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Fixed-angle scattering in quantum field theory*

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We have examined the high-energy, fixed-angle behavior of exclusive scattering amplitudes in the context of renormalizable field theories. We find that renormalization-group techniques, asymptotic conformal invariance, and reasonable conjectures about the existence of the zero-mass limit of such theories allow one to understand simple power laws for fixed-angle scattering of the type proposed by Brodsky and Farrar and Matveev *et al.* Since our techniques rely on the existence of a smooth infrared limit of the underlying quantum field theory, they are unfortunately not powerful enough to discuss theories with vector mesons. Consequently, it remains moot whether the classical examples of asymptotically free theories really manifest simple fixed-angle scaling.

I. INTRODUCTION

In models where the physical hadrons are loosely bound collections of fundamental constituents it is possible to give heuristic arguments for simple scaling behavior[1,2] of exclusive scattering amplitudes at high energy and fixed center-of-mass scattering angle:

$$A(s, \theta_{\text{c.m.}}) \sim S^{-n} F(\theta_{\text{c.m.}}).$$

The index n depends on the process and is related to the minimum number of constituents which participate in the process. In broad outline, this picture is consistent with the available experimental information and the question naturally arises as to whether such scaling behavior can be extracted in a reliable way from quantum field theory.

One would like to tackle this problem using the renormalization-group techniques which have proven so effective in studying the asymptotic behavior of semileptonic processes. There are, however, two new elements which make the discussion considerably more complicated: Firstly, all particles are on the mass shell (unlike, say, electroproduction where one of the participating particles is very far off the mass shell) so that the detailed nature of the infrared singularities of the theory is important. Secondly, it makes a great deal of difference to the fixed-angle asymptotic behavior whether the physical particles are directly describable as fundamental fields or whether they must be regarded as bound states. Both of these problems can be finessed in discussions of electroproduction, but must be faced up to squarely in the fixed-angle scattering problem. They are very difficult problems and we have found only partial solutions to them (and so only partial justifications of the dimension-counting scaling rules). Since the problem is so interesting we take the liberty of presenting incomplete results in the hope that it will stimulate somebody to do better.

In Sec. II we discuss how the renormalization group can say something about fixed-angle scattering. We find simple results only for theories with tame infrared behavior (thereby excluding theories with fundamental vector mesons from consideration). In Sec. III we discuss the modifications to the simple picture which arise when the physical particles are bound states. We show how, with the help of conformal invariance, one can extract (almost) the information about the bound-state wave function needed to discuss the fixed-angle limit. In Sec. IV we turn our attention to scalar bound states and explicitly display the interesting limits of the bound-state wave function. In Secs. V and VI we apply our accumulated wisdom to a study of meson form factors and meson-meson scattering and discuss the extent to which the simple dimension-counting rules apply.

II. RENORMALIZATION GROUP AND FIXED-ANGLE SCATTERING

To discuss high-energy on-mass-shell processes, we shall make use of the "improved" renormalization-group equations.[3,4] To obtain these relations, it is necessary to consider n-particle Green's functions

$$\Gamma^{(n)}(p_1 \cdots p_n; g, m, \mu)$$

which have *two* mass parameters, m and μ: μ describes the off-mass-shell point at which renormalization subtractions are carried out and m is a parameter related to, but not identical to, the physical mass. The crucial point is that for $m = 0$, one obtains the Green's functions of the zero-mass theory and μ then is just the mass parameter needed to define the renormalization subtractions in the zero-mass theory. The "improved" renormalization-group equations are

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+n\gamma(g)+\hat{\gamma}(g)m\frac{\partial}{\partial m}\right]\Gamma^{(n)}=0 \tag{2.1}$$

and they have the solution

$$\Gamma^{(n)}(\lambda\vec{p};g,m,\mu)=\lambda^{4-n}\exp\left[n\int_1^{\lambda}\frac{d\lambda'}{\lambda'}\gamma(\bar{g}(\lambda'))\right]\times\Gamma^{(n)}(\vec{p};\bar{g}(\lambda),m(\lambda),\mu)\,, \tag{2.2}$$

where

$$\lambda\frac{d}{d\lambda}g(\lambda)=\beta(g(\lambda)),\quad g(0)=g$$

$$\lambda\frac{d}{d\lambda}m(\lambda)=[-1+\hat{\gamma}(\lambda)]m(\lambda)\,.$$

In the event that $\beta(g)$ has an asymptotic fixed point g_0 and $\hat{\gamma}(g_0)<+1$, $m(\lambda)$ goes to zero as $\lambda\to\infty$ and we have the simple asymptotic solution

$$\Gamma^{(n)}(\lambda\vec{p};g,m,\mu)\to\lambda^{4-n[1-\gamma(g_0)]}\Gamma^{(n)}(\vec{p};g_0,0,\mu)\,. \tag{2.3}$$

We shall assume that both conditions are met so that we may in fact make use of this asymptotic solution.

To discuss on-mass-shell processes we simply replace momentum arguments λp by $\lambda p+\lambda^{-1}r$, with $p^2=r^2=0$. Then, in $\Gamma(\lambda p+\lambda^{-1}r)$, as λ grows, all the momentum invariants grow like λ^2 except for the external masses, which remain fixed. The on-mass-shell high-energy limit is then

$$\Gamma(\lambda p+\lambda^{-1}r;g,m,\mu)\to\lambda^{4-nd}\,\Gamma(p+\lambda^{-2}r;g(\lambda),m(\lambda),\mu)\to\lambda^{4-nd}\,\Gamma(p;g_0,0,\mu)\,, \tag{2.4}$$

where $d=1-\gamma(g_0)$. The high-energy behavior is a power determined only by the anomalous dimension of the fields so long as $\Gamma^{(n)}(p;g_0,0,\mu)$ is finite. This would be automatic if all possible partial sums $(\sum_1^j p_i)$ of momenta were *non*lightlike. We can and do arrange that all partial sums involving two or more momenta be nonlightlike (nonexceptional momenta). This is easily seen to mean that the original massive-particle amplitude is being considered in the limit where all particle energies go to infinity at the same rate while all center-of-mass scattering angles remain fixed. The individual p_i are, however, automatically on the light cone and the question is whether this introduces any singularities.

The answer to this question depends strongly on the theory under consideration. Singularities associated with the usual sort of multiparticle threshold (in s, t, etc.) are avoided by the nonexceptional-momentum requirement. Singularities associated with thresholds in the external particle mass cannot be avoided and correspond to the real decay of a zero-mass particle into other zero-mass particles. The only dangerous decay is into two particles, since decay into three or more particles has vanishing phase space. If the two-particle decay (into collinear massless particles) has vanishing matrix element, then the mass threshold singularity does not occur. This is the case in $\lambda\phi^4$ theories (since only even numbers of mesons participate in any process) and in Yukawa theories (when the fermion part of the matrix element for collinear massless decay vanishes), but not in ϕ^3 or vector-meson theories. (It should be noted that in the Yukawa case it is the zero-mass S-matrix element which is free of singularity—the Green's function has singularities that are removed by supplying external fermion spin projections.) Therefore, barring the effect of more exotic pinch-type singularities, it would appear that the zero-mass S matrix exists for Yukawa and $\lambda\phi^4$ theories. Since we have no systematic way of examining the more exotic singularities, we cannot prove this, but we shall assume it to be true henceforth. If we are right, then the asymptotic behavior of amplitudes at other than exceptional-momentum points is given by the renormalization-group powers, as in Eq. (2.4) and nothing else. If we are wrong, then, in Eq. (2.4), the amplitude $\Gamma(p;g_0,0,\mu)$ has singularities (probably only logarithmic to any finite order) which may well succeed, when summed to all orders, in modifying the renormalization-group power. Since we are trying to *predict* asymptotic behavior, we want to minimize this possibility and we restrict our attention henceforth to Yukawa theories. (At this point we might remark that although the zero-mass problem is very difficult for a general amplitude in a gauge theory, it is much simpler for amplitudes having no *charged* external lines. Since the modern gauge theories of strong interactions more or less require the physical particles to be gauge neutral, there is some hope that our techniques may be applicable. We hope to return to this point in a future publication.)

To summarize: In Yukawa and $\lambda\phi^4$ theories, the high-energy, fixed-angle behavior of on-mass-shell scattering amplitudes is $(\text{energy})^P\times$(function of center-of-mass scattering angles), where P is constructed from the anomalous dimensions of the fields participating in the amplitude, and, for canonical dimensions, is just the naive dimensional power. Thus, for canonical dimensions $A(\pi\pi\to\pi\pi)\to s^0 f(\theta)$, $A(NN\to NN)\to s^{-1}f(\theta)$, etc. Furthermore, it should be noted that $f(\theta)$ is a scattering amplitude computed in the zero-mass theory. Since symmetry breaking is usually the fault of mass terms we are therefore saying that the fixed-angle limit should possess the full symmetry of the underlying field theory. This has dramatic consequences which we do not wish to explore here.

The power-behavior result is encouraging, but the actual powers, if dimensions are nearly canonical, are much too small to have anything to do with the real world. The question then naturally arises whether this numerical situation remains true when the external particles are actually bound states rather than described directly by fundamental fields.

III. THE RENORMALIZATION-GROUP PROPERTIES OF BOUND-STATE AMPLITUDES

In the preceding section we have analyzed the asymptotic behavior of on-mass-shell amplitudes with the aid of the renormalization group. In theories in which the zero-mass singularities could be controlled this behavior was determined by the dimensions of the elementary fields involved. It is apparent, however, that this has little direct relevance to the real world, since hadrons are believed to be bound states rather than elementary particles. We therefore analyze, in this section, the properties of bound-stated amplitudes. The new ingredient that enters into the construction of such amplitudes is the bound-state wave function. It, however, is determined by the Bethe-Salpeter equation, whose kernel is simply a Green's function with appropriate topological properties. We therefore are able to establish, for a large class of bound states, that the bound-state wave functions and thereby the resulting amplitudes obey a renormalization-group equation. One could hope, therefore, to employ the methods developed in the previous section to discuss the asymptotic behavior of such amplitudes. On attempting to do this, one discovers that the zero-mass singularities of amplitudes involving bound states can be quite significant. Indeed they can, and do, modify by powers the asymptotic behavior one expects from naive dimension counting. Furthermore, knowledge of the scaling behavior of the bound-state wave function which follows from the renormalization group alone is not sufficient to determine these singularities. One must, in fact, possess quite detailed information on the behavior of this wave function as various momenta become lightlike, information which is not determined by overall scaling. To obtain such information we investigate the constraints of conformal transformations. This will be employed in the following section to analyze the zero-mass singularities of amplitudes involving bound states.

Let us consider the bound-state wave function of a scalar meson. We shall for the moment, consider fermion-antifermion bound states. In other words, the meson is a dynamical pole in the $J = 0$ fermion-antifermion scattering amplitude. The wave function is defined to be

$$\Gamma_{\alpha\beta}(p;q_1q_2;gm\mu) = \text{const}\times\int d^4x\, e^{iq\cdot x}\langle p|(\bar{\psi}_\alpha(x)\psi_\beta(0))_+|0\rangle\,, \tag{3.1}$$

where we have exhibited the dependence of Γ upon the coupling constants of the theory (represented by g), the mass parameters (represented by m) and the renormalization point. The mass of the bound state is $p^2 = m_B{}^2$.

The above bound-state wave function is a solution of a homogeneous Bethe-Salpeter equation[5] (see Fig. 1):

$$\Gamma_{\alpha\beta}(p;q_1q_2;gm\mu) = \int \frac{d^4k}{(2\pi)^4}\Gamma_{\alpha'\beta'}(p;k-p,k;gm\mu) \times T_{\alpha\beta,\alpha'\beta'}(k-p,k;q_1,q_2;gm\mu)\,. \tag{3.2}$$

The kernel of this equation, T, is the two-particle irreducible fermion-antifermion Green's function with two external propagators attached.

Let us now apply the homogeneous renormalization-group operator

$$D = \mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \tilde{\gamma}(g)m\frac{\partial}{\partial m} \tag{3.3}$$

to this equation. One might be concerned that this is not allowed, since Eq. (3.2) is only valid when $p^2 = m_B{}^2$, and m_B, of course, is a function of the dynamical parameters μ, g, and m. However, a physical parameter such as a bound-state mass is, of course, independent of where one chooses to perform the renormalization. In other words, all physical parameters, including m_B, must satisfy the homogeneous renormalization-group equation

$$Dm_B(g,m,\mu) = 0$$

so that one can apply D to Eq. (3.2) with impunity.

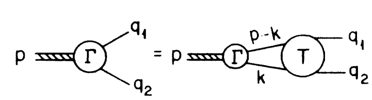

FIG. 1. The Bethe-Salpeter equation for the bound-state wave function.

One thereby derives, symbolically, that

$$D\Gamma=(D\Gamma)\otimes T+\Gamma\otimes DT\,.$$

However, T itself satisfies the homogeneous renormalization-group equation

$$DT=0\,. \tag{3.4}$$

The fact that T is not the full Green's function, but only the two-particle irreducible part, does not invalidate this equation. The requirement of two-particle irreducibility is a topological requirement on the Feynman graphs that contribute to T that does not affect the fact that it is primitively convergent. Thus, it, like the full amplitude, has a skeleton graph expansion in terms of the primitively divergent propagators and vertices of the theory. Since these obey an appropriate renormalization-group equation one establishes that the full Green's function does. The absence of anomalous dimensions in Eq. (3.4) is a consequence of the fact that T is "half amputated," i.e., that it contains two external fermion propagators. It therefore follows that $D\Gamma$ obeys the *same* homogeneous Bethe-Salpeter equation as does Γ. If, as seems reasonable, the solution of the Bethe-Salpeter equation is unique, then $D\Gamma$ must be proportional to Γ, i.e., $D\Gamma=\alpha\Gamma$.

The value of α will of course depend on how the bound-state wave function is normalized. Although this normalization is totally arbitrary, and can have no ultimate physical significance, it will affect the value of α. It is clear that one should choose a normalization which makes sense in the zero-mass theory, since in the renormalization-group approach, the zero-mass theory governs asymptotic behavior. We therefore normalize Γ by requiring that at $p^2=m_B^{\,2}$ the fermion-antifermion four-point Green's function $\Gamma^{(4)}(p_1p_2p_3p_4)$ be equal to

$$\Gamma(p;p_1p_2)\Gamma_{\theta\theta}(p)\Gamma(p;p_3p_4)\,, \tag{3.5}$$

where $\Gamma_{\theta\theta}$ is the "propagator" of the operator $\theta=\bar\psi\psi$. This two-point function shares the bound-state pole, and, like $\Gamma^{(4)}$, is renormalized in the new improved manner so as to have a well-defined zero-mass limit.

Applying D to Eq. (3.5) and remembering that $D\Gamma^{(4)}=-4\gamma_\psi\Gamma^{(4)}$ and $D\Gamma_{\theta\theta}=-2\gamma_\theta\Gamma_{\theta\theta}$ (γ_ψ and γ_θ being the anomalous dimensions of the fermion field and of θ), we derive that $\alpha=\gamma_\theta-2\gamma_\psi$ so that

$$\left\{\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\hat\gamma(g)m\,\frac{\partial}{\partial m}+[2\gamma_\psi(g)-\gamma_\theta(g)]\right\}\Gamma_{\alpha\beta}(p;q_1q_2)=0\,. \tag{3.6}$$

According to the above normalization Γ has physical dimension four so that we derive in the standard fashion, using dimensional analysis and integrating Eq. (3.6), that

$$\Gamma(\lambda p;\lambda q_1\lambda q_2;gm\mu)=\lambda^{-4}\Gamma(p;q_1q_2;g(\lambda),m(\lambda),\mu)\times\exp\left\{\int_1^\lambda\frac{d\lambda'}{\lambda'}[2\gamma_\psi(g(\lambda'))-\gamma_\theta(g(\lambda'))]\right\}. \tag{3.7}$$

Thus, when λ approaches infinity, we relate the bound-state wave function evaluated on-shell, $\lambda^2p^2=m_B^{\,2}$, and with large momenta to the bound-state wave function evaluated in the zero-mass theory [as long as $m(\lambda)$ vanishes in this limit]. If, say, the theory possesses an ultraviolet-stable fixed point $g_0\neq 0$ [and $\hat\gamma(g_0)<1$], then

$$\Gamma_{\alpha\beta}(\lambda p;\lambda q_1\lambda q_2;gm\mu)\underset{\lambda\to\infty}{\sim}\lambda^{-4+2\gamma_\psi-\gamma_\theta}\,\Gamma_{\alpha\beta}(p;q_1q_2;g_00\mu)\,. \tag{3.8}$$

A consistenty check on Eq. (3.8) is provided by considering the large-q limit of Eq. (3.1) with the aid of the operator-product expansion. The above dependence is an immediate consequence of the fact that for short distances we have

$$\bar\psi(x)\psi(0)\underset{x\to 0}{\approx}\bar\psi\psi(0)(x^2)^{-(2\gamma_\psi-\gamma_\theta)/2}.$$

This argument also clarifies the special role played by the composite field $\bar\psi\psi$. As an interpolating field for the bound state $\bar\psi\psi$ has, of course, no special significance—any operator with the same quantum numbers would be equally good. The reason that the dimension of $\bar\psi\psi$ controls the large-momentum behavior of the bound-state wave function is that it is the operator of *lowest* dimension for the quantum numbers of the bound state.

It should also be pointed out that in writing Eq. (3.8) in the first place, we are making an implicit assumption that the zero-mass limit of $\Gamma_{\alpha\beta}$ exists. This is not really guaranteed by anything we have said so far. In fact, if we think about how one would treat this problem by directly writing down the full operator-product expansion for $\bar\psi(x)\psi(0)$, it is not too difficult to see that our assumption is correct, provided that the anomalous dimensions of the twist-two operators of spin greater than zero are all greater than the anomalous dimension of the operator of spin zero. In the simple model we are studying this in the case, although one could imagine circumstances in which it is not.

One is now in a position to apply the renormalization group to amplitudes involving bound states. These amplitudes are constructed from Γ and from two-particle irreducible Green's functions. In Fig. 2 we illustrate this construction, where the amplitude T' represents a two-particle irreducible amputated Green's function. Since we have seen that Γ obeys the renormalization-group equation, where the bound state has an effective anomalous dimension equal to $\gamma_\theta(g)$, it trivially follows that all amplitudes in which it appears will also satisfy the renormalization-group equation.

Therefore, in the standard fashion, one can attempt to calculate the large-momentum behavior of bound-state amplitudes by employing the renormalization group to relate them to the zero-mass theory. Consider, for example, the scalar-meson electromagnetic form factor $F_\mu(q; p_1^2 = p_2^2 = m_B^2)$. This can be constructed from the bound-state wave function and the amplitude T_μ, which is the amputated current four-fermion Green's function, two-particle irreducible in the meson channels (see Fig. 3).

It thus satisfies the renormalization-group equation, from which one derives $[(\lambda p_i)^2 = m_B^2]$

$$F_\mu(\lambda q; \lambda p_1 \lambda p_2; g, m, \mu) \underset{\lambda\to\infty}{\sim} \lambda^{-3-2\gamma_\theta} F_\mu(q; p_1 p_2; g_0, m(\lambda), \mu).$$

Now if the form F_μ has a finite limit as the internal mass parameters vanish $[m(\lambda) \to \lambda^{-1+\bar{\gamma}(g_0)}]$ and the external bound-state masses vanish ($p_i^2 = m_B^2/\lambda^{-2}$), then one derives for the invariant form factor

$$F_\mu(q; p_1 p_2) = (p_1 - p_2)_\mu F(q^2)$$

that

$$F(q^2) \to (-q^2)^{-[2+\gamma_\theta(g_0)]}. \tag{3.9}$$

However, the existence of such a zero-mass limit for the form factor, as well as for any other bound-state amplitude, is highly unlikely. This is because the bound-state wave function $\Gamma_{\alpha\beta}(p; q_1 q_2)$ in the zero-mass theory is a homogeneous function of q_1^2 and q_2^2 of degree $2 + \gamma_F - \frac{1}{2}\gamma_\theta$. This means that

$$\Gamma_{\alpha\beta}(p; q_1 q_2; g_0 0 \mu) = (q_1^2 q_2^2)^{-[1+(\gamma_F/2)-(\gamma_\theta/4)]} F(q_1^2/q_2^2),$$

with the function f undetermined by overall scaling. If the anomalous dimensions are small, this implies that the singularities of $\Gamma_{\alpha\beta}$ as either q_1^2 or q_2^2 vanishes could be quite severe. Certainly one expects a more singular behavior than the $1/\not{q}$ singularity which would appear if $\Gamma_{\alpha\beta}$ described the coupling of an elementary scalar meson to two fermions. This can then lead to zero-mass singularities for amplitudes constructed out of Γ (as in Fig. 2), even in theories, such as Yukawa theories, where these are absent in the on-mass-shell amplitudes of the fundamental particles. These singularities will, in general, lead to power divergences of the zero-mass bound-state amplitudes. Thus, for example, $F_\mu(q; p_1 p_2; g_0 m(\lambda)\mu)$ could diverge like a power of λ as the internal and external masses vanish. As a consequence, the naive power-counting behavior of Fig. 2 will be altered by powers of q^2.

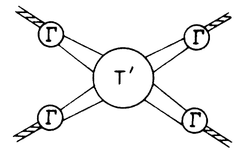

FIG. 2. The four-particle bound-state amplitude.

The only statement one can make with certainty, at this point, is that Eq. (3.9) provides a lower bound on the possible falloff of the form factor. To say more one must have information regarding the behavior of the bound-state wave function as the fermion masses vanish.

For the applications we have studied it appears that the necessary extra information can be extracted from the conformal invariance of the zero-mass theory at the fixed point. Since the meson bound-state wave function is essentially a three-point function, it is completely determined in the zero-mass limit by conformal invariance and we shall exploit this fact in what follows. Were we to study three-body bound states (the nucleon, for instance) we would need more powerful methods, based on the operator-product expansion.

The conformal properties of the bound-state wave function can be established from the conformal Ward identities satisfied by the Green's functions of the theory.[6] It has been shown that renormalizable, zero-mass theories, with the couplings set equal to fixed points of the renormalization group, are conformally invariant. (Gauge theories require special treatment. This will be explained below.) Actually a broken conformal Ward identity holds even when $\beta \neq 0$ and the masses are finite. However, these equations, unlike the scale invariance Ward identities, cannot be written as simple differential equations for the Green's functions, and appear to be useless. Since we are only concerned, however, with the properties of the bound-state wave function in the zero-mass, fixed-point theory, we will restrict our attention to exact con-

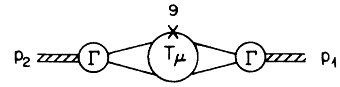

FIG. 3. The bound-state electromagnetic form factor.

formal invariance.

The conformal Ward identiies have the following form for the one-particle irreducible Green's functions $G^{(n)}(p_1 \cdots p_{n-1})$:

$$K_\mu G^{(n)} = \sum_{i=1}^{n} K_\mu^{(i)}(p_i) G^{(n)}(p_1 \cdots p_{n-1}) = 0\,, \qquad (3.10)$$

where $K_\mu^{(i)}(p)$ is the conformal differential operator appropriate to a field of spin J_i and dimension d_i:

$$K_\mu^{(i)}(p) = -2\left(d_i + p\cdot\frac{\partial}{\partial p}\right)\frac{\partial}{\partial p^\mu} + p_\mu \frac{\partial^2}{\partial p^2} + \Sigma_{\mu\nu}^{J_i}\frac{\partial}{\partial p_\nu}\,, \qquad (3.11)$$

where $\Sigma_{\mu\nu}^{J_i}$ is the Lorentz transformation matrix of a field with spin J_i (i.e., $\Sigma_{\mu\nu}^{0}\phi = 0$, $\Sigma_{\mu\nu}^{1/2}\psi = \frac{1}{4}i[\gamma_\mu, \gamma_\nu]\psi$, etc.).

The kernel of the Bethe-Salpeter equation for the bound state is not a full one-particle irreducible Green's function and so it need not in general satisfy Eq. (3.10). However, as we pointed out in discussing the renormalization-group behavior of the bound-state wave function, we have arranged things so that the kernel has a skeleton expansion and is constructed by integrating products of the fundamental primitively divergent Green's functions of the theory. The conformal generator, Eq. (3.11) has a distributive property which guarantees that if the primitive Green's functions satisfy Eq. (3.10), then so does any amplitude constructed via the skeleton expansion. The Bethe-Salpeter equation for the zero-mass, fixed-point version of the bound-state wave function of course has a kernel constructed out of the primitively divergent Green's functions of the massless fixed-point theory. Since these objects are conformal-invariant then so is the kernel, and by an argument similar to the one used in discussing renormalization-group properties of the wave function, so is the bound-state wave function itself. We shall use this fact in the next section to derive the explicit asymptotic form of the meson wave function.

Although we have focused attention on fermion-antifermion scalar bound states in a Yukawa theory, it is clear that the results have much wider applicability. The renormalization-group equation derived for the bound-state wave function will be valid for any bound state or theory, as long as the kernel of the Bethe-Salpeter equation which generates the bound state satisfies such an equation. Thus a fermion-antifermion bound state of spin J will satisfy Eq. (3.6) if we replace γ_θ by the anomalous dimension of the lowest-dimension operator with the quantum numbers of the bound state (i.e., the operator $\bar\psi\gamma_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\cdots\overleftrightarrow{\partial}_{\mu_n}\psi$). Similarly, a three-fermion bound-state wave function (say, the nucleon wave function in a quark model) $\Gamma(p_1q_1q_2q_3)$ will satisfy

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} + \hat\gamma(g)m\frac{\partial}{\partial m} + 3\gamma_\psi(g) - \gamma_3(g)\right]\times\Gamma(p_i q_1 q_2 q_3) = 0\,,$$

where γ_3 is the anomalous dimension of the operator ψ^3.

An exception to this list is a scalar bound state of two scalar mesons in a ϕ^4 theory. Since the four-point scalar Green's function is primitively divergent the two-particle irreducible kernel of the Bethe-Salpeter equation will not satisfy the standard renormalization group equation. A modified equation could probably be derived for this kernel, and thus for the scalar wave function. However, it is unclear whether the analog of β in such an equation would share the fixed points of β. We must therefore exclude from consideration all bound states which can appear in two scalar-meson channels. If we consider a quark model with a neutral singlet Yukawa coupling, we need only exclude singlet mesons. In a gauge theory which does not involve scalar mesons no such problem arises.

These wave functions will then be conformally invariant in the zero-mass theory at the fixed point of the renormalization group. Conformal invariance for such bound states will, however, be much less informative. A three-body bound-state wave function is essentially a four-point function, which is not totally determined by conformal invariance. Gauge theories require special treatment. The conventional gauge-fixing term that one must add to a Lagrangian which is invariant under gauge transformations of the second kind is not conformally invariant. Thus the standard Feynman rules for gauge theories do not give conformally invariant results for gauge-dependent amplitudes, even in the zero-mass, fixed-point theory. This difficulty can be overcome by working in a special nonlocal gauge,[7] where the vector-meson propagator in position space is given by

$$D_{\mu\nu}(x,y) = \left(\frac{\kappa}{4} - 1\right)\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\ln(x-y)^2 - \frac{1}{2}\frac{(x-y)^2}{(x-B)^2(y-B)^2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\frac{(x-B)^2(y-B)^2}{(x-y)^2}.$$

If the gauge parameter B_μ is taken to transform like a position four-vector under conformal transformations, then D_μ is conformally covariant. In the limit $B_\mu \to 0$ one recovers the usual Fermi-type gauges (if $\kappa = 0$, then as $B_\mu \to 0$ one recovers the Feynman gauge). With this choice of gauge the bound-state wave function will again be conformally invariant in the zero-mass, fixed-point theory. The price that one pays, however, is that the wave function depends on an additional four-vector, B_μ. Thus a two-body bound-state wave function will not be completely determined by conformal invariance and we will require additional input.

Asymptotically free theories again require special treatment. This is because the ultraviolet-stable fixed point of such theories is at zero coupling. One cannot, therefore, discuss the asymptotic properties of the bound-state wave function by examining the fixed-point theory. Here, different methods, based on the operator-product expansion, will prove more useful.

IV. MESON WAVE FUNCTION

The net result of the arguments of last section is that the form of a two-body bound-state wave function, $\Gamma(q_1^2, q_2^2)$, is completely determined in the limit $q_1^2, q_2^2 \to \infty$ (q_1^2/q_2^2 arbitrary) by the constraints of the conformal invariance. Since it is essentially this limit which governs the interesting practical applications, we would like to work it out explicitly for the meson wave function.

We are interested in computing

$$\begin{aligned}\Gamma_{\alpha\beta}(p,x) &= \langle\pi(p)|T(\psi_\alpha(x)\bar\psi_\beta(0))|0\rangle \\ &= \delta_{\alpha\beta}(x^2)^{-(2\gamma+\gamma_\theta)/2}g_1(p\cdot x, x^2p^2) \\ &\quad + [\not p, \not x]_{\alpha\beta}(x^2)^{-(2\gamma+\gamma_\theta)/2}g_2(p\cdot x, x^2p^2).\end{aligned} \tag{4.1}$$

In writing Γ out in terms of explicit covariants we have (a) rejected solutions with odd numbers of γ matrices in the high-energy or zero-mass limit (one must have either even or odd numbers of γ matrices and perturbation theory suggests we should choose even, and (b) explicitly displayed the correct $x\to 0$ behavior required by the short-distance expansion.

We require the zero-mass fixed-point version of this three-point function in order to determine the high-energy behavior of the wave function. The conformal-invariant three-point function is of course defined for all p^2, while the bound-state wave function is of interest only for $p^2=0$. Thus we should construct a conformal-covariant three-point function which has a smooth $p^2\to 0$ limit.

The full conformal-covariant three-point function is obtained by solving[6]

$$K_\mu(x)\Gamma(p,x) = \bar K_\mu(p)\Gamma(p,x),$$

where

$$K_\mu(x) = 2x_\mu(d + x\cdot\partial) - x^2\partial_\mu + \tfrac{1}{2}[\gamma_\mu, \not x],$$

$$\bar K_\mu(p) = 2\left(d_\theta + p\cdot\frac{\partial}{\partial p}\right)\frac{\partial}{\partial p^\mu} - p_\mu\frac{\partial^2}{\partial p^2},$$

$$d = \tfrac{3}{2}+\gamma, \quad d_\theta = 3+\gamma_\theta.$$

Since we want Γ at $p^2=0$, and by assumption the $p^2\to 0$ limit is smooth, it is enough to set $p^2=0$ inside Eq. (4.1) and ignore all terms proportional to p_μ (since they arise from differentiating the p^2x^2 dependence of the g_i). We are left with three independent equations for $g_i(p\cdot x)$, the coefficients respectively of $[\gamma_\mu, \not x]$, γ_μ, $x_\mu[\not p, \not x]$:

$$\tfrac{1}{2}g_1 + (p\cdot x)g_2 = -2i\hat d g_2 - 2ip\cdot x g_2',$$

$$d_\theta g_1 + 2p\cdot x g_2 = -2i[dg_1' + (p\cdot x)g_1''],$$

$$(d_\theta+1)g_1 + 2p\cdot x g_2' = -2i(d+1)g_2' - 2ip\cdot x g_2''.$$

These equations have the solution

$$g_1(p\cdot x) = \int_0^1 du[u(1-\omega)]^{(d_\theta-3)/2}e^{iup\cdot x},$$

$$g_2(p\cdot x) = \frac{i(2-d+\frac{1}{2}d_\theta)}{d_\theta-1}\int_0^1 du[u(1-u)]^{(d_\theta-1)/2}e^{iup\cdot x},$$

which yields, when Fourier transformed,

$$\Gamma_{\alpha\beta}(q_1q_2) = \delta_{\alpha\beta}\int_0^1 du\frac{[u(1-u)]^{(d_\theta-3)/2}}{[q_1^2u+q_2^2(1-u)]^{2+(d_\theta-2d)/2}} + \frac{(2-d+\frac{1}{2}d_\theta)}{d_\theta-1}[\not p, \not q_1-\not q_2]_{\alpha\beta}\int_0^1 du\frac{[u(1-u)]^{(d_\theta-1)/2}}{[q_1^2u+q_2^2(1-u)]^{3+(d_\theta-2d)/2}}.$$

The most important feature of this expression is that while Γ falls like $\lambda^{-[2+(d_\theta-2d)/2]}$ when both q_1^2 and q_2^2 are growing like λ, it falls like $\lambda^{-[2+(d_\theta-2d)/2]}$ when only *one* of the two momentum variables is growing like λ (as long as $d<3$). For canonical dimensions, the two powers are $(q^2)^{-2}$ and $(q^2)^{-1}$, respectively, so that it is the "corner" $q_1^2\to\infty$, fixed q_2^2 which will dominate asymptotic behavior. In fact if we evaluate Γ in the region where $q_1^2/q_2^2\to\infty$, we find

$$\Gamma_{\alpha\beta}(q_1q_2) \to \left(\frac{1}{q_1^2}\right)^{(d_\theta-1)/2}\left(\frac{1}{q_2^2}\right)^{5/2-d}\times\left(1+\frac{[\not p, \not q_1-\not q_2]}{2q_1^2}\right).$$

Finally, we remark that the calculational technique we use is, strictly speaking, accurate only in the limit that q_1^2 and q_2^2 are *both* large compared to the internal mass scale. Thus we can assert that Γ behaves like $(q_1^2)^{-(d_\theta-1)/2}$ for growing q_1^2 and fixed q_2^2 only for q_2^2 *large* compared to internal masses. It turns out that to discuss form factors, it is necessary to know how Γ behaves for growing q_1^2 and q_2^2 fixed and *not* necessarily large. We will *assume* that the large-q_1^2 behavior for fixed q_2^2 is insensitive to the value of q_2^2, so that our asymptotic evaluation is adequate.

By way of illustration, we have solved the Bethe-Salpeter equation for the ladder approximation to a Yukawa theory. The solution is in fact given by Eq. (4.2), as we have argued on general grounds that it must be. Details are given in the Appendix.

V. FORM FACTORS

In this section we shall study the asymptotic behavior of bound-state form factors using the results derived above. We shall first discuss the simplest of all cases—namely, the electromagnetic form factor of a scalar-meson bound state of two fermions in a nongauge theory. The electromagnetic form factor of such a bound state, $V_\mu(q;p_1p_2)$, can be calculated in terms of the bound-state wave function $\Gamma_{\alpha\beta}$ as follows (see Fig. 4):

$$\begin{aligned}V_\mu(q;p_1p_2) &= (p_1-p_2)_\mu F(q^2)\\ &= \int\frac{d^4k}{(2\pi)^4}\mathrm{tr}[\Gamma(p_1;-p_2+k,k)\Gamma_\mu(q;p_1+k,-k+p_2)\Gamma(p_2;-k,k+p_2)\Gamma_2(k)]\\ &\quad+\int\frac{d^4k_1d^4k_2}{(2\pi)^8}\mathrm{tr}[\Gamma(p_1;-\tfrac{1}{2}p_1+k_1,-\tfrac{1}{2}p_1-k_1)\\ &\qquad\times T_\mu(q;\tfrac{1}{2}p_1-k_1,\tfrac{1}{2}p_1+k_1,\tfrac{1}{2}p_2-k_2,\tfrac{1}{2}p_2+k_2)\Gamma(p_2;-\tfrac{1}{2}p_2+k_2,-\tfrac{1}{2}p_2-k_2)]\,.\end{aligned}\tag{5.1}$$

Here Γ_μ is the 1PI (one-particle irreducible) vertex function of the electromagnetic current for the fermions, Γ_2 is the inverse fermion propagator, and T_μ is the 1PI connected fermion-current vertex function. T_μ is also, of course, two-particle irreducible in the bound-state channels (see Fig. 5). We have explicitly separated out the disconnected part of T_μ, which is given by the first term in this equation.

Since all the amplitudes in Eq. (5.1) satisfy the renormalization group, it follows that so does V_μ. In fact, one easily derives that

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g)\frac{\partial}{\partial g}+\hat{\gamma}(g)m\frac{\partial}{\partial m}-2\gamma_\theta(g)\right]V_\mu(q;p_1p_2;gm\mu)=0\,.$$

Solving this equation one would then derive that

$$F(q^2)\sim\left(\frac{1}{q^2}\right)^{d_\theta-1}=\left(\frac{1}{q^2}\right)^{2+\gamma_\theta}\tag{5.2}$$

if the zero-mass fixed-point limit of V_μ were to exist. This, of course, is the naive dimension-counting result, the bound state behaving like a field of dimension d_θ. However, this is wrong. The vertex function does contain zero-mass singularities, and these will cause the form factor to vanish less rapidly.

The reason we have separated in Fig. 4 the contributions to Γ_μ coming from the disconnected and the connected graphs is that these have quite different zero-mass singularities. The zero-mass singularities of the totally disconnected graphs are easily analyzed. We are interested in possible divergences for these graphs in the zero-mass theory, when the external masses are of order λ^{-2} as $\lambda\sim(q^2)^{1/2}\to\infty$. The singularities arise because of the threshold singularities in p_1^2 and p_2^2 when

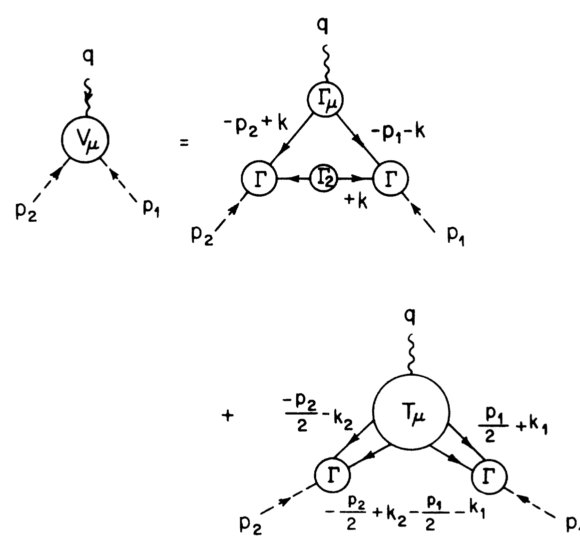

FIG. 4. The connectedness structure of the bound-state form factor.

these vanish. Focusing on the singularity which arises as $p_2^2 \sim \lambda^{-2} \to 0$, we see that such a singularity will occur when $K^2 \sim 0$, $(K+p_2)^2 \sim 0$. The strength of this singularity is easily derived by power counting. The bound-state wave function of one meson contributes a factor of $(\lambda^2)^{2+d_\theta/2-d_\psi}$. The bound-state wave function of the other meson contributes a factor of $(\lambda^2)^{5/2-d_\psi}$ arising from the vanishing of *one* of its legs. A factor of $(\lambda^2)^{d_\psi-3/2}$ arises from the inverse fermion propagator, and the phase space for this threshold yields a factor of λ^{-4}. All in all, this means that V_μ behaves like $(\lambda^2)^{1+d_\theta/2-d_\psi}$ in the zero-mass fixed-point theory when the external masses vanish like λ^{-2}. If this is the dominant zero-mass singularity, the net effect is to change the behavior indicated in Eq. (5.2) to

$$F(q^2) \to \left(\frac{1}{q^2}\right)^{d_\theta/2+d_\psi-2} = \left(\frac{1}{q^2}\right)^{1+\gamma_\theta/2+\gamma_\psi}. \qquad (5.3)$$

It is easy indeed to verify that the connected contributions, included in T_μ, cannot contain such a strong zero-mass singularity. The disconnected contribution dominates since it contains a fermion line common to both bound-state wave functions, whose mass-shell singularity is thereby doubly enhanced.

An alternative way of deriving the above result is by a direct investigation of the various contributions to V_μ as $q^2 \to \infty$, keeping $p_i^2 = m_p^2$. The disconnected contribution in Fig. 4 is easily analyzed. Here the large momentum, q, flowing into the graph, must pass either through *one* of the legs, $p_i \pm k$, or through all internal legs. In the latter case, we of course recover the naive behavior, for large q^2, as given in Eq. (5.2). The dominant contribution, however, arises when the large momentum flows through the line carrying momentum $-p_1-k$ (or p_2-k), so that $q^2 \sim (p_1+k)^2 \to \infty$. First, we get a contribution of $(1/q^2)^{1+\gamma_\theta/2}$ from the bound-state wave function. This then must be multiplied by the large-q^2 behavior of the 1PI current fermion vertex. With the insertion of an inverse fermion propagator, with momentum $(-p_1-k)$, this vertex is essentially given by

$$\left\langle p_2 - k \left| \int d^4x\, e^{iq\cdot x}\, T(J_\mu(x)\psi(0)) \right| 0 \right\rangle.$$

In the large-q limit we can use the operator-product expansion, which is dominated for small x^2 by the operator ψ itself: $J_\mu(x)\psi(0) \sim \not{x}\gamma_\mu\psi/x^4$. Thus, since J_μ has no anomalous dimension, the 1PI vertex will only differ from its canonical behavior owing to the inverse fermion propagator, and thus can be replaced with $\gamma_\mu[(p_1+k)^2]^{-\gamma_\psi} \sim \gamma_\mu/q^{2\gamma_\psi}$. This result could also be established using the renormalization-group equation for the vertex plus the fact that this vertex has *no* zero-mass singularities. Multiplying V_μ by $(p_1-p_2)_\mu$ we therefore obtain from this region that

$$q^2F(q^2) \sim \int d^4k\, f(k,p_1p_2)\, q\cdot k/(q^2)^{1+\gamma_\theta/2+\gamma_\psi}$$
$$\sim q^2/(q^2)^{1+\gamma_\theta/2+\gamma_\psi}$$

in accord with our previous analysis. In fact, the dominant region analyzed here is exactly the same region which gave rise, when all momenta were scaled down by $1/(q^2)^{1/2}$, to the dominant zero-mass singularity.

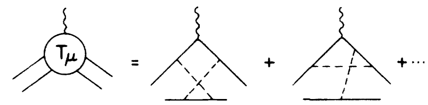

FIG. 5. Graphs that contribute to the one-particle irreducible fermion current vertex function.

The fact that the contributions coming from the connected graphs represented by T_μ fall off faster for large q^2 is even more transparent following the above line of reasoning, even if we concentrate on a region for which the large momentum q flows out of T_μ through only one fermion leg (other possibilities are obviously suppressed). This is because the operator ψ can no longer contribute to the operator-product expansion of T_μ, for its contribution is not two-particle irreducible in the bound-state channels. This is illustrated in Fig. 6. Consequently, the leading contribution will arise from a dimension-four-or-greater operator (e.g., $J_\mu\psi = \phi\gamma_\mu\psi/x^4$) and thus will be less singular on the light cone and give rise (for canonical dimensions) to a term behaving like $1/q^4$ or faster in $F(q^2)$.

The extension of the above results to gauge theories, even at a nonzero fixed point of β where we are able to determine the asymptotic behavior of the bound-state wave function, is not straightforward. The zero-mass singularities of gauge theories are so strong that they destroy the above arguments. In fact, in gauge theories the connected amplitude T_μ can give rise to zero-mass singularities of comparable strength to the dis-

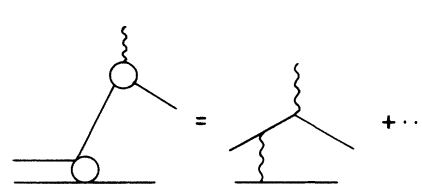

FIG. 6. The contribution of the operator ψ to T_μ.

connected contributions. This is easily seen using the ultraviolet analysis, since in the presence of gauge fields the operator-product expansion of J_μ and ψ can contain operators of the form $x\cdot A\not{x}\gamma_\mu\psi/x^4$, which are of comparable strength to $\not{x}\gamma_\mu\psi/x^{-4}$ and can arise from two-particle irreducible graphs (for example, Fig. 6 contains such a term). Thus the analysis of bound-state form factors in gauge theories, including asymptotically free theories, is much more involved, and will be discussed elsewhere.

The reader might wonder why we have gone through such contortions. After all, the form factor is constructed from amplitudes which, at the fixed point in the zero-mass theory, are explicitly conformally invariant. Thus one would expect the form factor itself to be conformally invariant, and therefore one should be able to determine the large-q^2 behavior by imposing conformal invariance on the zero-mass theory. This was the method we employed to determine the asymptotic behavior of the bound-state wave functions without having to analyze specific Feynman diagrams. However, it is easily verified that if one demands that the zero-mass form factor be conformally invariant, one does not recover the q^2 behavior, Eq. (5.3), derived above. This apparent paradox is resolved by the realization that even though V_μ is constructed from conformally invariant ingredients, Γ_μ, F, and Γ_2 say, it does not satisfy the conformal Ward identity. In fact, if one applies the conformal operator to V_μ, i.e., $[K_\mu(q)+K_\mu(p_1)+K_\mu(p_2)]V_\mu$, one finds that the integration by parts required to show that this yields zero is not allowed owing to the increased infrared singularities generated by the bound-state wave function. Thus there exists in bound-state amplitudes an infrared conformal anomaly. It would be very nice if one could control this anomaly, since the use of conformal invariance would probably greatly simplify the above discussions. Most inportant it would allow us to treat gauge theories directly.

VI. MESON-MESON SCATTERING

The generic amplitude for "π-π" scattering is displayed in Fig. 2 where four pion bound-state wave functions are tied together by an eight-quark scattering amplitude which is two-particle irreducible in each single external meson channel. Our "renormalization-group" analysis has told us how the bound-state wave function behaves when one or both leg masses is large and also how the fundamental field scattering amplitude behaves for large energy in all channels. Both pieces of information will turn out to be essential.

Fixed-angle scattering is notoriously complicated to analyze, in part because of the many possible connectedness structures of the eight-quark amplitude appearing in Fig. 2, each of which seems to have different asymptotic behavior. Several possibilities are shown in Figs. 7(a)–7(c). Figures 7(a) and 7(b) essentially correspond to the parton interchange model and Fig. 7(c) corresponds to the "pinch" graph considered by Landshoff.[8]

It would be supererogatory on our part to redo the entire two-body scattering problem. Much wisdom has already been accumulated by parton-model enthusiasts[9] in the study of particular diagrams, and there is a consensus that the "pinch" diagram of Landshoff is the winner. Our aim will be to show that the detailed information contributed by the renormalization group on the manner in which bound-state wave functions fall off at large mass, as well as on the way asymptotic logarithms sum to powers, can render these parton model calculations more precise.

Let us begin by discussing Landshoff's pinch graph, since it is a convenient place to introduce notation that we will need. The typical graph we are concerned with is shown in Fig. 7(c). In the fixed-angle high-energy limit, the momenta p and q_1, taken in the center-of-mass frame, grow like $\sqrt{s}$. We adopt Landshoff's notation throughout, so

$$p\cdot q = p'\cdot q = 0\,,$$

$$p^2 = p'^2 = -q^2 + m^2 = \tau\,,$$

$$p\cdot p' = \lambda\tau\,,$$

$$s\sim 2\tau(1+\lambda),\quad t\sim -4\tau,\quad u\sim 2\tau(1-\lambda)\,,$$

where m is the pion mass and $\tau\to\infty$ with λ fixed. The eight momenta k_i, k_i' are parametrized in

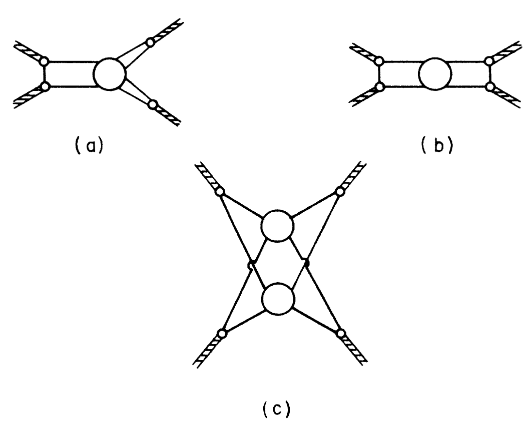

FIG. 7. Possible dominant configurations for large-angle scattering: (a) and (b) parton-exchange graphs, (c) the Landshoff graph.

terms of 16 scalar parameters as follows:

$$\begin{pmatrix} k_1 \\ k_1' \end{pmatrix} = \begin{pmatrix} x_1 \\ 1-x_1 \end{pmatrix} (p+q) \pm y_1 \frac{(p'+\lambda q)}{[\tau(\lambda^2-1)]^{1/2}} \pm \kappa_1 n \pm \frac{z_1}{2\tau} q\,,$$

$$\begin{pmatrix} k_2 \\ k_2' \end{pmatrix} = \begin{pmatrix} x_2 \\ 1-x_2 \end{pmatrix} (p-q) \pm y_2 \frac{(p'-\lambda q)}{[\tau(\lambda^2-1)]^{1/2}} \pm \kappa_2 n \pm \frac{z_2}{2\tau} q\,,$$

$$\begin{pmatrix} k_3 \\ k_3' \end{pmatrix} = \begin{pmatrix} x_3 \\ 1-x_3 \end{pmatrix} (p'+q) \pm y_3 \frac{(p+\lambda q)}{[\tau(\lambda^2-1)]^{1/2}} \pm \kappa_3 n \pm \frac{z_3}{2\tau} q\,,$$

$$\begin{pmatrix} k_4 \\ k_4' \end{pmatrix} = \begin{pmatrix} x_4 \\ 1-x_4 \end{pmatrix} (p'-q) \pm y_4 \frac{(p-\lambda q)}{[\tau(\lambda^2-1)]^{1/2}} \pm \kappa_4 n \pm \frac{z_4}{2\tau} q\,,$$

where n is a spacelike unit vector orthogonal to p, p', q. The virtue of this parametrization is that for finite x_i, y_i, z_i, κ_i in the center-of-mass frame each k_i is a finite fraction of the corresponding p_i plus a finite orthogonal vector. Next, we record the masses of the k_i in the limit of large s:

$$\begin{aligned}
k_1^2 &= -x_1 z_1 + m^2 x_1^2 - y_1^2 - \kappa_1^2\,,\\
k_1'^2 &= (1-x_1) z_1 + m^2(1-x_1)^2 - y_1^2 - \kappa_1^2\,,\\
k_2^2 &= -x_2 z_2 + m^2 x_2^2 - y_2^2 - \kappa_2^2\,,\\
k_2'^2 &= (1-x_2) z_2 + m^2(1-x_2)^2 - y_2^2 - \kappa_2^2\,,\\
k_3^2 &= -x_3 z_3 + m^2 x_3^2 - y_3^2 - \kappa_3^2\,,\\
k_3'^2 &= (1-x_3) z_3 + m^2(1-x_3)^2 - y_3^2 - \kappa_3^2\,,\\
k_4^2 &= -x_4 z_4 + m^2 x_4^2 - y_4^2 - \kappa_4^2\,,\\
k_4'^2 &= (1-x_4) z_4 + m^2(1-x_4)^2 - y_4^2 - \kappa_4^2\,.
\end{aligned}$$

The parametrization has of course been chosen so that these masses are finite in the high-s limit for finite values of the parameters x_i, y_i, z_i, κ_i. Finally, the momenta k_i satisfy one constraint:

$$\sum_{i=1}^{4} k_i + p + p' = 0\,.$$

This means that, in the large-s limit one may integrate over the 16 parameters x_i, y_i, z_i, κ_i independently if we supply the constraint[8]

$$\frac{1}{\sqrt{stu}}\,\delta(x_1+x_2-1)\delta(x_3+x_4-1) \times\delta(x_1-x_2+x_3-x_4)\delta(\kappa_1+\kappa_2+\kappa_3+\kappa_4)\,.$$

Now we can discuss the asymptotic behavior of the graph. First of all, the three x_i δ functions leave one x to be integrated over, which we shall call ξ. In the high-s limit, the two four-quark amplitudes become $\Gamma^{(4)}(\xi p_1, \ldots, \xi p_4)$ and $\Gamma^{(4)}((1-\xi)p_1, \ldots, (1-\xi)p_4)$, which are just fixed-angle high-energy quark-quark scattering amplitudes. If the appropriate spin projection operators are supplied, our renormalization-group discussion yields,

$$\Gamma(\xi p_1, \ldots, \xi p_4) \to (\xi\sqrt{s})^{4(1-d)}\Gamma(\hat{p}_1, \ldots, \hat{p}_4; g_0, m=0, \mu) = (\xi\sqrt{s})^{4(1-d)} f(\theta)\,,$$

where d is the anomalous dimension at the fixed point, g_0, of the Fermi field, $\hat{p}_i = \lim p_i/\sqrt{s}$ is a lightlike vector, θ is the center-of-mass scattering angle of the problem, and $f(\theta)$ describes the angular dependence of the zero-mass quark-quark scattering amplitude at the fixed point. $f(\theta)$ is of course not known and will make the angular dependence of the overall meson-meson amplitude unknown.

Next we must concern ourselves with the energy dependence of the bound-state wave functions. Barring a divergence in the y_i, z_i, κ_i integrations (which does not occur) the bound-state wave functions, $\Gamma_{\alpha\beta}(p_i; k_i k_i')$, are needed for finite k_i^2, $k_i'^2$ but large k_i, k_i'. Since

$$\Gamma_{\alpha\beta} = \delta_{\alpha\beta} g_1(k_1^2, k_1'^2) + [\not{k}_1, \not{k}_1']_{\alpha\beta} g_2(k_1^2, k_1'^2)\,,$$

it is apparent that the largest contribution will come from g_2 since the term $[\not{k}_1, \not{k}_1']$ can grow like $\sqrt{s}$. Consider the $[\not{k}_1, \not{k}_1']$ term. According to the parametrization of Eq. (6.1), the leading contribution in the center-of-mass frame is

$$[\not{k}_1, \not{k}_1'] = -\left[\not{p}_1, \kappa_1 \not{n} + y_1 \frac{(\not{p}' + \lambda \not{q})}{[\tau(\lambda^2-1)]^{1/2}}\right]\,.$$

This is the only place where y_1, as opposed to y_1^2, appears in the calculation, so that the y_1 term is odd and automatically integrates to zero. The same cannot be said of the k_1 term since the κ_i integrations are constrained by $\delta(\kappa_1+\kappa_2+\kappa_3+\kappa_4)$. Therefore, what survives is

$$[\not{k}_i, \not{k}_i'] = -\kappa_i[\not{p}_i, \not{n}] \to k_i\sqrt{s}[\hat{\not{p}}_i, \not{n}]\,.$$

The $\hat{\not{p}}$ can be regarded as acting either to the left or the right since $\hat{\not{p}}$ anticommutes with $\not{n}$ and supplies the spin projection which we needed in the previous paragraph to guarantee the existence of the zero-mass limit of the quark-quark scattering amplitude. Because of the factor of $\sqrt{s}$ it is clearly favorable to pick the g_2 covariant at each vertex.

The overall integration to be done is then

$$\left(\frac{1}{s}\right)^{4(d-1)} \frac{(\sqrt{s})^4}{\sqrt{stu}} F(\theta) \int d\xi[\xi(1-\xi)]^{-4(d-1)} \int \prod_{i=1}^{4} dy_i dz_i d\kappa_i \delta\left(\sum \kappa_i\right)\kappa_i g_2(k_i^2, k_i'^2)\,, \tag{6.1}$$

where $F(\theta)$ is a product of two zero-mass quark-quark scattering amplitudes and the various x_i δ functions have evaluated the x_i such that

$$k_i^2 = -\xi z_i + m^2\xi^2 - y_i^2 - \kappa_i^2\,,$$

$$k_i'^2 = (1-\xi)z_i + m^2(1-\xi)^2 - y_i^2 - \kappa_i^2\,.$$

To discuss this integral it is convenient to write the κ_i δ function as $\int d\alpha\, e^{i\alpha\Sigma\kappa_i}$. The amplitude then becomes

$$\left(\frac{1}{s}\right)^{4(d-3/2)}\frac{F(\theta)}{\sqrt{stu}}\int d\xi[\xi(1-\xi)]^{-4(d-1)}\int d\alpha\phi^4(\alpha)\,, \tag{6.2}$$

where

$$\phi(\alpha) = \int d\kappa dy dz\, \kappa\, e^{i\kappa\alpha} g_2(k^2, k'^2)\,.$$

To appreciate the relevant features of the integral over g_2, we choose the form of Eq. (4.2), augmented by a mass to put thresholds in the right place:

$$g_2(k^2, k'^2) = \int_0^1 du \frac{[u(1-u)]^{(\bar{d}-1)/2}}{[k^2(1-u) + k'^2u - M^2 + i\epsilon]^3}$$

$$= \int_0^1 du \frac{[u(1-u)]^{(\bar{d}-1)/2}}{\{z(u-\xi) - (y^2+\kappa^2+M^2) - m^2[\xi^2(1-u) + (1-\xi)^2u] - i\epsilon\}^3}\,.$$

If ξ is outside the interval between 0 and 1, the z integration contour can always be closed in either the upper or lower half plane in such a way as to avoid all singularities. By the same token, when $0 \le \xi \le 1$, it is only at $u = \xi$ that the z integration does not, by contour deformation, give zero. Thus the z and u integrations cast up a factor of $[\xi(1-\xi)]^{(\bar{d}-1)/2}$ at each vertex. The remaining integrals are all splendidly convergent and yield a function ϕ which is smooth in ξ and falls off exponentially in α so long as the reasonable threshold condition $2M > m$ is met. The final answer for the amplitude is

$$\left(\frac{1}{s}\right)^{4(d-3/2)}\frac{F(\theta)}{\sqrt{stu}}\int_0^1 d\xi[\xi(1-\xi)]^{2(1-2d+\bar{d})}a(\xi)\,,$$

where $a(\xi)$ is a smooth function of ξ. For nearly canonical dimensions, $2d \sim \bar{d}$, and the ξ integration converges. Since $d \sim \frac{3}{2}$, we find that the overall power behavior of the amplitude is nearly $s^{-3/2}$, while the angular distribution is *a priori* undetermined.

Note that the ξ dependence of the integrals over g_2 was essential to render the ξ integration in Eq. (6.2) finite. What was critical was the power of $u(1-u)$ in the Deser-Gilbert-Sudarshan (DGS) representation of g_2 [Eq. (4.2)] which in turn governs the falloff of $g_2(k^2, k'^2)$ when $k^2 \to \infty$ with k'^2 fixed. As discussed in Sec. IV, the conformal argument gives the power falloff in k^2 with k^2 large and k'^2 fixed and large compared to the internal mass scale. We have assumed that the same power obtains for k'^2 *not* large, and so long as that is so, the essential features of the previous analysis remain unchanged. The interesting point about this set of graphs is that the bound-state wave functions are needed near the mass shell only and the renormalization group comes in only to control the growth of the internal quark scattering amplitude.

The off-mass-shell behavior of the wave function *does* play a crucial role in the class of graphs shown in Fig. 7(b). Although they are relatively less important than the "pinch" graphs, we shall discuss them in order to illustrate how the wave function enters. There are two independent loop integrations to do, and we will parametrize them as follows:

$$k_1 = x(p+q) + \frac{y}{2\tau}q + \kappa\,,$$

$$k_2 = (1-x)(p+q) - \frac{y}{2\tau}q - \kappa\,,$$

$$k_3 = -k_1 + 2q = -k_1 + (p_1 - p_2)\,,$$

where κ is a two-dimensional spacelike vector perpendicular to p, q (similar definitions for the other loop). The corresponding masses are

$$k_1^2 = -xy + m^2x^2 + \kappa^2 + O(s^{-1})\,,$$

$$k_2^2 = (1-x)y + m^2(1-x)^2 + \kappa^2 + O(s^{-1})\,,$$

$$k_3^2 = -4(1-x)(\tau - m^2) + 2y + k_1^2\,.$$

Thus, for finite x, y, κ in the limit $s \to \infty$, k_1^2 and

$k_2{}^2$ remain finite while $k_3{}^2$ grows.

The line with momentum k_2 is supplied with an inverse fermion propagator: since we have defined the bound-state wave function to include the fermion propagator there is a double-counting problem when two bound-state wave functions are jointed directly together. In the large-s limit, the leading contribution of the inverse propagator is $(1-x)\not{p}_1 = \sqrt{s}(1-x)\hat{\not{p}}_1$. The fixed lightlike vector $\hat{p}_1$ eventually serves as a spin projection operator on the internal quark-quark scattering amplitude.

The internal four-fermion scattering amplitude is evaluated in the high-s limit with the following momentum arguments:

$$\Gamma(xp_1, (1-x)p_1 - p_2, x'p_4, (1-x')p_4 - p_3) \sim (\sqrt{s})^{4(1-d)}\Gamma(x\hat{p}_1, (1-x)\hat{p}_1 - \hat{p}_2, x'\hat{p}_4, (1-x')\hat{p}_4 - \hat{p}_3; g_0, m = 0, \mu) .$$

This simple scaling law of course only holds so long as the zero-mass lines are acted on by the appropriate spin projection operator, provided, in fact, by the inverse propagator.

Finally, the bound-state wave function at the p_2 vertex falls off like $(s)^{-(\bar{d}-1)/2}$ since one leg mass is fixed while the second is growing like s. Here, again, our assumption that the power behavior of the wave function in the limit $q_1{}^2 \to \infty$, $q_2{}^2$ fixed may be evaluated in the zero-mass theory comes into play. There are two such factors; one associated with the left-hand loop and one with the right-hand loop. One may easily verify that the two covariants in the bound-state wave function behave the same way. Once all of these explicit factors of s have been factored out, it is not too difficult to verify, by arguments similar to those used already for the pinch graph, that the x, y, κ integrations converge.

The result of these arguments is that the overall power behavior of the graph is

$$(\sqrt{s})^2(\sqrt{s})^{4(1-d)}(s^{-1})^{(\bar{d}-1)/2} = (s^{-1})^{(\bar{d}-1)/2+2(d-3/2)} ,$$

while the angular dependence is unknown and related to that of the fundamental fermion-fermion fixed-angle scattering amplitude. For canonical dimensions, the asymptotic energy dependence is s^{-2} and the parton-interchange graph is therefore expected to be less important than the "pinch" graph.

In other theories and/or other processes, the relative importance of these various topological classes of two-body scattering graphs may well be different. Nevertheless, the technique illustrated here should suffice, if applied with sufficient care, to extract the dominant behavior.

VII. CONCLUSION

The net result of our discussion is that simple dimensional scaling of on-mass-shell processes seems just as reasonable in the context of quantum field theory as in the context of a naive constituent model. By this we mean that the semireliable features of bound-state wave functions provide a natural source of asymptotic powers over and above renormalization-group powers and that if anomalous dimensions are small, the net powers in field theory and in the constituent model agree with one another. When exotic infrared singularities are important (such as pinch singularities in meson-meson scattering) they affect both approaches in a similar fashion.

The main failing of our discussion is that it has nothing to say about vector-meson theories in general and asymptotically free theories in particular.[10] The basic difficulty is that whenever vector mesons are present, the zero-mass s matrix is very badly infrared divergent, whereas it is precisely this quantity which governs on-mass-shell asymptotic behavior. Therefore, until we can reliably sum these infrared singularities we cannot say anything about vector theories. Since, in four dimensions all asymptotically free theories involve vector mesons, we have therefore not said anything about such theories. This is unfortunate, since the vanishing of the asymptotic effective coupling constant would allow us to augment the general scaling statements by more specific statements about angular dependence and the like.

APPENDIX: THE BOUND-STATE WAVE FUNCTION IN THE LADDER APPROXIMATION

In this appendix we shall explicitly solve the Bethe-Salpeter equation for a scalar bound state in the ladder approximation[11] to a zero-mass Yukawa theory. This provides a concrete example of such a zero-mass bound-state wave function which can be compared with that constructed in Sec. IV using conformal invariance.

The Bethe-Salpeter equation for the bound state in the ladder approximation takes the form (see Fig. 8)

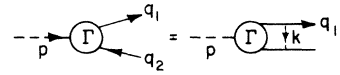

FIG. 8. The ladder approximation to the Bethe-Salpeter equation.

$$\not{q}_1 \Gamma(p; q_1, q_2)\not{q}_2 = \frac{\lambda}{\pi^2} \int \frac{d^4k}{k^2} \Gamma(p; q_1+k, q_2-k), \tag{A1}$$

where $p^2 = 0$, $\lambda = g^2/16\pi^2$ (g being the Yukawa coupling), and we have performed the Wick rotation. The matrix Γ must be even in γ matrices, so that

$$\Gamma = \phi_0(q_1^2, q_2^2) + [\not{q}_1, \not{q}_2]\, \phi_1(q_1^2, q_2^2). \tag{A2}$$

We can now transform Eq. (A1) into a set of coupled integral equations for ϕ_0 and ϕ_1. These simplify if we make the ansatz:

$$\phi_i(q_1^2, q_2^2) = \int_0^1 du \frac{g_i(u)}{[q_1^2 u + q_2^2(1-u)]^{D+i}},$$

with $g_i(u)$ and D to be determined. With this ansatz Eq. (A1) implies the following equations for g_i:

$$\int_0^1 du \frac{1}{[q_1^2 u + q_2^2(1-u)]^D}\left[g_0(u) - \frac{q_1^2+q_2^2}{q_1^2 u + q_2^2(1-u)} g_1(u)\right] = \frac{2\lambda}{D(2-D)} \int_0^1 du \frac{g_1(u)}{[q_1^2 u + q_2^2(1-u)]^D},$$

$$\int_0^1 du \frac{1}{[q_1^2 u + q_2^2(1-u)]^D}\left[(q_1^2+q_2^2)g_0(u) - \frac{(q_1^2-q_2^2)^2}{q_1^2 u + q_2^2(1-u)} g_1(u)\right] = \frac{2\lambda}{(D-1)(2-D)} \int_0^1 du \frac{g_0(u)}{[q_1^2 u + q_2^2(1-u)]^{D-1}}. \tag{A3}$$

If we set $q_1^2 = q_2^2$ in Eq. (A3) we will determine D, since one then derives that

$$(D-1)(2-D) = \lambda, \quad D = \tfrac{1}{2}[3 + \sqrt{(1-4\lambda)^{1/2}}\,].$$

Inserting this value of D into Eqs. (A3) and integrating by parts so as to yield common denominators, one shows that these equations are satisfied as long as

$$g_1(u)' = D(1-2u)g_0(u),$$
$$Dg_0(u) = 4(D-1)g_1(u) + (1-2u)g_1(u)'.$$

The solution of these equations is

$$g_0 = c[u(1-u)]^{D-2},$$
$$g_1 = c\frac{D}{D-1}[u(1-u)]^{D-1},$$

so that

$$\Gamma(p; q_1 q_2) = \int_0^1 du \frac{c[u(1-u)]^{D-2}}{[q_1^2 u + q_2^2(1-u)]^D} \times \left[1 + \frac{D[\not{q}_1, \not{q}_2]u(1-u)}{(D-1)[q_1^2 u + q_2^2(1-u)]}\right]. \tag{A4}$$

This is precisely the bound-state wave function derived in Sec. IV if we identify D as

$$D = 2 + \tfrac{1}{2}\gamma_\theta = 2 + \tfrac{1}{2}d_\theta - d_\psi \tag{A5}$$

and if γ_ψ were zero.

This is altogether reasonable since in the ladder approximation β is zero, there being no coupling-constant renormalization, so that the zero-mass theory should be conformally invariant for all g. Furthermore, the fact that there is no renormalization of the fermion propagator in the ladders implies that $\gamma_\psi = 0$. On the other hand, the operator $\theta = \bar\psi\psi$ requires renormalization in this approximation, the relevant graphs being shown in Fig. 9. To lowest order in g^2 it is trivial to calculate the anomalous dimension of θ. One finds

$$\gamma_\theta(g^2) = -g^2/8\pi^2,$$

consistent with Eq. (A5), which implies that

$$\gamma_\theta(g^2) = (1 - g^2/4\pi^2)^{1/2} - 1$$
$$= -\frac{g^2}{8\pi^2} + O(g^4). \tag{A6}$$

Although we have not calculated γ_θ directly (from, say, the graphs of Fig. 9) there is no question but that Eq. (A6) is indeed γ_θ in the ladder approximation.

The bound-state wave function can also be explicitly constructed in the ladder approximation to massless QED.[12] The equation to be solved is

$$\not{q}_1 \Gamma(p; q_1 q_2)\not{q}_2 = -\frac{i\lambda}{\pi^2}\int d^4k \gamma^\mu \Gamma(p; q_1+k, q_2-k)\gamma^\nu D_{\mu\nu}(k), \tag{A7}$$

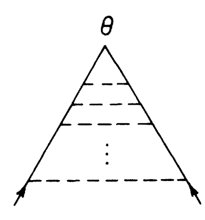

FIG. 9. The ladder approximation to the anomalous dimension of $\theta = \bar\psi\psi$.

where $D_{\mu\nu}(k)$ is the massless vector-meson propagator. This we choose in an arbitrary Fermi-type gauge

$$D_{\mu\nu}(k) = -\frac{i}{k^2}\left(g_{\mu\nu} - \epsilon\frac{k_\mu k_\nu}{k^2}\right).$$

We make the same ansatz as above for the form of Γ, which reduces this equation to the following equations for $g_i(u)$:

$$\int_0^1 du \frac{1}{[q_1^2u + q_2^2(1-u)]^D}\left[g_0(u) - \frac{q_1^2+q_2^2}{q_1^2u+q_2^2(1-u)}g_1(u)\right] = \frac{\frac{1}{2}\lambda\epsilon}{D(D+2)(D-2)}\int_0^1 du\frac{g_1(u)}{[q_1^2u+q_2^2(1-u)]^D},$$

$$\int_0^1 du \frac{1}{[q_1^2u + q_2^2(1-u)]^D}\left[(q_1^2+q_2^2)g_0(u) - \frac{(q_1^2-q_2^2)^2g_1(u)}{q_1^2u+q_2^2(1-u)}\right] = \frac{\frac{1}{2}\lambda(4-\epsilon)}{(D-1)(2-D)}\int_0^1 du\frac{g_0(u)}{[q_1^2u+q_2^2(1-u)]^{D-1}} \tag{A8}$$

These equations then determine D,

$$D = \tfrac{1}{2}\{3 + [1 - 4\lambda(4-\epsilon)]^{1/2}\},$$

and $g_i(u)$,

$$g_0(u) = C[u(1-u)]^{\beta-1/2},$$

$$g_1(u) = \frac{2D}{2\beta+1}C[u(1-u)]^{\beta+1/2},$$

where

$$\beta = \tfrac{1}{2}D - 1 - \frac{(D-1)\epsilon}{2(4-\epsilon)(D+2)}.$$

Now once again one can easily check that $D = 2 + \frac{1}{2}\gamma_\theta$. This is to be expected since in this approximation $\beta = 0$ and $\gamma_\psi = 0$, and the above value of D then follows from the fact that the wave function should obey the renormalization-group equation. The fact that the wave function *is not* given by the conformally invariant form derived in Sec. IV is also to be expected owing to the fact that here we are dealing with a gauge theory. The arguments presented in Sec. IV to the effect that, as $q_1^2/q_2^2 \to 0$, Γ should behave as

$$\Gamma \sim \left(\frac{1}{q_1^2}\right)^{1+\gamma_\theta/2}\left(\frac{1}{q_2^2}\right)^{1-\gamma_\psi}$$

do not apply here since we are not summing a gauge covariant set of graphs. Indeed one finds from the above that in this limit (if $\frac{3}{2} \geq \beta > -\frac{1}{2}$)

$$\Gamma(p; q_1, q_2) \to \left(\frac{1}{q_1^2}\right)^{\beta+1/2}\left(\frac{1}{q_2^2}\right)^{3/2+\gamma_\theta/2-\beta}.$$

The main lesson to be drawn from this calculation is that the behavior of the Γ calculated in a non-gauge-covariant approximation is very sensitive to the value of the gauge parameter. Thus it clearly is meaningless to use such an approximation for purposes of determining asymptotic behavior. One could derive quite different results, say, for the asymptotic behavior of bound-state form factors in the ladder approximation by changing gauge.

Note added in proof. A discussion of on-shell applications of the renormalization group, similar to our Sec. II, has been given by G. C. Marques, Phys. Rev. D 9, 386 (1974).

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[10]In a recent paper, T. Appelquist and E. Poggio [Phys. Rev. D 10, 3280 (1974)] have treated much the same

problem as we do. They examine an asymptotically free theory by looking at ϕ^3 in *six* space-time dimensions, thereby avoiding the vector-meson difficulty.

[11]Ladder models have been examined extensively in the context of scalar field theory. We might cite papers by M. Ciafaloni and P. Menotti: Phys. Rev. 173, 1575 (1968); Nuovo Cimento Lett. 6, 545 (1973).

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