to decouple X and η in lowest order in $1/\lambda$ we go to the kink fixed coordinate system by another change of integration variables:

$$\chi(\rho, t) = \eta(\rho + X(t), t) ,$$
$$\hat{p} = p + \int d\rho\, \pi \frac{\partial \chi}{\partial \rho} . \tag{8}$$

The final result gives

$$S_{fi} = \int \mathfrak{D}X \mathfrak{D}P \mathfrak{D}\chi \mathfrak{D}\pi\, \delta\left(\int d\rho\, \pi \frac{\partial \phi^0}{\partial \rho}\right) \delta\left(\int d\rho\, \chi \frac{\partial \phi^0}{\partial \rho}\right) \times \exp\left[i \int dt \left(p\dot{X} + \int dx\, \pi \dot{\chi} - H\right)\right] \Psi_f^*[X(\infty), \chi(\infty, x)] \Psi_i[X(-\infty), \chi(-\infty, x)] , \tag{9}$$

$$H = M_0 + \frac{\left(p + \int \pi \frac{\partial \chi}{\partial \rho} d\rho\right)^2}{2M_0(1 + \xi/M_0)^2} + \int d\rho \left[\tfrac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{\partial \chi}{\partial \rho}\right)^2 - \frac{1}{2}\left(1 - \frac{3\phi^0(\rho)^2}{\lambda^2}\right)\chi^2 + \frac{1}{\lambda^2}\phi^0(\rho)\chi^3 + \frac{1}{4\lambda^2}\chi^4\right] . \tag{10}$$

To leading order in $1/\lambda$ the term $\int \pi(\partial\chi/\partial\rho)d\rho$ does not contribute and H separates into H_{0k} and $H_{0\,\text{field}}$, where

$$H_{0k} = M_0 + \frac{p^2}{2M_0} ,$$
$$H_{0\,\text{field}} = \int d\rho \left[\tfrac{1}{2}\pi^2 + \frac{1}{2}\left(\frac{\partial \chi}{\partial \rho}\right)^2 - \frac{1}{2}\left(1 - 3\frac{\phi^{02}}{\lambda^2}\right)\chi^2\right]. \tag{11}$$

H_{0k} is the correct nonrelativistic Hamiltonian, and the quantization of $H_{0\,\text{field}}$ by expanding χ and π in terms of eigenfunctions of the corresponding wave equation [i.e., $-\partial^2\chi/\partial\rho^2 - (1 - 3\phi^{02}/\lambda^2)\chi = \omega^2\chi$] has been discussed in Ref. 4. From the δ condition introduced through formula (4) it follows that the mode of $\omega = 0$ should be dropped, in agreement with Ref. 4. It is then straightforward in principle to determine (10) order by order in perturbation theory, and the first-order correction to M_0 is given by $\langle 0 | H_{0\,\text{field}} | 0 \rangle$, which agrees with the result of Ref. 4. Unfortunately, however, this separation of kink variables is not Lorentz invariant. Without explicitly taking pair creation of kinks into account, we cannot be sure that Lorentz invariance will be restored by considering higher orders in $1/\lambda$.

In order to have Lorentz-invariant separation we use a boosted version of formula (2). It involves a parameter θ which later on will be related to the canonical momentum of kink coordinates. We shall use light-cone coordinates to avoid the difficulty of pair creation of kinks, namely

$$V_\pm = \frac{V_0 \mp V_1}{\sqrt{2}} , \quad \tau = \frac{t - x}{\sqrt{2}} , \quad \sigma = \frac{t + x}{\sqrt{2}} , \tag{12}$$

and specify the kink position by $X_-(\tau)$, which we denote by Y. Thus, we write instead of formula (2)

$$\phi^0(u) = \lambda \tanh(u/\sqrt{2}) , \quad u = \frac{e^{\theta(\tau)}}{\sqrt{2}}[\sigma - Y(\tau)] \tag{13}$$

and instead of (4)

$$1 = \int \prod_t [\mathfrak{D}Y(\tau)\mathfrak{D}\theta(\tau)\delta(F_1)\delta(F_2)] J_1 J_2 , \tag{14}$$

where

$$F_1 \equiv \frac{e^\theta}{\sqrt{2}} \int d\sigma \left(\frac{d}{du} u \frac{d\phi^0(u)}{du}\right) \phi(\sigma, \tau) ,$$
$$F_2 \equiv \sqrt{2}\, e^{-\theta} \int d\sigma\, u \left(\frac{\partial \phi}{\partial \sigma}\right)^2 ,$$
$$J_1 = \prod_\tau \frac{\partial F_1}{\partial \theta(\tau)} ,$$
$$J_2 = \prod_\tau \frac{\partial F_2}{\partial Y(\tau)} .$$

We insert this formula into (3), exchange orders of integrations, and let $\phi(\sigma, \tau) = \phi^0(u) + \chi(u, \tau)$. In this case we go directly to kink fixed coordinates (u, τ) since no transformation similar to (7) is needed. One gets

$$S_{fi} = \int \mathfrak{D}Y \mathfrak{D}P \mathfrak{D}\chi\, \delta(F_1)\delta(F_2) J_1 \Psi_f^* \Psi_i \exp\left(i \int d\tau L\right), \quad L = \dot{Y}P + \frac{\mathfrak{M}^2}{2P} + \int du \frac{\partial \chi}{\partial \tau}\frac{\partial \chi}{\partial u} \tag{15}$$

where

$$J_1 = -\int u\,du\left(\frac{d}{du}u\frac{d\phi^0}{du}\right)\frac{\partial\chi}{\partial u},$$
$$P = J_2 = -\frac{e^\theta}{\sqrt{2}}\int du\left(\frac{\partial\phi^0}{\partial u}+\frac{\partial\chi}{\partial u}\right)^2, \tag{16}$$
$$\mathfrak{M}^2 = \int du\left(\frac{\partial\phi^0}{\partial u}+\frac{\partial\chi}{\partial u}\right)^2\left\{\int du\left[\left(\frac{\partial\phi^0}{\partial u}-\frac{\partial\chi}{\partial u}\right)^2-\left(\frac{\partial\chi}{\partial u}\right)^2-\left(1-3\frac{\phi^{02}}{\lambda^2}\right)\chi^2+\frac{2}{\lambda^2}\phi^0\chi^3+\frac{1}{2\lambda^2}\chi^4\right]\right\}$$

and $\mathfrak{D}\theta J_2 = \mathfrak{D}P$ was used.

In order to see the Hamiltonian formalism more clearly we expand χ in terms of orthonormal functions $f_n(u)$, $h_n(u)$ ($n=0,1,\ldots$) which have the properties

$$f_0(u) = \frac{d}{du}u\frac{d\phi^0}{du}\bigg/\left[\int du\left(\frac{d}{du}u\frac{d\phi^0}{du}\right)^2\right]^{1/2},$$
$$f_n(u) = f_n(-u),$$
$$h_n(u) = -h_n(-u), \tag{17}$$
$$\chi(u,\tau) = \sum_{n=0}^{\infty} q_m(\tau)f_m(u) + \tfrac{1}{2}\sum_{m,n} p_m(\tau)(A^{-1})_{mn}h_n(u),$$

where

$$A_{nm} = \int du\,\frac{\partial h_n}{\partial u} f_m .$$

It is then easy to see that $\mathfrak{D}\chi\,\delta(F_1)\delta(F_2)J_1$ can be replaced by $\prod_m dp_m dq_m \delta(q_0)\delta(p_0 - p_0(q_i,p_i))$ so that the q_0 and p_0 integrations can be done. The Lagrangian is given by

$$L = \dot{Y}P + \sum_{i=1}^{\infty}\dot{q}_i p_i + \frac{\mathfrak{M}^2(p_i,q_i)}{2P}, \tag{18}$$

where $\mathfrak{M}^2$ is a function of p_i and q_i ($i=1,\ldots,\infty$), the explicit form of $\mathfrak{M}^2$ being calculated by inserting (17) into (16). The Feynman path integral (15) is then the functional integration in the phase space spanned by Y, P, q_i, and p_i, and it can be shown to be relativistically invariant.

It is instructive to write down the corresponding Schrödinger equation from which Feynman path integral is derived:

$$-i\frac{\partial}{\partial\tau}\Psi = -\frac{\mathfrak{M}^2(p_i,q_i)}{2P}\Psi. \tag{19}$$

This equation is the Klein-Gordon equation expressed in terms of light-cone variables (12), and $\mathfrak{M}^2$ is the mass operator of the extended particle. The coordinates q_i represent the internal degrees of freedom of the particle, and the mass spectrum is simply obtained by the eigenvalues[5] of $\mathfrak{M}^2$.

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[3]This solution has been known in the study of superconductivity. See, e.g., P. G. de Gennes, *Superconductivity of metals and alloys* (Benjamin, New York, 1966).

[4]R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4130 (1974); J. Goldstone and R. Jackiw, Phys. Rev. D 11, 1486 (1975).

[5]This problem together with the renormalization will be discussed in a separate paper.