PHYSICAL REVIEW D VOLUME 11, NUMBER 10 15 MAY 1975

Infrared singularities and massive fields*

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(Received 30 September 1974)

We examine some problems associated with the low-momentum behavior of gauge theories and other renormalizable field theories. Our main interest is in the infrared structure of unbroken non-Abelian gauge theories and how this is affected by the presence of other heavy fields coupled to the massless gauge fields. It is shown in the context of a simple model of gauge mesons coupled to massive fermions that the heavy fields decouple at low momenta except for their contribution to renormalization effects. This result is used to discuss the mass-shell structure of the fermion propagator. The decoupling theorem is then stated for a general renormalizable theory and applied to some interesting examples. One is a more general gauge theory which makes use of the Higgs mechanism and attempts to unify the elementary particle forces. Another is the connection of the linear and nonlinear σ models in the limit $m_\sigma \rightarrow \infty$.

I. INTRODUCTION

Developments in gauge theories over the past few years[1] have led to a reexamination of the old question of infrared behavior in quantum field theory. In this paper, we make some remarks about this problem and establish one result which should prove useful for future work.

We will examine the infrared behavior of Green's functions in perturbation theory and make use of renormalization-group methods when possible to go beyond perturbation theory. Our primary interest is in theories which contain an unbroken Yang-Mills gauge group (massless, self-coupled fields) along with other massive fields. The mass of the fields can arise through the Higgs mechanism or simply be present in the Lagrangian. Our interest in such theories is partly motivated by the fact that asymptotic freedom[2] seems to require the existence of an unbroken Yang-Mills subgroup.[3]

We will discuss the role of the heavy fields in the infrared behavior of the theory. We will show that the only role of the heavy fields in the low-momentum behavior of graphs without external heavies is their contribution to coupling-constant and field-strength renormalization. The heavy fields effectively decouple and the low-momentum behavior of the theory is described by a renormalizable Lagrangian consisting of the massless fields only. This result is used to discuss the behavior of the effective coupling constant of the renormalization group at low momenta and then the mass-shell structure of the heavy field propagator.

The decoupling theorem applies not only to theories with massless fields but in fact to any renormalizable theory with different mass scales. At momenta small compared to the larger masses, the dynamics is determined by the light sector of the theory. The effective low-energy renormalizable Lagrangian may or may not contain interactions. In the case of theories with massless non-Abelian gauge bosons it certainly does. An example of the other situation is the linear σ-model in the limit $m_\sigma \rightarrow \infty$. Here, the chiral symmetry forbids the existence of renormalizable interactions involving only the pion field. In these cases, interactions will be found only by keeping terms proportional to inverse powers of the large mass. This will be discussed in Sec. IV.

It is interesting to compare the infrared structure of non-Abelian gauge theories to the infrared structure in quantum electrodynamics. In order to have some points of comparison, we will devote Sec. II to a few comments about infrared behavior in quantum electrodynamics. This section should be skimmed or skipped altogether by the learned. Section III is the main part of the paper. It is devoted to the elucidation of the decoupling theorem in the context of a model with massless Yang-Mills fields coupled to massive spin-$\frac{1}{2}$ fields. This result is used to discuss the mass-shell structure of the fermion propagator and to compare this behavior with the analogous problem in quantum electrodynamics. Section IV is a discussion of the decoupling theorem for a more general class of renormalizable theories. Particular attention is paid to a Higgs model which attempts to unify all the elementary particle forces and to the linear σ model.

II. A REVIEW OF QUANTUM ELECTRODYNAMICS

The most familiar infrared problem in quantum electrodynamics is the Bloch-Nordsieck problem[4]—the calculation of transition probabilities that are free of infrared divergences. By now it has been shown that this can be done to arbitrary order in perturbation theory,[5] and this fact serves as the foundation for the important task of calculating the radiative corrections to scattering experiments. A coherent-state formalism has been developed by many authors[6] which provides even firmer theoretical support for the perturbative results.

The observation underlying the coherent-state formalism is that, because of the emission and reabsorption of massless particles, the renormalized electron propagator $\tilde{S}'_F(p)$ does not have an isolated first-order pole at $p^2=m^2$, but rather a branch-point singularity. Perturbation theory to nth order gives the behavior

$$(p^2-m^2)^{-1}\left[\ln(p^2-m^2)\right]^n,$$

and it can be shown that the logarithms sum to the exact infrared behavior

$$(p^2-m^2+i\epsilon)^{-1+\gamma(\alpha)}, \tag{2.1}$$

where $\gamma(\alpha)=\alpha/\pi$ holds to lowest order in α in the Feynman gauge.[7,8,9] The singularity structure (2.1) means that the conventional reduction formalism must be modified. This is precisely what is done in the coherent-state formalism.

We will summarize without proof the origin of the above power-law behavior in order to show why such power laws do not necessarily emerge in the non-Abelian theory. Following Bogoliubov and Shirkov,[8] the behavior of the electron propagator can be expressed in a renormalization-group equation. This equation becomes particularly simple in the limit $m^2-p^2 \ll m^2$. The near mass-shell equation is particularly simple because the only graphs that survive in this limit are those in which photons are emitted and reabsorbed without interacting with each other and, more importantly, without self-energy insertions. The self-energy insertions are suppressed since the conventional charge renormalization at $k^2=0$ defines the exact photon propagator to behave like $1/k^2+O(1/m^2)$ for $k^2\ll m^2$, and it is this region of k^2 that determines the $p^2\to m^2$ behavior of the electron propagator. Consequently, graphs with self-energy insertions are suppressed by factors of $(p^2-m^2)/m^2$ relative to the dominant graphs. Diagrams with photon-photon scattering insertions are suppressed for the same reason. That is, these insertions are of order $(k/m)^p$ $(p\geq 1)$ for $k\ll m$.

As a consequence of this low-energy behavior, the renormalization-group equation for the electron propagator in the $p^2-m^2\ll m^2$ limit becomes free of charge-renormalization effects (the effective β function is zero) and it turns into a simple scaling equation. The equation asserts that the logarithmic terms of the electron propagator exponentiate, and the resulting anomalous dimension $\gamma(\alpha)$ can be calculated as perturbation expansion in $\alpha\simeq 1/137$.[9] All of this simplicity is due to the absence of self-coupled massless fields in the theory.

It is also interesting to consider QED with $m=0$. Green's functions can still be defined providing the wave-function renormalization of the photon propagator as well as the electron propagator is performed off shell.[10] The mass-shell behavior of the Green's functions can be discussed as in the $m\neq 0$ situation. The behavior is more complicated now since coupling-constant renormalization must be taken into account in the renormalization-group equations. However, since the origin is an infrared-stable fixed point, the Green's functions exhibit free-field infrared behavior up to logarithmic corrections.

A Yang-Mills theory is similar to $m=0$ quantum electrodynamics in most ways. Green's functions exist in perturbation theory and properly defined transition probabilities can presumably be calculated as a perturbation expansion in some renormalized coupling constant (see Ref. 10). It parts company with $m=0$ QED when the perturbation expansion is resummed using the renormalization group since the origin is now ultraviolet stable. The Yang-Mills theory with other massive fields has some features of both $m=0$ and $m\neq 0$ electrodynamics. The decoupling theorem says that the massive fields play no role in the low-momentum dynamics (as in the $m\neq 0$ case). The self-coupled massless fields, of course, play an important role (as in the $m\neq 0$ case). We now turn our attention to such a model.

III. THE DECOUPLING THEOREM

This section is devoted to the decoupling theorem and some of its consequences. We choose to work with a specific model in order to be as concrete as possible, but we emphasize again that the theorem is quite general. The model contains a set of massless gauge fields $A_{\alpha\mu}(x)$ (the light fields) coupled to a set of massive spin-$\frac{1}{2}$ fields $\Psi_n(x)$ (the heavy fields). This is a prototype for a possible realistic model of the strong interactions. The Lagrangian is

$$\mathcal{L}(x)=-\tfrac{1}{4}F_{\alpha\mu\nu}F_{\alpha}^{\mu\nu}-\bar{\Psi}\gamma_\mu D^\mu\Psi-\bar{\Psi}m\Psi-\delta m\bar{\Psi}\Psi, \tag{3.1}$$

where

$$F_{\alpha\mu\nu}=\partial_\mu A_{\alpha\nu}-\partial_\nu A_{\alpha\mu}-gC_{\alpha\beta\gamma}A_{\beta\mu}A_{\gamma\nu} \tag{3.2}$$

and

$$(D_\mu\Psi)_n = \partial_\mu \Psi_n - (t_\alpha)_{nm}\Psi_m A_{\alpha\mu}. \quad (3.3)$$

The Lie algebra of the gauge group is

$$[t_\alpha, t_\beta] = iC_{\alpha\beta\gamma} t_\gamma. \quad (3.4)$$

The theory is quantized in the usual way and the Feynman rules can be generated. We will work in the Landau gauge so that the vector-meson propagator is transverse and so that a change in the renormalization point does not have to be accompanied by a change of gauge.[11] Regularization[12] and renormalization also follow conventional lines. The mass counterterm δm is adjusted to make the one-particle-irreducible (1PI) fermion self-energy vanish at $\not{p} = m$ to each order in perturbation theory. In order to avoid infrared divergences, the remaining counterterms must be adjusted to effect subtractions at off-shell Euclidean points. We will introduce a single new mass scale μ to characterize these points.

The gauge meson propagator, for example, is of the form

$$D_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2}\, d\left(\frac{k^2}{\mu^2}, \frac{m^2}{\mu^2}, g_\mu\right), \quad (3.5)$$

with the normalization condition $d(-1, m^2/\mu^2, g_\mu) = 1$. The coupling constant g_μ is defined by this normalization convention along with a similar condition on the 1PI three-vector-meson vertex. The appropriate counterterm can be adjusted to normalize it, for example, at the symmetric Euclidean point $p^2 = q^2 = r^2 = -\mu^2$. Its general form at a symmetric point $p^2 = q^2 = r^2 \equiv k^2$ is[13]

$$i\Gamma^{\mu\nu\lambda}_{\alpha\beta\gamma} = G\left(\frac{k^2}{\mu^2}, \frac{m^2}{\mu^2}, g_\mu\right)[(p-q)^\mu g^{\nu\lambda} + (q-r)^\nu g^{\lambda\mu} + (r-p)^\lambda g^{\mu\nu}]C_{\alpha\beta\gamma}$$
$$+ \text{ terms involving three powers of the momentum}, \quad (3.6)$$

with $G(-1, m^2/\mu^2, g_\mu) = 1$. The terms not explicitly written down are superficially convergent to each order in perturbation theory.

The remaining part of the renormalization program, including the proof of Ward-Slavnov identities, can be developed following Ref. 11. The only further piece of the renormalization program we will discuss in detail is the wave-function renormalization of the fermion. Following Bogoliubov and Shirkov,[8] we take

$$\tilde{S}_F'(p) = \frac{a(p^2)\not{p} + b(p^2)m}{p^2 - m^2}, \quad (3.7)$$

with Z_2 adjusted so that, say, $a(p^2/\mu^2, m^2/\mu^2, g_\mu)$ is normalized to 1 at $p^2 = -\mu^2$. The functions $a(p^2)$ and $b(p^2)$ become infrared divergent as $p^2 \to m^2$, containing arbitrarily high powers of $\ln(p^2 - m^2)$. We will return to a discussion of this mass-shell behavior.

Before getting into the decoupling theorem, it is useful to have recorded the renormalization-group equation[14] of the theory for general momentum. The effective coupling constant is defined as

$$g_k \equiv g_k\left(\frac{k^2}{\mu^2}, \frac{m^2}{\mu^2}, g_\mu\right)$$
$$\equiv g_\mu\, G\left(\frac{k^2}{\mu^2}, \frac{m^2}{\mu^2}, g_\mu\right) d^{3/2}\left(\frac{k^2}{\mu^2}, \frac{m^2}{\mu^2}, g_\mu\right), \quad (3.8)$$

where $g_k(-1, m^2/\mu^2, g_\mu) = g_\mu$. It satisfies the differential equation of the renormalization group

$$k^2 \frac{\partial}{\partial k^2} g_k = \beta\left(\frac{m^2}{-k^2}, g_k\right), \quad (3.9)$$

where

$$\beta\left(\frac{m^2}{-k^2}, x\right) = \frac{\partial}{\partial y} g_k\left(y, \frac{m^2}{-k^2}, x\right)\Big|_{y=1}. \quad (3.10)$$

To discuss physics when all momenta are large compared to m (and Euclidean, of course), it is useful to take $\mu \gg m$. The analysis of Kinoshita[10] can then be applied to show that in each order of perturbation theory m can be scaled to zero without encountering any infrared singularities. The conventional discussion of the short-distance behavior using the $m \to 0$ limit of Eq. (3.9) can then be carried out.

We now consider the low-momentum behavior of the theory and establish the decoupling theorem. For any 1PI Feynman graph with external vector mesons only but containing internal fermions, we will establish the following fact. When all the external momenta are small relative to m, then apart from coupling-constant and field-strength renormalization the graph will be suppressed by some power of momentum m relative to a graph with the same number of external vector mesons but no internal fermions. When the graph under consideration is ultraviolet convergent (the graph and all its subgraphs are superficially convergent) the proof is straightforward and we shall treat this case first.

When ultraviolet divergences are present, then each graph is accompanied by counterterms which make it finite. By adjusting the counterterms so that the normalization mass is on the order of the external momenta ($\ll m$), we claim that the renormalization effects we have referred to will be automatically absorbed into g_μ and the field strength. Then *any* graph with external vector mesons only and internal fermions will be suppressed. The entire effect of a shift from $\mu \gg m$ (appropriate for discussing high-momentum behavior) to $\mu \ll m$ appropriate for low momenta is, of course, just a rescaling of g_μ and field strength. The effective coupling strength g_k is defined to be invariant under this rescaling.

To begin with, we will consider graphs with $n \geq 5$ external boson legs and which contain no divergent subgraphs. An arbitrary 1PI Feynman graph with n external legs has degree of divergence $d = 4 - n$. Thus, if there are no internal fermion lines and all the external momenta are on the order of k, the amplitude behaves like k^{4-n}. Now consider a graph, still with n external bosons lines but which contains internal fermion lines. For $k \ll m$, we will show that all such graphs are suppressed relative to k^{4-n}. We focus on any one of the internal fermion loops. Any such loop can be severed from the parent graph by cutting enough vector-meson lines. Suppose that the *minimum* number of severings required for the fermion loop under consideration is f.

We will consider the cases $f \geq n$ and $f < n$ separately. In the case $f \geq n$, we consider the vector-meson lines attached *directly* to the fermion loop. Suppose that it is possible to sever the loop from the parent graph by cutting F ($\geq n$) of these lines. These F lines are the external lines of a subgraph S which, as m goes to infinity, behaves like m^{4-F}. This follows from power counting and the fact that because of the mass of the fermion all the momenta flowing into S can be scaled to zero without encountering singularities. Scaling m to infinity corresponds to shrinking the subgraph S to a point and to get the behavior of the overall graph G, we must yet analyze the reduced graph which contains a single vertex of F lines in addition to the usual vertices. The degree of divergence of the reduced graph is $d_R = F - n$. In fact, since we are considering the case $f \geq n$, this overall degree of divergence of the reduced graph is the maximum degree of divergence to be found anywhere in the reduced graph. This apparent divergence is, of course, cut off by the shrunken graph S which, at high enough momentum, ceases to act like a point. Thus, the behavior of the entire graph G, for $k \ll m$, is $m^{4-F} m^{F-n} = m^{4-n}$, which is strongly suppressed relative to k^{4-n}.

Next suppose that $f < n$. A set of f lines which can be severed to remove the fermion loop from the graph is the external lines of some subgraph S' which contains the fermion loop. As m goes to infinity, we can apply the result of case 1 to the subgraph S' to conclude that it behaves like m^{4-f}. The degree of divergence of the reduced graph obtained by shrinking S' to a point is $d_{R'} = f - n < 0$. If there are no fermion loops in the reduced graph, it is proportional to k^{f-n} so that the entire graph behaves like $m^{4-f} k^{f-n}$, which, since $f \geq 5$, is suppressed relative to k^{4-n}. If the reduced graph contains other fermion loops, the graph will be even further suppressed.

We next consider the general case of 1PI graphs and subgraphs with two or more external vector mesons. The degree of divergence of any such graph is $d = 4 - n$, which, in the case of $n = 2$ or $n = 3$, is reduced to zero by making use of the transverse structure (3.5) of the propagator and the tensor structure of the three-point vertex. Accompanying any graph is a set of counterterms which effectively reduces the degree of divergence of the superficially divergent graphs one step further, to -1. Furthermore, because the subtractions are performed at a momentum scale $\mu \ll m$, the loop momenta are cut off at momenta small compared to m. For comparison purposes, 1PI graphs with n (≥ 2) external vector-meson lines and with *no* internal fermions behave like k^{4-n} up to logarithms.

We again focus on any fermion loop in the graph and consider the $f \geq n$ case and $f < n$ case separately. In the first case, the above argument can be repeated with the following modification. If $F \leq 4$, then the $k \ll m$ behavior of S *with its counterterms* will be

$$k^{4-F} \times \text{terms of order } k/m \text{ or } \mu/m. \tag{3.11}$$

The first factor comes from the transversality of the propagator and the tensor structure of the three-point function. The second factor is present because the counterterm subtracts the logarithmic divergence at a low mass μ relative to m. Schematically,

$$\left[\ln\frac{\Lambda}{m} + O\left(\frac{k}{m}\right)\right] - \left[\ln\frac{\Lambda}{m} + O\left(\frac{\mu}{m}\right)\right] = O\left(\frac{k}{m}, \frac{\mu}{m}\right). \tag{3.12}$$

It is now easy to repeat the rest of the argument to show that this second factor produces a suppression of at least k/m or μ/m relative to the behavior of graphs without internal fermions. The case $f < n$ can be carried through as before by making use of the $f \geq n$ case.

An immediate consequence of the decoupling theorem is that order by order in perturbation theory the general β function of the theory (Eq. 3.9) reduces to the β function of the pure Yang-Mills theory in the low-momentum limit

$$\beta\left(\frac{m^2}{-k^2}, g_k\right) \xrightarrow[(\mu^2/m^2),\,(k^2/m^2) \to 0]{} \beta_{\text{YM}}(g_k). \tag{3.13}$$

The entire renormalization-group formalism can thus be applied to this low-momentum region with the heavy fields omitted. This will be discussed further in the next section.

The decoupling theorem can be applied to the problem of mass-shell behavior as well as low-momentum behavior. As an example, we examine the mass-shell structure of the fermion propagator. As pointed out in Sec. II, this is a soluble problem in QED because charge-renormalization effects drop out of the renormalization-group scaling equations. In the Yang-Mills theory, what can be shown is that the only graphs that contribute near the mass shell are those with no fermion loops, so that the mass-shell structure is essentially governed by the light sector of the theory.

The general structure of the propagator is given by Eq. (3.7). In probing the limit $m^2 - p^2 \ll m^2$, it is useful to perform the wave-function renormalization subtraction on the fermion close to its mass shell. Thus, suppose this subtraction is performed at $p^2 = m^2 - \mu^2$ with $\mu^2 \ll m^2$ rather than at $p^2 \ll m^2$ as done before. Gauge invariance will then require subtracting the vector-meson–fermion–fermion vertex near the fermion mass shell as well. Now, note that the proof of the decoupling theorem did not depend on where these subtractions were performed since fermion lines entered only as closed loops.

With the new renormalization point, it can be seen that any graph G without fermion loops gives a contribution to, say, $a(p^2)$ of the form

$$a_G\left(\frac{p^2 - m^2}{\mu^2},\ g_\mu\right) + O\left(\frac{p^2 - m^2}{m^2}\right) + O\left(\frac{\mu^2}{m^2}\right).$$

The decoupling theorem ensures that graphs with fermion loops are suppressed in this limit simply because the momentum flowing into them is small compared to the mass of the fermion. This is because one is very close to threshold and because the subtractions are also being performed close to threshold. Thus, if the sum of the perturbation expansion has the same structure as each term in the expansion, the function $a(p^2)$ becomes a function of the dimensionless ratio $(p^2 - m^2)/\mu^2$ and the coupling constant g_μ in this limit. Now the renormalization-group scaling analysis can be applied to this function with the behavior of g_k governed by the β function of the pure Yang-Mills theory. A power-law behavior, for example, would emerge if the pure Yang-Mills theory possessed an infrared-stable fixed point.

IV. CONCLUDING REMARKS

In this section, we will make a few remarks about extensions and other applications of the decoupling theorem.

(1). As pointed out in the introduction, the decoupling theorem applies to any renormalizable field theory in which different mass scales exist. In particular, the internal heavy lines need not occur only in closed loops. The theorem has recently been applied to a unified model of the strong, electromagnetic, and weak interactions in which an overall simple gauge group SU(5) is broken down to the group $SU(2)\times U(1)\times SU(3)_{\rm color}$ through the ordinary Higgs mechanism.[15,16] The mass scale associated with this breakdown is very large, on the order of 10^{17} GeV. At laboratory momenta (of order $10^{2\pm1}$ GeV), the superheavy vector and scalar meson fields associated with the breakdown decouple except for their contribution to renormalization. The resulting three direct-product groups communicate with each other only through superheavy exchange so that they are effectively decoupled. A consequence of this is that at laboratory momenta, one has three effective coupling constants, each obeying its own renormalization-group equation. These equations can be used to trace the behavior of each of the effective coupling constants down through the momentum region where each of them is small. This includes momenta small enough so that the $SU(3)_{\rm color}$ coupling constant has become much larger than the weak and electromagnetic couplings. Thus, the decoupling theorem plays an important role in understanding a possible unification of elementary particle forces.

(2). We have so far applied the decoupling theorem to situations in which the effective low-energy theory [derived by throwing away terms of order $1/m$(heavy) or smaller] is described by a conventional renormalizable Lagrangian with interactions. Just on dimensional grounds, one can see that this effective theory will never be nonrenormalizable, but it can be noninteracting. A simple example of this situation is the Yukawa theory in which the mass of the scalar field is taken much larger than the fermion mass. In such cases, interactions will appear only to order $1/m$(heavy). These effective interactions will, on dimensional grounds, have a nonrenormalizable structure. The apparent nonrenormalizable divergences are, of course, cut off at momenta on the order m(heavy).

Perhaps the most interesting example of this situation can be found in the σ model.[17] What happens here is that the chiral symmetry forces the interaction to vanish to leading (zeroth) order in $1/m_\sigma^2$. Nonvanishing contributions are found only when terms of order k^2/m_σ^2 are kept. These are the usual nonrenormalizable interactions of the nonlinear σ model.[18]

(3). We return to the Yang-Mills theories to make one last point. We have noted that without knowledge of the infrared behavior of the pure Yang-Mills theory, the mass-shell structure of a heavy field coupled to the gauge field is unknown. Thus, in any theory with an unbroken non-Abelian subgroup, nothing is known about the mass spectrum of the theory. The only calculations that are completely trustworthy in such a theory are those that are insensitive to mass-shell structure.

ACKNOWLEDGMENTS

We would like to acknowledge helpful conversations with our colleagues in Cambridge and elsewhere. We are especially grateful to Steve Weinberg for his comments and helpful suggestions.

*Work supported in part by the National Science Foundation under Grant No. GP40397X.

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[9]The renormalization-group approach tells us only that the sum of all the logarithms is of the form (2.1). The exponent must be explicitly calculated. As far as we can tell, the question of whether or not there exist higher-order corrections to $\gamma(\alpha)$ is an open one although much of the coherent-state formalism is based on the assumption that there are not. Soloviev has argued that there is no fourth-order contribution (L. D. Soloviev, Dokl. Akad. Nauk. SSSR 110, 203 (1956) [Sov. Phys. Dokl. 1, 536 (1957)]).

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