Dynamical symmetry breaking in non-Abelian gauge field theories*

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The possibility of dynamically induced spontaneous symmetry breaking is investigated in a general non-Abelian gauge theory with fermions. We present arguments which demonstrate how an asymptotically free gauge theory with fermions can lead to massive vector mesons without the presence of canonical scalar fields. In particular, the asymptotic freedom of the theory is exploited in an investigation of the relevant Bethe-Salpeter equation beyond the conventional ladder approximation. Differences between the Abelian and the non-Abelian gauge theories are discussed.

I. INTRODUCTION

Spontaneous-symmetry-breaking solutions in non-Abelian gauge theories have received considerable interest in recent years. In all models[1] (for weak and strong interactions) that have been studied so far, the presence of canonical scalar fields in the Lagrangian is necessary for the generation of masses for the vector (Yang-Mills) mesons via the Higgs mechanism. However, there is no suggestion from experiments that scalar particles play any important role in weak or electromagnetic interactions. Furthermore, in strong interactions, it is difficult to retain the desirable feature of asymptotic freedom in the presence of canonical scalar fields.

Recently the alternative possibility, namely, spontaneous symmetry breaking without introducing scalar fields into the Lagrangian, has been revived by Jackiw and Johnson,[2] and Cornwall and Norton.[3] They studied the case of Abelian gauge theories. There, the mass of the vector meson emerges when $\Pi(q^2)$ in the vacuum-polarization tensor

$$\Pi_{\mu\nu}(q)=(g_{\mu\nu}q^2-q_\mu q_\nu)\Pi(q^2)$$

acquires a simple pole at zero momentum transfer with residue $g^2\lambda^2$. Then the complete vector-meson propagator $D_{\mu\nu}(q)$ in the Landau (transverse) gauge becomes, for small q^2, with $\pi(q^2)\sim g^2\lambda^2/q^2$,

$$D_{\mu\nu}(q)=-i\,(g_{\mu\nu}-q_\mu q_\nu/q^2)\,\frac{1}{q^2-q^2\Pi(q^2)}$$
$$\simeq -i\,(g_{\mu\nu}-q_\mu q_\nu/q^2)\,\frac{1}{q^2-g^2\lambda^2}.$$

Such a pole arises from a massless bound excitation in the fermion-antifermion channel, when the fermion acquires a mass term (an additional mass term in case the fermion has a bare mass already) which breaks some symmetry of the Lagrangian. To see this more explicitly, let us look at the following Ward identity:

$$q_\mu\Gamma^\mu_T(p,p+q)=TS^{-1}(p+q)-S^{-1}(p)\bar{T}$$
$$\underset{q\to 0}{\longrightarrow}\lambda P(p^2),$$

where S is the fermion propagator and Γ^μ_T is the fermion-fermion-meson proper vertex with T symmetry [$T=\gamma_5=-\bar{T}$ for chiral symmetry[2] and $T=\bar{T}=\tau_2$ for O(2) symmetry[3] in the Lagrangian]. A fermion mass term which does not commute with T implies the left-hand side to be nonzero as $q\to 0$. Thus Γ^μ_T must have a pole. This pole generates via the Dyson-Schwinger equation the mass of the vector meson discussed above.

In this work we extend this approach of symmetry breaking to the case of non-Abelian gauge theories. The motivation is clear. It is the non-Abelian case that is physically most interesting. What is more, it is the only known asymptotically free[4] renormalizable field theory which has properties vastly different from those of the Abelian case.

We consider an asymptotically free non-Abelian gauge theory with fermions,[4,5] where dynamical symmetry breaking occurs in such a way that the vertices remain proportional to the structure constants f^{abc} of the gauge group. In this case, it is consistent to assume that all the vector mesons can have a common mass generated by dynamical symmetry breaking, and the only Goldstone bound-state pole present in momentum q in the limit $q_\mu\to 0$ is the one in the fermion-antifermion channel. The resulting Bethe-Salpeter equation for the corresponding bound-state wave function $B^{abc}(p,p+q)$ is very similar to that in the Abelian case, except here the vertices have asymptotically free properties.

In fact, because of asymptotic freedom, we are able to obtain an asymptotic solution for $B(p, p+q)$ in the forward direction (i.e., $q_\mu \sim 0$) with large p^2 (group indices suppressed)

$$B(p^2) \underset{p^2\to\infty}{\sim} (\ln p^2)^{-A},$$

where A is a number determined by the gauge group and fermion representations. Assuming that the Green's function with a mass operator and a meson operator insertion, $\Delta\tilde{\Gamma}(q)$, does not have a pole in q_μ, then A is an explicitly determined number independent of the coupling constant g. [The function $\Delta\tilde{\Gamma}(q)$ arises in the discussion of the asymptotic behavior of proper vertex functions.] The asymptotic solution for $B(p^2)$ is in contrast to the Abelian case, where the Bethe-Salpeter equation can be solved only in the ladder approximation.[25] To our knowledge this is the first time where a Bethe-Salpeter equation can be treated beyond the ladder approximation in any relativistic renormalizable field theory. Thus asymptotic freedom not only makes non-Abelian gauge theory physically interesting in the realm of strong interactions, but it also provides the hope that nonperturbative calculations are indeed feasible, despite its formal, non-Abelian complications.

In the process of solving for $B(p^2)$, we shall also demonstrate the consistency of asymptotic freedom and the massiveness of vector mesons. This dynamical generation of mass is preferable to the Higgs mechanism if one desires asymptotic freedom and the absence of scalar particles.

We also briefly discuss how a Goldstone bound excitation in the two Yang-Mills meson sector can arise. This may generate masses for the vector mesons themselves in the absence of both fermions and scalar mesons. Throughout, we do not consider any physical bound states, a study of which is an interesting project by itself.

In Sec. II we derive the Ward identity for proper vertices in a non-Abelian gauge theory with fermions. In Sec. III we discuss how symmetry breaking can generate poles in $\Pi(q^2)$. To simplify the discussion, we consider, for the moment, only the case where the vector mesons' masses are all equal. Then it follows from the Ward-Takahashi (WT) identity that the only Goldstone bound-state pole present in any channel with momentum q at $q_\mu \sim 0$ is the one in the fermion-antifermion channel. The resulting Goldstone bound state decouples from all physical processes. In Sec. IV we obtain a solution for the forward fermion-antifermion bound-state wave function $B(p^2)$ for large p^2; the renormalization equation, Wilson's operator-product expansion, and asymptotic freedom supplement the Bethe-Salpeter equation such that together they enable us to obtain an asymptotic solution for $B(p^2)$. An illustrative model where the vector mesons' masses can be set to be equal is provided. In Sec. V we observe that, if the pure multimeson channels also have Goldstone bound-state poles (i.e., via the WT identity, the vector mesons do not have equal masses), the leading contributions of such poles to the mass generation of vector mesons are identically zero. This seems to indicate that the Goldstone pole in the fermion-antifermion channel plays a dominant and direct role in the mass generation, even in the presence of pure meson channel Goldstone poles. In this section we also argue that the symmetry-breaking fermion mass $\Sigma^V(p)$ vanishes asymptotically as $\Sigma^V(p)\sim(\ln p^2)^{-a}$, independent of the presence or absence of pure multimeson channel poles. It implies via the WT identity that the asymptotic behavior, $B(p^2)\sim(\ln p^2)^{-A}$, as obtained earlier, remains unchanged even when the vector mesons' masses are unequal. This extends the validity of the discussions in Sec. IV. Section VI contains the conclusion. In the Appendix higher-order contributions to the Bethe-Salpeter kernel discussed in Sec. IV are examined. They do not contribute to the leading behavior of $B(p^2)$.

II. PRELIMINARIES

In working with gauge theories it is convenient to introduce a condensed notation.[6,8] The summation convention will be extended to include integration, and a simple general index $i, j, k, \ldots$ may stand for all relevant indices and variables. Whenever a more explicit notation is required group indices will be denoted by $a, b, c, \ldots$ and Lorentz indices by $\lambda, \mu, \ldots$; e.g.; a gauge field will be denoted by $A_\mu^a(x)$ or, in the condensed notation, by A_i.

We consider a general system of fermions interacting with non-Abelian gauge fields. We do not commit ourselves to any specific model and, at least for the moment, we leave the underlying group unspecified. Explicitly, the Lagrangian is given by

$$\mathcal{L} = -\tfrac{1}{4} G_{\mu\nu}^a G^{\mu\nu a} + \bar\psi_n\left(i\not\partial - M - [g]^{ab}\gamma^\mu T_{nm}^b A_\mu^a\right)\psi_m, \tag{2.1}$$

where

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - [g]^{ab} f^{bcd} A_\mu^c A_\nu^d. \tag{2.2}$$

f^{abc} are the structure constants of the (compact, semisimple) group and $[g]^{ab} = g_a \delta^{ab}$ the corresponding gauge coupling-constant matrix. $\mathcal{L}$ is invariant under the simultaneous transformation

$$A_{\mu a} \to A_{\mu a} + \left(\left[\frac{1}{g} \right]_{ab} \partial_\mu + t^b_{ac} A_{\mu c} \right) \omega^b,$$

$$\psi_n \to \psi_n - i\, T^a_{nm}\, \psi_m\, \omega^a, \tag{2.3}$$

$$\bar{\psi}_n \to \bar{\psi}_n + i\, \bar{\psi}_m\, \bar{T}^a_{mn}\, \omega^a.$$

We have defined the (real, antisymmetric) matrices $(t^a)_{bc} \equiv f_{bac}$. The group matrices T^a belonging to the fermion representation may contain parts proportional to γ^5 as well as 1. We can take T^a to be Hermitian. $\bar{T}^a$ is defined by $\gamma_0 \bar{T}^a = T^a \gamma_0$, i.e., it is equal to T^a with all γ^5's replaced by $-\gamma^5$.

As is well known the generating functional of Green's functions is given by[7]

$$W[J,\eta,\bar{\eta}] = \int [dA\, d\psi d\bar{\psi}] (\det M) \times \exp\left[\int i(S - \tfrac{1}{2} F_a{}^2 + J_i A^i + \bar{\eta}_i \psi_i + \eta_i \bar{\psi}_i) \right] d^4x. \tag{2.4}$$

where $\det M$ is the Faddeev-Popov determinant and F_a is a gauge-fixing term. By performing gauge transformations on the variables of integration with ω^a restricted by

$$\omega_a = [M^{-1}]_{ab} \Lambda_b, \quad \Lambda_b \text{ arbitrary} \tag{2.5}$$

and setting the variation equal to zero we obtain the Ward-Takahashi (WT) identities for the generating functional of Green's functions[8]

$$\left\{ -F_a\left(\frac{1}{i} \frac{\delta}{\delta J} \right) + J_i \left(t^b_{ij} \frac{1}{i} \frac{\delta}{\delta J_j} + \Lambda^b_i \right) \left[M^{-1}\left(\frac{1}{i} \frac{\delta}{\delta J} \right) \right]_{ba} - \left(i\bar{\eta}_i T^b_{ij} \frac{\delta}{i\delta\bar{\eta}_j} - i\eta_i \bar{T}^b_{ji} \frac{\delta}{i\delta\eta_j} \right) \left[M^{-1}\left(\frac{1}{i} \frac{\delta}{\delta J} \right) \right]_{ba} \right\} W = 0, \tag{2.6}$$

where

$$\Lambda^a_i = \left[\frac{1}{g} \right]^{ab} \partial_\mu \delta^4(x - x_a), \quad i = (\mu, b, x). \tag{2.7}$$

We assume that the gauge-defining term F_a is linear in the fields and involves only A_i, i.e.,

$$F_a[A] = F_{ai} A_i. \tag{2.8}$$

Then M is given by

$$M_{ab}[A] = F_{ai} (\Lambda^b_i + t^b_{ij} A_j). \tag{2.9}$$

Repeated differentiation of Eq. (2.6) with respect to the external sources $J, \bar{\eta}, \eta$ gives the series of WT identities relating the connected Green's functions of the theory.

However, it will be more convenient to look at WT identities involving *proper* (one-particle-irreducible) vertices. Therefore, writing $W = e^{iZ}$, we introduce the first Legendre transform,[9]

$$\Gamma[\mathcal{A}, \Psi, \bar{\Psi}] = Z[J, \eta, \bar{\eta}] - J_i\, \mathcal{A}_i - \bar{\eta}_i \Psi^i - \bar{\Psi}_i \eta^i, \tag{2.10}$$

where the new variables $\mathcal{A}_i, \Psi_i, \bar{\Psi}_i$ are defined as follows:

$$\mathcal{A}_i \equiv \frac{\delta Z}{\delta J_i}, \quad \Psi_i \equiv \frac{\delta Z}{\delta \bar{\eta}_i}, \quad \bar{\Psi}_i \equiv -\frac{\delta Z}{\delta \eta_i}. \tag{2.11}$$

As is well known Γ is the generating functional of all one-particle-irreducible vertices. From the definitions (2.10) and (2.11) it follows that

$$\frac{\delta\Gamma}{\delta\mathcal{A}_i} = -J^i, \quad \frac{\delta\Gamma}{\delta\Psi_i} = \bar{\eta}_i, \quad \frac{\delta\Gamma}{\delta\bar{\Psi}_i} = -\eta_i. \tag{2.12}$$

We also define the ghost propagator with the gauge fields having the vacuum expectation values $\mathcal{A}_i$:

$$G_{ba}[\mathcal{A}_i] \equiv M^{-1}{}_{ba}\left[\mathcal{A}_i + D_{ij} \frac{\delta}{\delta\mathcal{A}_j} \right] 1, \tag{2.13}$$

where D is the boson propagator.

Using the relations (2.10)–(2.12) and the functional identity

$$f\left[\frac{\delta}{\delta J} \right] \exp(iV[J]) = \exp(iV[J]) f\left[\frac{\delta V}{\delta J} + \frac{\delta}{\delta J} \right],$$

which holds for arbitrary functionals f and V, it is straightforward to rewrite Eq. (2.6) in terms of Γ and derive the following equation for the WT identities in terms of proper vertices[10]:

$$-F_a[\mathcal{A}] - \frac{\delta\Gamma}{\delta\mathcal{A}_i} B_{ia} + \frac{\delta\Gamma}{\delta\Psi_i} B'_{ia} - \frac{\delta\Gamma}{\delta\bar{\Psi}_i} B''_{ia} = 0, \tag{2.14a}$$

where

$$B_{ia} \equiv \Lambda_{ia} + t^a_{ij}\, \mathcal{A}_j - t^b_{ij} D_{kj} G_{bc} \frac{\delta}{\delta\mathcal{A}_k} G^{-1}{}_{ca},$$

$$B'_{ia} \equiv iT^a_{ij} \Psi_j - i\, T^b_{ij} S_{kj} G_{bc} \frac{\delta}{\delta\bar{\Psi}_k} G^{-1}{}_{ca}, \tag{2.14b}$$

$$B''_{ia} \equiv i\, \bar{T}^a_{ji} \bar{\Psi}_j + i\, \bar{T}^b_{ji} S_{jk} G_{bc} \frac{\delta}{\delta\Psi_k} G^{-1}{}_{ca}.$$

D_{ij} and S_{ij} stand for the boson and fermion propagators, respectively. Reducible vertices, e.g., complete propagators, will be represented diagrammatically as shaded circles, whereas unshaded circles will represent proper vertices, i.e., derivatives of Γ. Other diagrammatic conventions are shown in Fig. 1. B_{ai} has then the explicit representation shown in 1(b), similarly for B'_{ai} and B''_{ai}. To make things more transparent we will use the condensed notation of Fig. 1(c). Equation (2.14a) for the WT identities of proper vertices has then the form shown in Fig. 2(a).

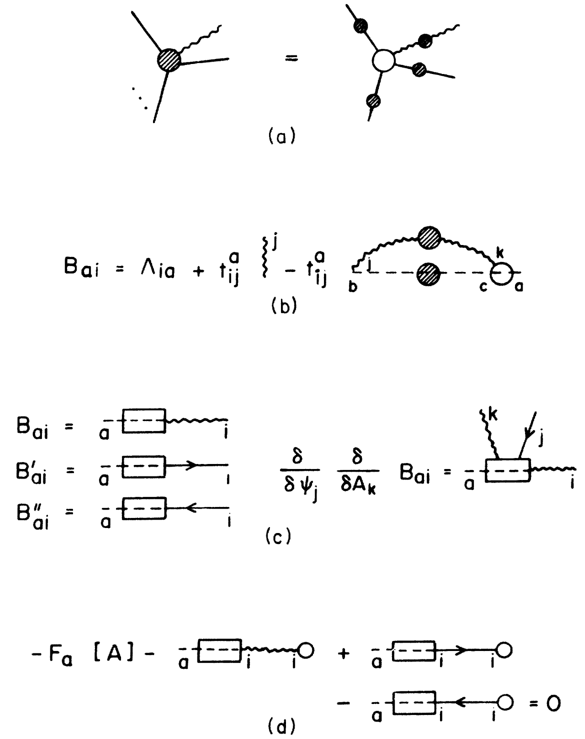

FIG. 1. (a) Diagrammatic notations showing the various particle lines and the relation between proper and improper vertices. In this work straight lines are fermion propagators; wavy, dashed, and dotted lines are meson, ghost, and bound-state propagators, respectively. (b) Diagrammatic representation of the quantity B_{ai} occurring in (2.14). (c) Further diagrammatic notations concerning the quantities B_{ai}'s and their derivatives. (d) The generalized Ward identity for proper vertices, Eq. (2.14), in graphical form.

III. SYMMETRY BREAKING AND MASS GENERATION

A. Dynamical symmetry breaking

Consider the Lagrangian (2.1), which is invariant under the transformation (2.3).

Because of the non-Abelian nature of the system, dynamical symmetry breaking may be introduced in more than one way. The basis of the subsequent discussion will be Eq. (2.14).

Differentiating this equation with

$$\frac{\delta}{\delta\overline{\Psi}_k}\frac{\delta}{\delta\Psi_j}\bigg|_{\psi=\overline{\psi}=a=0}$$

we obtain Fig. 2(b). Written out explicitly it has the form shown in Fig. 2(c). By Lorentz invariance the second term on the left-hand side in Fig. 2(c) is proportional to q^μ and the equation has the structure

$$\begin{aligned} q_\mu[\delta^{ad}+\Lambda^{ad}(q^2)]\,\Gamma^\mu_{dcb}(q,p,p+q) &= [T^a S^{-1}(p+q)]_{cb}-[S^{-1}(p)\overline{T}^a]_{cb}\\ &\quad+\Lambda^{adc}(q,p+q)\,S^{-1}{}_{db}(p+q)-\Lambda^{adb}(q,p)\,S^{-1}{}_{dc}(p). \end{aligned} \tag{3.1}$$

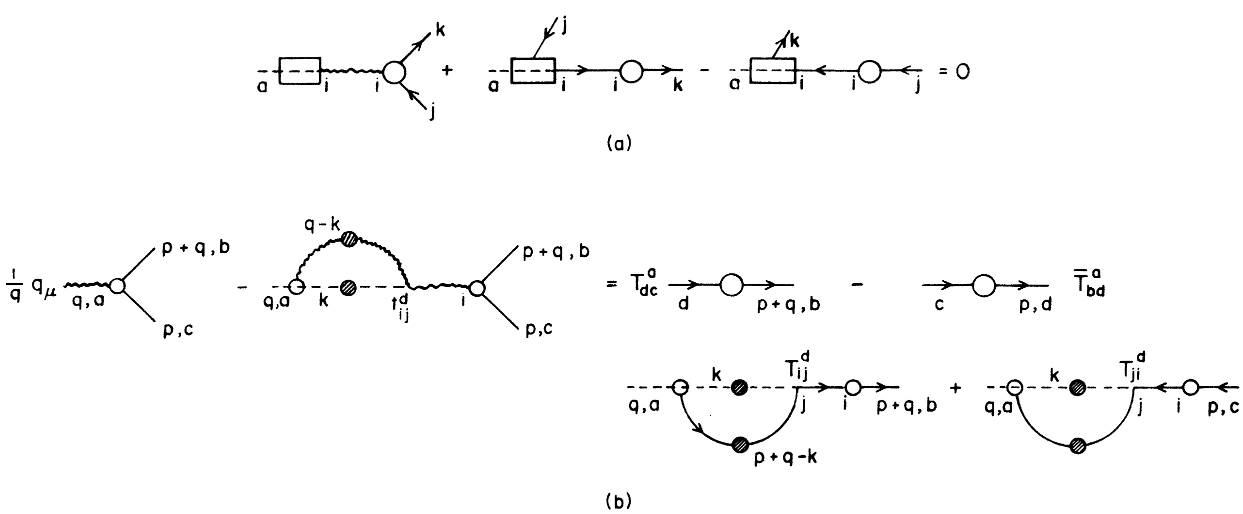

FIG. 2. (a) Ward identity for the meson-fermion-antifermion proper vertex. (b) More explicit representation of (a).

The functions Λ^{ab}, Λ^{abc} are finite quantities in the renormalized theory.

Dynamical symmetry breaking may be introduced by assuming

$$TS^{-1}(p) - S^{-1}(p)\bar{T} \neq 0. \tag{3.2}$$

The WT identity (3.1) now implies a pole in Γ^{abc}_μ as $q_\mu \to 0$ and gives the residue in terms of (3.2). The pole will be attributed to a bound massless excitation (Goldstone boson) in the fermion-antifermion channel (Fig. 3).

The existence of this massless bound state will, of course, result in poles in other Green's functions of the theory. However, because of gauge invariance, the residues of all these vertex functions will be restricted and interrelated by WT identities. In particular, consider the WT identity satisfied by the 3-point vector-meson vertex. It is obtained by differentiating Eq. (2.14) and is shown diagrammatically in Fig. 4. As in the case of Eq. (3.1), this equation, when written out explicitly, has the structure

$$\begin{aligned} q_\mu(\delta^{ad}+\Lambda^{ad}(q^2))\,\Gamma^{dcb}_{\mu\lambda\nu}(q,p,p+q) &= i\,[t^a D^{-1}{}_{\lambda\nu}(p+q)]_{cb} - i[D^{-1}{}_{\lambda\nu}(p)\,t^a]_{cb} \\ &\quad + \Lambda^{\lambda\mu}_{adc}(p,q)D^{-1\,db}_{\mu\nu}(p+q) - \Lambda^{\mu\nu}_{abd}(p,q)D^{-1\,dc}_{\lambda\mu}(p), \end{aligned} \tag{3.3}$$

where $D^{ab}_{\mu\nu}(p)$ is the meson propagator. Throughout this work, we retain the real completely antisymmetric property of the structure constants $(t^a)_{bc} = f_{bac}$ even in the presence of symmetry breaking.

Then, with $\Lambda^{\mu\lambda}_{adc}(p,q) = f_{adc}\Lambda^{\mu\lambda}(p,q)$, the right-hand side of Eq. (3.3) becomes, in the limit $q_\mu \to 0$,

$$[\Pi^{\lambda\nu}(p), t^a]_{cb} + (g^{\mu\nu}p^2 - p^\mu p^\nu)\Lambda^{\mu\lambda}(p)\,[\Pi(p^2)t^a]_{cb} - (g^{\mu\lambda}p^2 - p^\mu p^\lambda)\Lambda^{\mu\nu}(p)[\,t^a\Pi(p^2)]_{cb}. \tag{3.4}$$

For the moment let us neglect the Λ functions in Eq. (3.3), which can then be written as

$$\lim_{q_\mu\to 0} q^\mu\,\Gamma^{acb}_{\mu\lambda\nu}(q,p,p+q) = (g_{\lambda\nu}p^2 - p_\lambda p_\nu)f_{cab}\,(\Pi_c(p^2) - \Pi_b(p^2))\ . \tag{3.5}$$

Here we have taken $\Pi_{cd}(p^2)$ to be diagonal, $\Pi_{cd}(p^2) = \delta_{cd}\Pi_c(p^2)$, without loss of generality. Dynamical symmetry breaking introduces masses for the vector mesons so that $\Pi_c(p^2 \sim 0) = \mu_c{}^2/p^2$. If the vector-meson masses are different, then $\Pi_c(p^2) - \Pi_b(p^2) \neq 0$, and hence Eq. (3.4) implies that $\Gamma^{acb}_{\mu\lambda\nu}(q,p,p+q)$ has a pole in q as $q_\mu \to 0$. The inclusion of the Λ functions does not change this conclusion. There will be massless bound-state poles in other proper vertices as well, as required by WT identities. This is a rather general situation of dynamical symmetry breaking. It is quite complicated, involving a set of coupled Bethe-Salpeter equations for the various massless bound-state wave functions.

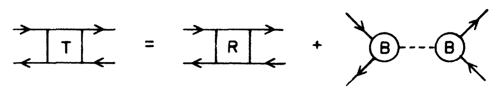

FIG. 3. The massless bound state in the fermion-antifermion channel. T is one-particle-irreducible.

From Lorentz invariance, we see that the massless scalar pole contributions to $\Gamma^{abc}_\mu(q,p,p+q)$, $\Gamma^{abc}_{\mu\lambda\nu}(q,p,p+q)$, the ghost-ghost-meson vertex function $\gamma^{abc}_\mu(q,p,p+q)$ and the four-meson vertex function $\Gamma^{abcf}_{\mu\nu\lambda\rho}(q,p,p+q+r,r)$ must be of the following forms:

$$\Gamma^{abc}_\mu(q,p+q)\big|_{\text{pole}} = q_\mu I^{ad}(q^2)\frac{i}{q^2}B^{dbc}(p,p+q), \tag{3.6}$$

$$\Gamma^{abc}_{\mu\nu\lambda}(q,p+q)\big|_{\text{pole}} = q_\mu I^{ad}(q^2)\frac{i}{q^2}P^{dbc}_{2\,\nu\lambda}(p+q,p), \tag{3.7}$$

$$\gamma^{abc}_\mu(q,p+q)\big|_{\text{pole}} = q_\mu I^{ad}(q^2)\,\frac{i}{q^2}\,P^{dbc}_4(p,p+q), \tag{3.8}$$

$$\Gamma^{abcf}_{\mu\nu\lambda\rho}(q,p+q+r,r)\big|_{\text{pole}} = q_\mu I^{ad}(q^2)\frac{i}{q^2}P^{dbcf}_{3\nu\lambda\rho}(p,p+q+r,r), \tag{3.9}$$

where the bound-state wave functions B and P_i are assumed to be nonzero as $q_\mu \to 0$.

There can also be present other poles, such as in $\Gamma^{acb}_{\mu\lambda\nu}(q,p,p+q)$,

$$\Gamma^{acb}_{\mu\lambda\nu}(q,p,p+q)\big|_{\text{pole}} = \frac{i\,q_\mu}{q^2}\,[I^{ad}(q^2)P^{dbc}_{2\nu\lambda}(p+q,p) + I^{ad}(q^2)q\cdot p\,R^{dbc}_{\nu\lambda}(p+q,p)]. \tag{3.10}$$

However, such poles (the last term on the right-

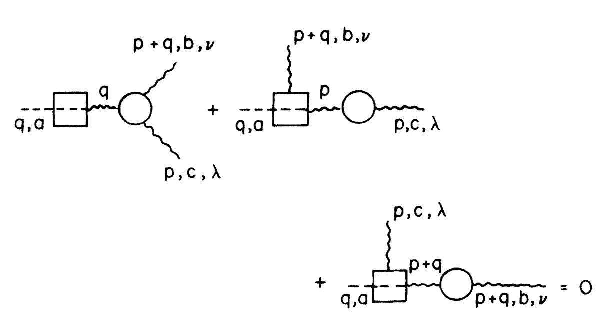

FIG. 4. Ward identity for the 3-point vector-meson proper vertex.

hand side) are not required by the WT identity and dynamical symmetry breaking. These poles (which will be called second-order poles) may be present only when $q_\mu \neq 0$ as $q^2 \to 0$. We will not discuss the existence or absence of all second-order poles. In our discussions (e.g., on the Bethe-Salpeter equations) we take the limit $q_\mu \to 0$ where these poles are negligible, if present.

In the weak-coupling limit, it has been shown[11] that the Bethe-Salpeter equations do not have a solution for the bound-state wave function $P^{dcb}_{\lambda\nu}(p, p+q)$, but can admit a second-order wave function $R^{dcb}_{\lambda\nu}(p, p+q)$. Consistency with the WT identity (3.3) then requires that all the vector mesons must have a common mass (induced by dynamical symmetry breaking). We do not know if the general case (i.e., unequal masses) exists in the asymptotic free limit or not. Here we consider, for the moment, only the case where all vector mesons' masses are equal. Then $\Gamma^{abc}_{\mu\lambda\nu}(q, p, p+q)$ does not have a pole in q at $q=0$. This is true also for the ghost–ghost–vector-meson vertex; this follows from the Ward-Slavnov identity satisfied by this vertex which can be obtained from (2.6). In fact, just from gauge group symmetry, Bose symmetry, and Lorentz invariance, it is straightforward to see that $\Gamma^{abc}_{\lambda\mu\nu}$ cannot have a scalar, massless simple pole when the vector mesons (a, λ), (b, μ), and (c, ν) have the same mass μ. However, the vector meson does acquire a mass because the mass generation mechanism exhibited below (Sec. III B) is operative as long as Γ^{abc}_{μ} has a pole. The same holds for all higher vertices that contain only external vector-meson lines, in particular for the 4-point vertex the WT identity for which is given by Fig. 5(a).

As $q_\mu \to 0$, the only pole left is the fermion-antifermion Goldstone pole, and the set of Bethe-Salpeter equations reduces to the simple one discussed in Sec. IV. The situation is then exactly analogous to that in the Abelian case treated in Refs. 2 and 3 where, from gauge invariance, we again have $q_\mu \Gamma^{\mu\lambda\cdots\nu} = 0$ [Fig. 5(b)] for all vertices containing only external photon lines. The only difference in our case is that, as we shall see,

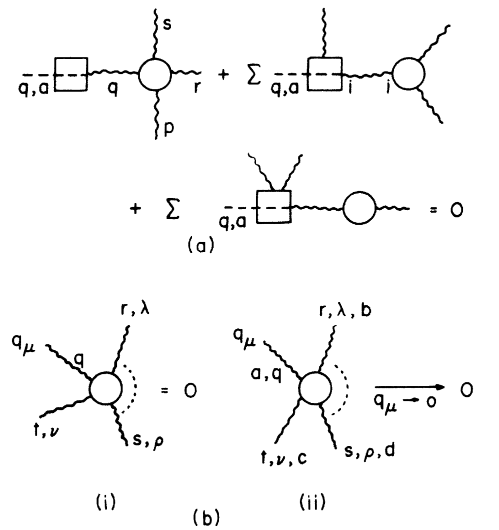

FIG. 5. (a) Ward identity for the 4-point vector-meson proper vertex. (b) Ward identity for n-point vector-meson vertex: (i) in the Abelian case, (ii) in the non-Abelian case in the limit $q_\mu \to 0$.

the substitution of a non-Abelian vector meson makes the theory asymptotically free and allows a treatment of the Bethe-Salpeter equation beyond the ladder approximation.

B. Vector-meson mass generation

Consider now the vacuum-polarization tensor $\Pi^{ab}_{\mu\nu}(q)$, the Dyson-Schwinger equation for which, derived by standard methods, has the form shown in Fig. 6. As $q_\mu \to 0$, the pole part of $\Pi(q^2)$ in $\Pi^{ab}_{\mu\nu}(q)$ is given by Eq. (3.11) (see Fig. 7)

$$\begin{aligned}\Pi^{ab}_{\mu\nu}(q)|_{\text{pole}} &= q^\mu I^{ad}(q^2)\frac{i}{q^2}\left(-q^\nu I^{db}(q^2)\right) \\ &= -i\frac{q_\mu q_\nu}{q^2}\lambda^{ad}\lambda^{db},\end{aligned} \tag{3.11}$$

where $I^{ab}(q^2)|_{q=0} \equiv \lambda^{ab}$. Therefore, nonvanishing $B^{abc}(p, p+q)$, $R^{abc}_{\lambda\nu}(p, p+q)$, etc., provide a nonvanishing mass matrix for the vector meson. In the unequal-masses case, the P_i poles should be added to Eq. (3.11) as well.

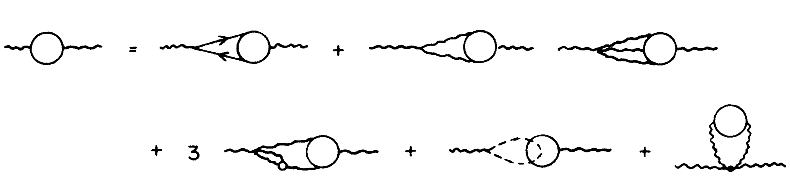

FIG. 6. Dyson-Schwinger equation for the vacuum-polarization tensor $\Pi_{\mu\nu}$.

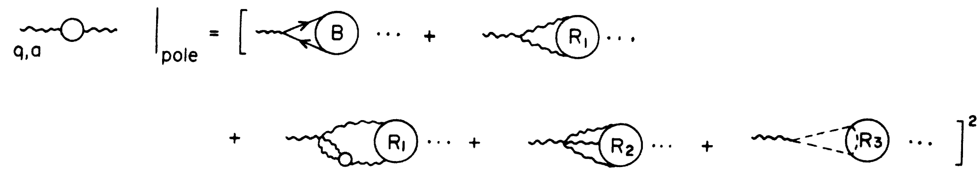

FIG. 7. Graphical representation of (3.11).

C. Decoupling of Goldstone boson

The parts proportional to $q^\mu q^\nu$ in the meson propagator do not contribute when acting on on-mass-shell vertices. This follows from the WT identities as in the Abelian case. Therefore, the demonstration of the decoupling[12] is exactly analogous to that in Ref. 2 and need not be repeated here.

IV. THE GOLDSTONE BOUND EXCITATIONS

In Secs. II and III we assumed that the theory possesses an asymmetric solution and then proceeded to examine the consequences of this assumption. This was accomplished with the help of the general framework of WT identities and Schwinger-Dyson equations that relate the various Green's functions of the theory in a manner consistent with the requirements of gauge invariance. We next turn to the question of whether the assumption can actually be implemented, i.e., whether a symmetry breaking solution can exist. The Bethe-Salpeter equations and the operator-product expansion in conjunction with the asymptotically free behavior of non-Abelian field theory will be our main tools in discussing this problem.

We consider the case where symmetry breaking is introduced with a common vector-meson mass. Then only the fermion-meson proper vertex function can have a pole in the single-meson channel. (See Sec. V for the removal of this constraint.)

To obtain the fermion-antifermion bound-state function $B(p, p+q)$, we reexpress the fermion-meson vertex function in the Bethe-Salpeter form in Fig. 8(a), where the Bethe-Salpeter (2-fermion irreducible) kernels exclude the one-meson intermediate state. Only the fermion-antifermion-meson vertex function has a massless scalar pole as $q\to 0$. Therefore, equating the pole terms of Fig. 8(a), we obtain, using Eq. (3.6), Fig. 8(b). Thus $B(p,p)$ must obey this (forward) homogeneous Bethe-Salpeter equation. Clearly, the function $B(p,p)=B(p^2)$ obeys multiplicative renormalization so we can consider the renormalized version of Fig. 8(b),

$$B(p^2)=\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\,S(k)B(k^2)S(k)K(k,p)\,, \tag{4.1}$$

where the group indices are suppressed. $S(k)$ is the full fermion propagator.

In this section we will show that the bound-state wave function satisfying the Bethe-Salpeter Eq. (4.1) has a solution

$$B(p^2)\underset{p^2\to\infty}{\simeq}(\ln p^2)^{-A}[1+O((\ln p^2)^{-1})]\,,\quad A>\tfrac{1}{2} \tag{4.2}$$

and that, furthermore, this solution is obtained by simply considering the simplified form of Eq. (4.1)

$$B(p^2)=\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\,S(k)B(k^2)S(k)\times\Gamma^\mu(k,p)D_{\mu\nu}(k-p)\Gamma^\nu(k,p)\,, \tag{4.3}$$

where Γ^μ is the full proper fermion-meson vertex function and $D_{\mu\nu}$ is the full meson propagator. Note that this is not a ladder approximation (see Fig. 9). Equation (4.3) and its solution, Eq. (4.2), follow directly as a consequence of the fact that non-Abelian theories are ultraviolet stable as the effective coupling constant approaches zero (asymptotic freedom). If the function $B(p,p)$ were a Green's function one could automatically establish a result of the type Eq. (4.2) by using renormaliza-

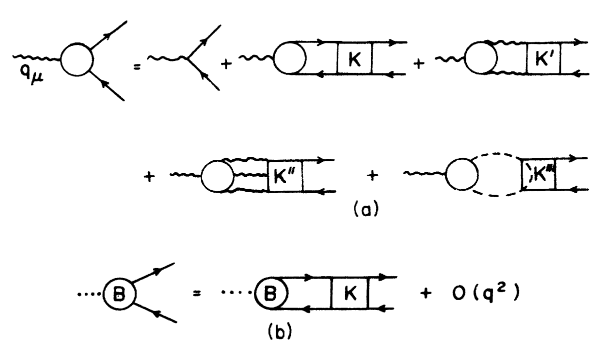

FIG. 8. (a) Dyson-Schwinger equation for the fermion-antifermion-meson proper vertex. (b) The zero-mass scalar pole part of (a).

tion-group Callan-Symanzik arguments.[13] In fact, by using this type of arguments together with the operator-product expansion[14] (in momentum space) we find that these are in agreement with Eqs. (4.2) and (4.3), and that these provide the power A which could not be otherwise simply obtained from the Beth-Salpeter analysis.

To find A and study Eq. (4.3), we require some of the known results of asymptotic free (renormalizable) theories. In the next two subsections, relevant properties of the non-Abelian gauge theory are summarized.

A. Asymptotic free theory[4,5,15]

The renormalized one-particle-irreducible (1PI) Green's functions $\Gamma^{(n)}(p_i)$, given by (with $Z_{[n]}$ as a product of the relevant renormalization constants)

$$\Gamma^{(n)}(p_i) = Z_{[n]}{}^{1/2}\Gamma^{(n)}_{\text{unrenormalized}}(p_i), \tag{4.4}$$

are finite functions of the renormalized charge g, the gauge parameter α, and the renormalization point μ. $\Gamma^{(n)}(p_i)$ then satisfies the renormalization-group equation[16]

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(g,\alpha)\frac{\partial}{\partial g}-\gamma^{[n]}(g,\alpha)+\eta(g,\alpha)\frac{\partial}{\partial\alpha}\right]\times\Gamma^{(n)}(g,\alpha,\mu)=0, \tag{4.5}$$

where

$$\beta(g,\alpha)=\mu\frac{\partial g}{\partial\mu}\Big|_{g_u,\alpha_u,\Lambda\text{ fixed}}, \tag{4.6}$$

$$\gamma^{[n]}(g,\alpha)=\frac{1}{2}\mu\frac{\partial}{\partial\mu}(\ln Z_{[n]})\Big|_{g_u,\alpha_u,\Lambda\text{ fixed}}, \tag{4.7}$$

$$\eta(g,\alpha)=-2\alpha\gamma_3(g,\alpha)=-2\alpha\left(\frac{1}{2}\mu\frac{\partial}{\partial\mu}\ln Z_3\right)\Big|_{g_u,\alpha_u,\Lambda\text{ fixed}} \tag{4.8}$$

Pure dimensional analysis implies that

$$\Gamma^{(n)}(\lambda p_i,g,\mu)=\mu^{4-n}\,\Gamma^{(n)}\left(\frac{\lambda p_i}{\mu}\right), \tag{4.9}$$

so that Eq. (4.5) can be rewritten as

$$\left[\lambda\frac{\partial}{\partial\lambda}-\beta(g,\alpha)\frac{\partial}{\partial g}-4+n+\gamma^{[n]}(g,\alpha)-\eta(g,\alpha)\frac{\partial}{\partial\alpha}\right]\times\Gamma^{(n)}(\lambda p_i,g,\mu,\alpha)=0. \tag{4.10}$$

This equation can be most effectively solved by introducing the effective coupling constant $\bar g(t,g,\alpha)$ and the effective gauge parameter $\bar\alpha(t,g,\alpha)$:

$$\frac{d}{dt}\bar g=\beta(\bar g,\bar\alpha),\quad \bar g(0,g,\alpha)=g \tag{4.11}$$

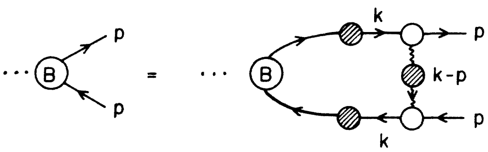

FIG. 9. Graphical representation of Eq. (4.3).

$$\frac{d}{dt}\bar\alpha=\eta(\bar g,\bar\alpha),\quad \bar\alpha(0,g,\alpha)=\alpha \qquad t=\ln\lambda. \tag{4.12}$$

If the above equations admit a solution such that

$$\lim_{t\to\infty}\bar g(t,g,\alpha)=g_\infty, \tag{4.13}$$

$$\lim_{t\to\infty}\bar\alpha(t,g,\alpha)=\alpha_\infty, \tag{4.14}$$

then we say that g_∞ (α_∞) is an ultraviolet fixed point. This fixed point is determined by the zeros of $\beta(g,\alpha)$ $[\eta(g,\alpha)]$; i.e., $\beta(g_\infty,\alpha_\infty)=\eta(g_\infty,\alpha_\infty)=0$ (there may exist more than one solution, of course). The fixed points g_∞ of the renormalization group will be ultraviolet (uv) stable if and only if $(d/dg)\beta(g,\alpha)|_{g_\infty}<0$, and similarly for the gauge fixed points α_∞. The asymptotic behavior of the Green's functions $\Gamma^{(n)}$ are then governed by the asymptotic values of the effective coupling constant and the effective gauge parameter. In particular, $\beta(g=0)=\eta(\alpha=0)=0$ is true. Thus, we conclude that the theory is asymptotically free if and only if

$$\frac{d}{dg}\beta(g\simeq 0)<0. \tag{4.15}$$

The theory is also asymptotically determined by the Landau gauge ($\alpha=0$) if

$$\frac{d}{d\alpha}\eta(\alpha\simeq 0)<0. \tag{4.16}$$

Since asymptotic freedom is a physically desirable feature, we will only consider theories that are asymptotically free. From Eqs. (4.8) and (4.10), we will note that if we begin in the Landau gauge ($\alpha=0$) where $\eta(g,\alpha)=0$, then, the proper vertex functions will remain in the Landau gauge under a change in renormalization point. Thus we may stay in the Landau gauge ($\alpha=0$) whenever convenient. We will come back to discuss the gauge dependence later.

B. The operator-product expansion for the $\psi\bar\psi$B vertex

Following Callan,[14] using a generalization of the Wilson operator-product expansion in momentum space, we would like to analyze the behavior of the vertex function $\Gamma_{\psi B\bar\psi}(p_1,p_2,p_3)$ in the following limits: (1) $p_1=p-q/2$, $p_2=-p-q/2$, $p_3=q$, all momenta large, with p larger than q (see Fig. 10). This will help us to study the Bethe-Salpeter kernel at certain asymptotic values of its momenta. (2) $p_1 = p - q/2$, $p_3 = p - q/2$, $p_2 = q$, where p is large and q is finite (and in fact the limit $q_\mu \to 0$).

In the first case we have

$$\Gamma_{\psi B\bar\psi}(p - q/2, -p - q/2, q) = \sum_\phi C^\phi_{\psi B}(p)\Gamma^\phi_{\bar\psi}(q), \tag{4.17}$$

where $\{\phi\}$ sums over a complete set of operators which make the expansion valid. Γ^ϕ_ψ is effectively a "one-point" ϕ-inserted Green's function. As $p^2 \to \infty$, the leading operator in the sum is, of course, the fermion operator ψ. $\Gamma_{\psi\bar\psi B}$ obeys the following renormalization-group equation (in the Landau gauge) [see Eq. (4.5)]:

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - 2\gamma_\psi - \gamma_B\right]\Gamma_{\psi B\bar\psi}(p, q) = 0\,. \tag{4.18}$$

Since Γ^ϕ_ψ, with ϕ as the fermion field operator, obeys

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - 2\gamma_\psi\right]\Gamma^{\phi=\psi}_{\bar\psi}(q) = 0 \tag{4.19}$$

we obtain, using Eqs. (4.17), (4.18), and (4.19),

$$\left[\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \gamma_B\right]C^{\phi=\bar\psi}_{\psi B}(p) = 0\,, \tag{4.20}$$

where

$$\begin{aligned}\gamma_\psi &= f_0 g^2 + O(g^4)\,,\\ \gamma_B &= c_0 g^2 + O(g^4)\,,\\ \beta(g) &= -\frac{b_0}{2}g^3 + O(g^5)\end{aligned} \tag{4.21}$$

are, respectively, defined to be [following Eqs. (4.6), (4.7), and (4.8)]

$$\gamma_\psi = -\frac{1}{2}\frac{\partial}{\partial \ln\Lambda}\ln Z_2\,, \tag{4.22a}$$

$$\gamma_B = -\frac{1}{2}\frac{\partial}{\partial \ln\Lambda}\ln Z_3\,, \tag{4.22b}$$

$$\beta(g) = -g\frac{\partial}{\partial \ln\Lambda}\left(\frac{Z_3^{3/2}}{Z_1}\right). \tag{4.22c}$$

Z_1, Z_2, and Z_3 are the vertex, the fermion propagator, and the vacuum polarization renormalization constants, respectively, and Λ is the ultraviolet cutoff.

When the theory becomes asymptotically free, the effective charge goes like

$$g_k^2 \underset{k^2\to\infty}{\simeq} \frac{2}{b_0 \ln k^2}\,. \tag{4.23}$$

Under these conditions Eqs. (4.18) to (4.20) can be readily solved to yield

$$\Gamma_{\psi B\bar\psi}(p\sim k, q\sim k) \underset{k\to\infty}{\propto} g_k(\ln k^2)^{-2f_0/b_0 - c_0/b_0} \tag{4.24}$$

and

$$\Gamma_{\psi B\bar\psi}(p - q/2, -p - q/2, q) \underset{\substack{p\gg q\\ q\to\infty}}{\simeq} g_p(\ln q^2)^{-2f_0/b_0}(\ln p^2)^{-c_0/b_0} \times\left[\text{constant} + O\left(\frac{\ln q^2}{\ln p^2}\right)\right]. \tag{4.25}$$

It should also be recalled that the asymptotic behavior of the vector-boson and fermion propagator is given by

$$\begin{aligned}S(k) &= i\frac{\not k}{k^2}S_0(\ln k^2)^{2f_0/b_0}\\ &= i\frac{k}{k^2}S_F(k)\,,\end{aligned} \tag{4.26a}$$

$$\begin{aligned}D^{ab}_{\mu\nu}(k) &= -i\delta^{ab}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2}d(k^2)\\ &= -i\delta^{ab}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2}d_0(\ln k^2)^{2c_0/b_0}\,,\end{aligned} \tag{4.26b}$$

such that

$$S_F(k)d^{1/2}(k^2)\Gamma_{\psi B\bar\psi}(k) \underset{k\to\infty}{\simeq} g_k\,. \tag{4.26c}$$

Let us now consider the second case $\Gamma_{\psi\bar\psi B}(p - q/2, -p - q/2, q)$ where p is large and q is finite. Introducing an operator-product expansion

$$\Gamma_{\psi\bar\psi B}(p - q/2, -p - q/2, q) \underset{p\to\infty}{\simeq} \sum_\phi C^\phi_{\psi\bar\psi}(p \pm q/2)\Gamma^\phi_B(q)\,, \tag{4.27}$$

we can again use the renormalization-group equations analogous to Eqs. (4.18) to (4.20).

Since q_μ remains finite, the zero on the right-hand side of Eq. (4.18) should be replaced by $\Delta\Gamma_{\bar\psi B\psi}(p, q)$, which is $\Gamma_{\bar\psi B\psi}$ with a mass operator insertion. We also have

$$\left[\mu'\frac{\partial}{\partial\mu'} + \beta(g)\frac{\partial}{\partial g} - \gamma_B - \gamma_\phi\right]\Gamma^\phi_B(q) = \Delta\bar\Gamma(q)\,, \tag{4.28}$$

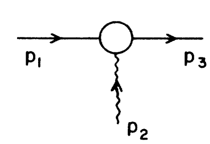

FIG. 10. The proper vertex with momenta p_i.

where $\Delta\tilde{\Gamma}(q)$ is $\tilde{\Gamma}_B^\phi(q)$ with a mass operator insertion; it then follows from Eq. (4.28) and Eq. (4.18), with $\Delta\Gamma_{\bar{\psi}B\psi}(p,q)$ on its right-hand side, that

$$\left[\mu'\frac{\partial}{\partial\mu'}+\beta(g)\frac{\partial}{\partial g}+\gamma_\phi+\frac{[\Delta\tilde{\Gamma}(q)-\Delta\Gamma'(q)]}{\Gamma_B^\phi(q)}\bigg|_{q\sim 0}\right]C_{\psi\bar{\psi}}^\phi(p)=0\,, \quad (4.29)$$

where

$$\Delta\Gamma\cong C_{\psi\bar{\psi}}^\phi(p)\Delta\Gamma'(q)\,,$$

and

$$\rho=[\Delta\tilde{\Gamma}(q)-\Delta\Gamma'(q)]/\Gamma_B^\phi(q)|_{q\sim 0}$$

is a constant. Since the leading term in the operator-product expansion obviously comes from ϕ being the meson field operator, we obtain

$$\Gamma_{\psi\bar{\psi}B}\underset{p\to\infty}{\simeq}\tilde{\Gamma}^{(0)}(q)(\ln p^2)^{+c_0/b_0-\rho-1/2-2f_0/b_0}+\tilde{\Gamma}^{(1)}(q)(\ln p^2)^{c_0/b_0-\rho-2f_0/b_0-3/2}+\cdots\,, \quad (4.30)$$

where we have used Eqs. (4.28) and (4.29), where $\tilde{\Gamma}^{(0)}(q)$ is $\Gamma_B^B(q)$. Clearly, if $\Gamma_{\psi\bar{\psi}B}^\mu(p,q)$ has a pole in the fermion-antifermion channel as $q\to 0$, as in Eqs. (3.1) and (3.2),

$$\lim_{q\to 0}\Gamma_{\psi\bar{\psi}B}(p,q)\neq 0\,,$$

then at least one of the $\tilde{\Gamma}^{(i)}(q)$ must have a pole as $q\to 0$. If $\tilde{\Gamma}^{(0)}(q)$ has a pole then we get[17]

$$B(p^2)\underset{p^2\to\infty}{\simeq}(\ln p^2)^{c_0/b_0-\rho-2f_0/b_0-1/2}\,. \quad (4.31)$$

Of course, we have assumed that $\tilde{\Gamma}_{\psi\bar{\psi}B}^\mu$ can have a pole. To demonstrate the existence of such a pole, we have to examine the Bethe-Salpeter equation.

C. Solution to the Bethe-Salpeter equation

We want to show that Eq. (4.2) is consistent with a solution to the Bethe-Salpeter equation (4.1).

Let us start by considering the no-loop, one-boson exchange part of the kernel. This is given by Eq. (4.3) where, for reasons of simplicity, we do not display group and spin indices.

We will discuss Eq. (4.3) in the Euclidean space (i.e., the standard Wick's rotation is performed). Because of the presence of mass terms, the usual infrared divergence of non-Abelian gauge theory is removed. For large momentum k, the mass terms can be neglected. Thus we can in effect drop the mass terms and introduce an effective lower cutoff λ^2. We further assume that λ^2 is large enough (and/or asymptotic behavior sets in early enough) that asymptotic behavior is valid inside the integral for $\lambda<k<\infty$. Since the integration is carried out with fixed momentum p, Eq. (4.25) for the proper vertex functions $\Gamma_\mu(p>k)$ and $\Gamma_\nu(k>p)$ is substituted into Eq. (4.3). The propagators are given by Eqs. (4.26) and (4.27). Putting all these together, we obtain, in the Landau gauge,

$$B(p^2)\underset{p^2\,\text{large}}{=}B_0\int_{\lambda^2}^{p^2}\frac{dk^2}{p^2}B(k^2)g_p^{\,2}\left[1+O\left(\frac{\ln k^2}{\ln p^2}\right)\right]+B_0\int_{p^2}^{\infty}\frac{dk^2}{k^2}B(k^2)g_k^{\,2}\left(\frac{\ln k^2}{\ln p^2}\right)^{2f_0/b_0}\left[1+O\left(\frac{\ln p^2}{\ln k^2}\right)\right], \quad (4.32)$$

where λ^2 is an effective cutoff due to the presence of mass terms (and the lack of infrared divergence). The solution we obtain from Eq. (4.32) is independent of λ^2, as should be the case if the assumption on λ^2 stands any chance of being correct. The constant B_0 is a number [it contains the factor $1/(2\pi)^4$] which depends on the Feynman rules, group properties, angular integrations, and the coefficient from the operator-product expansions. B_0 is left undetermined, for the moment.

Now Eq. (4.32) can be solved simply by reducing it into a differential equation. In particular, in the Landau gauge $f_0=0$, and we neglect for the moment the nonleading terms in Eq. (4.32). Then we obtain that, with $x=\ln p^2$,

$$\frac{d}{dx}\left[\frac{x}{1+x}\frac{dB}{dx}\right]=-\frac{dB}{dx}+\frac{B_0^{\text{LG}}}{b_0}\frac{B}{x}\,, \quad (4.33)$$

where B_0^{LG} is the constant B_0 evaluated in the Landau gauge. For $x\to\infty$, we see immediately that $B(x)=x^{-A}$ can be a solution of Eq. (4.33). Substituting it into Eq. (4.33) we obtain

$$A(A+1)x^{-A-2}=Ax^{-A-1}+\frac{B_0^{\text{LG}}}{b_0}x^{-A-1}\,,$$

so as $x\to\infty$, there will be a solution of the form (4.2) with

$$A=-\frac{B_0^{\text{LG}}}{b_0}\,.$$

We note that we have dropped lower-order terms for the proper vertex function in the solution of the Bethe-Salpeter equation. There may be present terms of the order $[(\ln p^2)/\ln k^2]^{+\epsilon}$ for $k^2>p^2$ which will also contribute to the leading term of $B(p^2)$. These we can easily include, bringing only a change in the coefficient B_0. For consistency, maybe we should also include the Goldstone excitation exchange into the kernel. This again changes only the coefficient B_0, giving the same logarithmic form for $B(p^2)$ in the asymptotic limit, with A being an undetermined number. As is clear, A must be positive in order to render the integral in the Bethe-Salpeter equation convergent. The bound-state wave function $B(p^2)$ must also satisfy the Bethe-Salpeter normalization condition[18] (for $q_\mu \to 0$)

$$\mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} \frac{d^4k'}{(2\pi)^4} B(k^2)\left[\frac{\partial}{\partial q_\mu} S(k+q)S(k)(2\pi)^4\delta^4(k-k') + S(k+q)S(k)K(k,k',q)S(k'+q)S(k')\right] B(k'^2) = 2q_\mu \,. \tag{4.34}$$

It is straightforward to see that for the integral to converge, A must be greater than $\frac{1}{2}$. So the existence of a scalar massless pole requires $A > \frac{1}{2}$. The normalization condition imposes a more stringent requirement than the convergence of the Bethe-Salpeter equation. We will discuss the constraint on A later.

Of course the solution discussed so far has been derived for the one-boson exchange diagram only. But one can show that all higher-loop contributions to the kernel are nonleading by powers of logs, in comparison with the one-boson exchange part of the kernel we just examined. This is a consequence of asymptotic freedom. In the Appendix we show this by evaluating the one-loop graphs and then give a general argument for more than one-loop graphs. We should stress that in leading asymptotic order, we obtained a consistent solution for the (forward) Bethe-Salpeter problem. This must be contrasted with the Abelian case where the existence of a solution hinges crucially on the ladder approximation and on a weak-coupling assumption. We should stress that one could proceed as in the Abelian case and get a solution $B \sim (p^2)^{-\epsilon}$. In view of our findings we see that the usual ladder approximation does not pick up the leading term of $B(p^2)$ in asymptotic free theories.

We note that in a gauge where $\alpha \neq 0$, there is an extra complication to the Bethe-Salpeter equation (4.3). For $\alpha \neq 0$, the vector-meson propagator will have an extra term proportional to $\alpha k_\mu k_\nu / k^4$ in addition to its transverse part. For $k \sim p$,

$$\Gamma_\mu(k,p)\big|_{\text{pole}} \sim \lambda B(k-p,p)\frac{(k-p)_\mu}{(k-p)^2}\,,$$

$$\Gamma_\nu(k,p)\big|_{\text{pole}} \sim \lambda B(k-p,p)\frac{(k-p)_\nu}{(k-p)^2}\,.$$

Thus in Eq. (4.32) the right-hand side of the Bethe-Salpeter equation will have an additional term due to this pole term.

This additional term cannot be neglected and the analysis of the Bethe-Salpeter equation (4.32) becomes much more complicated. In the Landau gauge, the pole parts of $\Gamma_\mu(k,p)$ and $\Gamma_\nu(k,p)$ do not contribute because of the transverse nature of the vector-meson propagator (4.26b).

D. Discussion

In principle, the power A of $B(p^2)$ should be obtainable from the Bethe-Salpeter equation. Unfortunately, we do not know how the calculate it. However, from Eq. (4.30), we obtain A if

$$\rho = \frac{\Delta\tilde{\Gamma}(q)}{\tilde{\Gamma}(q)}\bigg|_{q=0} \propto q\Delta\tilde{\Gamma}(q)\big|_{q=0}$$

vanishes. For the rest of the discussion, we assume that $\rho = 0$ [i.e., $\Delta\tilde{\Gamma}(q)$ does not have a pole in q]. The plausibility of this assumption is better seen if we use the new renormalization-group approach.[19] The functions $\Delta\tilde{\Gamma}$ and $\Delta\Gamma$ defined earlier are replaced by mass derivative terms $m_i\partial/\partial m_i$ operating on $\Gamma_{\bar{\psi}B\psi}(p,q)$ and $\Gamma_B^\phi(q)$, respectively. The mass derivative terms probably include both the symmetric and symmetry-breaking masses. Since they all vanish asymptotically (see Sec. V for the asymmetric mass term), the leading behavior of $C^\phi_{\bar{\psi}\psi}$ is given as if the extra term ($=\rho$) in Eq. (4.29) is zero.

The normalization condition of the Bethe-Salpeter wave function $B(p^2)$ requires

$$A = -c_0/b_0 + 2f_0/b_0 + \tfrac{1}{2} + j > \tfrac{1}{2}\,,$$

where

$$b_0 = \frac{1}{8\pi^2}\left[\tfrac{11}{3}C_2(G) - \tfrac{4}{3}T(R)\right], \tag{4.35a}$$

$$c_0 = -\frac{1}{16\pi^2}\left[(\tfrac{13}{3}-\alpha)C_2(G) - \tfrac{8}{3}T(R)\right], \tag{4.35b}$$

$$f_0 = \frac{1}{16\pi^2}\alpha C_2(R)\,, \tag{4.35c}$$

and ρ has been taken to be zero (the rest of the discussion on the gauge dependence can easily be modified in the case where the assumption that ρ is zero is not valid). The number j is an integer[17] [see Eq. (4.30), where $j = 0$ for the leading term]. $C_2(G)$ is the value of the quadratic Casimir operator for the adjoint representation of the gauge group G: $C_2(\mathrm{SU}(n)) = n$. $T(R)$ is defined by $N\,\mathrm{Tr}(\lambda^a\lambda^b)$

$= T(R)\delta^{ab}$ where λ^a are the matrices of the representation R of the gauge group G, and N is the number of such representations. From Eq. (4.35), A becomes

$$A = \frac{13C_2(G) - 8T(R) - 3\alpha[C_2(G) - 2C_2(R)]}{22C_2(G) - 8T(R)} + j + \tfrac{1}{2} > \tfrac{1}{2}. \quad (4.36)$$

In the Landau gauge, constraint (4.36) becomes

$$8(1+j)T(R) < (13 + 22j)C_2(G). \quad (4.37)$$

Recall that the asymptotic free condition is, following Eqs. (4.15), (4.21), (4.22), and (4.35),

$$T(R) < \tfrac{11}{4}C_2(G). \quad (4.38)$$

This means whenever we have a bound excitation, the asymptotic free condition is automatically satisfied; e.g., if G is SU(3), then one can accomodate up to 16 triplets of fermions with asymptotic freedom, but the maximum number of triplets is reduced according to Eq. (4.37) if we also want Goldstone bound excitations for vector-meson mass generations.

The condition (4.37) holds only in the Landau gauge. However, we have learned earlier from the renormalization-group equation that, following Eq. (4.8), we can consistently work in the Landau gauge where the renormalization-group equation is particularly simple. In this gauge the leading term $\tilde{\Gamma}^{(0)}(q)$ in Eq. (4.30) can have a Goldstone pole, and A is given by Eq. (4.36) with $\alpha = j = 0$. To see what happens if we start at some other gauge $\alpha \neq 0$, let us first recall that, from Eqs. (4.12) to (4.16), $\alpha = 0$ (the Landau gauge) is a uv-stable fixed point if Eq. (4.16) is true.

In the asymptotic free limit, $\beta(g,\alpha)$ to the lowest nontrivial order $[O(g^3)]$ is gauge independent. This gives the gauge-independent constraint (4.38). In the limit of small coupling constant $g \to 0$, we calculate $\eta(\alpha, g)$ to lowest nontrivial order $[O(g^2)]$. From the definitions (4.8), (4.21), (4.22), and (4.35), we obtain the following constraint from Eq. (4.13):

$$T(R) > \tfrac{13}{8}C_2(G). \quad (4.39)$$

For groups that satisfy Eqs. (4.38) and (4.39), if we start at a gauge with small but nonzero α, the effective gauge parameter $\bar{\alpha}$ approaches zero asymptotically so that the theory is determined by the Landau-gauge behavior to the leading order. For $j = 0$, however, conditions (4.37) and (4.39) mutually exclude each other, since Eq. (4.37) becomes

$$T(R) < \tfrac{13}{8}C_2(G). \quad (4.37')$$

To investigate Eq. (4.37a) we have to examine the gauge parameter behavior more closely.[20] To order $O(g^2)$,

$$\begin{aligned}\eta(g,\alpha) &= -\frac{2\alpha}{16\pi^2}\left[(\tfrac{13}{3} - \alpha)C_2(G) - \tfrac{8}{3}T(R)\right]g^2 \\ &= \frac{g^2}{8\pi^2}C_2(G)\left(\frac{13}{3} - \frac{8}{3}\frac{T(R)}{C_2(G)} - \alpha\right)\alpha \\ &= a(b-\alpha)\alpha,\end{aligned} \quad (4.40)$$

$$\frac{d\eta(g,\alpha)}{d\alpha} = a(b - 2\alpha). \quad (4.41)$$

When $b > 0$ [i.e., $\frac{13}{8}C_2(G) > T(R)$], $\alpha = b$ is a uv-stable point. This means if we begin with any gauge parameter $\alpha > 0$, the asymptotic behavior is determined by the gauge $\alpha = b > 0$[21]; that is, c_0 vanishes asymptotically, and f_0 becomes a positive number [see Eq. (4.35c)]. Again it is straightforward to see the normalization condition (4.34) is satisfied for $j = 0$.

To summarize, we have shown that the Bethe-Salpeter equation has a solution for the bound-state wave function $B(p^2)$ in the large-p^2 limit with the form $(\ln p^2)^{-A}$ provided A is greater than $\frac{1}{2}$. This implies that, from the operator-product expansion, the leading term [in Eq. (4.30)] $\tilde{\Gamma}^{(0)}(q)$ can have a pole as $q \to 0$. Then the operator-product expansion [Eq. (4.30)] tells us that $B(p^2)$ is given by[17]

$$B(p^2) \underset{p^2\,\text{large}}{\sim} (\ln p^2)^{c_0/b_0 - 1/2 - \rho}, \quad (4.42)$$

and the constraint (4.36a) must be satisfied. We notice that, although $B(p^2)$ is in general gauge dependent,[22] the constraint (4.37a) should be a gauge-independent statement. That $B(p^2)$ is gauge dependent should not be surprising since it is effectively a mass-inserted propagator which is well known to be gauge dependent. (One should mention that in recent investigations[23] of the effective-potential method, an alternative analysis of the symmetry-breaking problem, one finds the potential, at least at the one-loop level, to be gauge dependent too.) Of course what should be relevant is that physically meaningful objects be gauge independent. For instance, the Goldstone bound excitation, with a gauge-dependent coupling, decouples in all physical amplitudes, independent of the gauge.

We emphasize that the operator-product expansion tells us that, if a massless, meson pole exists in the fermion-antifermion channel of the vertex function $\Gamma_{\psi\bar{\psi}B}$, and if the existence of this pole is consistent with asymptotic freedom, then $B(p^2)$ is given by Eq. (4.31), for large p^2. To demonstrate the existence of such a pole consistent with asymptotic freedom, we have to examine the Bethe-Salpeter equation (4.1) for $B(P^2)$. Once we demonstrate the existence of such a solution with $A > \frac{1}{2}$, then it must be of the form (4.42).[17] For the formula (4.31) to be meaningful, it is crucial to note that the use of the renormalization-group equation to obtain the asymptotic behavior of Green's functions does not require the existence of a convergent perturbation theory (e.g., weak coupling approximation). For asymptotic free theories, it is sufficient to assume that perturbation theory yields an asymptotic expansion of the relevant functions. Thus one should not be surprised to find consistency between the presence of a pole in $\Gamma_{\psi\bar\psi B}$ and asymptotic freedom.

We repeat that we obtain the logarithmic form solution for $B(p^2)$ with large p^2 by assuming that the lower part ($\lambda > k > 0$) of the integral [Eq. (4.3)] is not dominant. We believe this assumption is justified *a posteriori* in the Landau gauge by the solution one obtains from the operator-product expansion (4.30) and (4.42).

We conclude this section with two remarks:

(1) We note that the following form of the effective coupling constant has been suggested in the literature[4]:

$$g_k{}^2 = \frac{g^2}{1 + \frac{1}{2} b_0 g^2 \ln k^2} \underset{k\to\infty}{\sim} \frac{2}{b_0 \ln k^2}\,.$$

The same asymptotic form can be reached by letting g^2 get large and keeping k away from unity. For string interactions, g^2 is probably quite large; hence, in this case at least, the asymptotic approximation used in this section is valid even for intermediate Euclidean momenta. Here one assumes that $\beta(g)$ has no other zero besides the one at $g = 0$.

(2) As an illustration, consider the following artificial model: an SU(2) gauge group with three doublets of fermions, whose propagators are given by

$$S_i{}^{-1}(p) = \not p - \Sigma_s(p) - \Sigma_i^V(p)\sigma_i \quad (i \text{ not summed}),$$

where $i = 1, 2, 3$, for the three fermion doublets and $[\sigma_i, \sigma_j] = i\epsilon_{ijk}\sigma_k$. The three independent symmetry-breaking masses $\Sigma_i^V(p)$ can be adjusted so that the three vector-meson masses are equal. This happens with $\Sigma_i^V(p)$ being all equal. In the case where second-order poles exist, there will be enough free parameters that a common vector-meson mass can be achieved much more easily.

V. DISCUSSION OF THE GENERAL CASE

Up to now we have discussed and analyzed in detail what happens if the vector mesons' masses are all equal. Here we briefly discuss some of the consequences when the vector mesons' masses are different. Now all the basic vertices of the theory, $\psi B\psi$, BBB, $BBBB$, and $\gamma B\gamma$, are assumed to contain a zero-mass bound-state excitation. Using the general Schwinger-Dyson equations relating these vertex functions, and the expressions for them in the Bethe-Salpeter form, we deduce that the various bound-state functions obey a set of coupled Bethe-Salpeter equations [neglecting $O(q^2)$ terms] (see Fig. 11) where B, P_2, P_3, P_4 are the two-fermion, two-meson, three-meson, and two-ghost bound-state wave functions, respectively; the kernels K_{11}, K_{12}, K_{13}, K_{14}, K_{22}, K_{23}, K_{24}, K_{33}, K_{34}, and K_{44} are the two-fermions, the two fermions to two mesons, the two fermions to three mesons, the two fermions to two ghosts, the two-mesons, the two mesons to three mesons, the two mesons to two ghosts, three-mesons, the three mesons to two ghosts, and the two-ghosts irreducible kernels, respectively. If there exists a zero-mass excitation pole in the 3-meson ver-

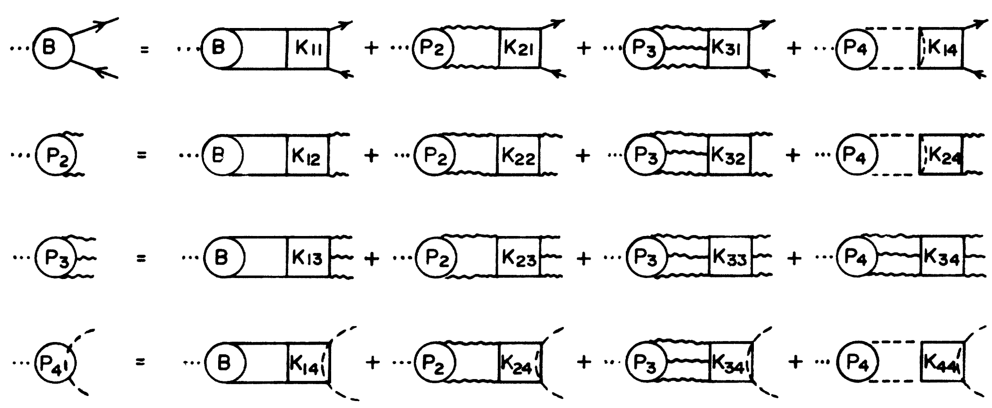

FIG. 11. Complete coupled set of Bethe-Salpeter equations for the bound-state wave functions B, P_i.

tex function, then following the steps of Sec. IV we can study, using the operator-product expansion methods, the behavior of the three-meson vertex function $\Gamma^{abc}_{\lambda\mu\nu}(q, \lambda p_1, \lambda p_2)$ as λ becomes asymptotically large. The result is

$$\Gamma^{abc}_{\lambda\mu\nu} \underset{\lambda\to\infty}{\sim} \lambda(\ln\lambda)^{-c_0/b_0-1/2}[u^{(0)}(q)+u^{(1)}(q)(\ln\lambda)^{-1}+\cdots] . \quad (5.1)$$

This suggests the wave function P_2 to have the form

$$P_2(\lambda p) \underset{\lambda\to\infty}{\sim} \lambda(\ln\lambda)^{-B} P_2(p) . \quad (5.2)$$

The asymptotic behavior of the other bound-state wave functions P_i and R_i can be obtained similarly if we can establish their existence. However, the investigation of the existence of poles in the various proper vertices via the complicated set of coupled Bethe-Salpeter equations is difficult. Instead of investigating this general situation, we discuss some consequences of the bound-state wave functions P_2 and P_3.

The presence of ghosts is nothing but a gauge effect. We take their masses to remain zero. Then the massless ghosts do not have the necessary binding energy to form a bound state. Of course, they may still couple to the Goldstone bound state via other channels, such as the two-meson channel. For the rest of this section, we neglect the ghost-ghost bound state. Alternatively, one may want to carry the discussion in a ghost-free gauge. For example, in the axial gauge, the only change in the WT identity derived in Sec. II is to drop the Λ functions.[24] To simplify the rest of the discussions, we neglect the Λ functions in the WT identity (2.14) since they are irrelevant to the argument.

We first show that the leading contributions of the bound-state wave functions P_2 and P_3 to the mass generation is zero. We then argue that the asymptotic behavior of the function $B(p^2)$ obtained earlier is not changed even in the presence of P_2 and P_3. These results seem to indicate that the fermion-antifermion channel Goldstone pole dominates in the study of dynamical symmetry breaking in non-Abelian gauge models.

Combining Eqs. (3.4) and (3.7)

$$\lambda_{ad} P^{dcb}_{2\lambda\nu}(p,p) = f_{cab}(\Pi^c_{\lambda\nu}(p) - \Pi^b_{\lambda\nu}(p)) , \quad (5.3)$$

where the right-hand side is symmetric under the interchange of b and c. Substituting Eq. (5.3) into the mass generation calculation of Eq. (3.13), we obtain that the leading contribution in the ultraviolet region is proportional to[11,24]

$$\sum_{c,b} f_{cab}(\Pi^c_{\lambda\nu}(p) - \Pi^b_{\lambda\nu}(p)) f_{clb} = 0 . \quad (5.4)$$

Thus the leading contribution of the two-meson channel Goldstone bound state to the mass generation is zero.

Similarly, we obtain from Eqs. (3.9) and (3.11) that

$$\begin{aligned} q_\mu \Gamma^{\mu\nu\lambda\rho}_{acdf}(q,p,s,r) &\underset{q\sim 0}{\sim} iI^{ah}(q^2) P^{\nu\lambda\rho}_{4hcdf}(q,p,s,r) \\ &= f_{abd}\Gamma^{cbf}_{\nu\lambda\rho}(p, s+q, r) \\ &\quad + f_{abc}\Gamma^{bdf}_{\nu\lambda\rho}(p+q, s, r) \\ &\quad + f_{abf}\Gamma^{cdb}_{\nu\lambda\rho}(p,s,r+q) . \end{aligned} \quad (5.5)$$

In the limit $q_u \to 0$, the right-hand side can be written as

$$\begin{aligned} I^{\nu\lambda\rho}_{adcf}(p,s,r) &= f_{abd} f_{cbf}\Gamma^{(cbf)}_{\nu\lambda\rho}(p,s,r) \\ &\quad + f_{abc} f_{bdf}\Gamma^{(bdf)}_{\nu\lambda\rho}(p,s,r) \\ &\quad + f_{abf} f_{cdb}\Gamma^{(cdb)}_{\nu\lambda\rho}(p,s,r) , \end{aligned} \quad (5.6)$$

where $\Gamma^{cbf}_{\nu\lambda\rho}(p,s,r) = f_{cbf}\Gamma^{(cbf)}_{\nu\lambda\rho}(p,s,r)$ so that the structure constant in $\Gamma^{cbf}_{\nu\lambda\rho}(p,s,v)$ is explicitly displayed. Because of symmetry breaking, $\Gamma^{(cbf)}_{\nu\lambda\rho}(p,s,r)$ is still dependent on the quantum numbers of the mesons (p,ν,c), (s,λ,b), and (r,f,ρ). [The superscript (cbf) is just for the purpose of identifying the quantum numbers of the individual mesons.] In the absence of symmetry breaking, or when the vector mesons' masses $\mu_c{}^2$, $\mu_b{}^2$, $\mu_f{}^2$ are equal, $\Gamma^{(cbf)}_{\nu\lambda\rho}(p,s,r)$ is independent of the quantum numbers c, b, and f, and the right-hand side of Eq. (5.6) consequently vanishes as expected.

Substituting Eq. (5.6) into the mass generation calculation of Eq. (3.11), Eq. (5.6) is multiplied to the bare four-meson vertex

$$f_{cdl}f_{lfh}(g^{\nu\rho}g^{\lambda\sigma} - g^{\nu\sigma}g^{\lambda\rho}) + f_{cfl}f_{ldh}(g^{\sigma\rho}g^{\lambda\nu} - g^{\rho\lambda}g^{\nu\sigma}) + f_{chl}f_{lfd}(g^{\rho\nu}g^{\lambda\sigma} - g^{\lambda\nu}g^{\rho\sigma}) . \quad (5.7)$$

The dominant contribution to mass generation from the three-meson channel pole is proportional to, for the first term of Eq. (5.7),

$$\sum_{c,d,f,l} I^{\nu\lambda\rho}_{adcf}(p,s,r) f_{cdl} f_{lfh} = 0 , \quad (5.8)$$

where the identity

$$f_{abd}f_{cdl} + f_{acd}f_{dbl} + f_{adl}f_{bcd} = 0 \quad (5.9)$$

has been repeatedly used; similarly for the other two terms of Eq. (5.7). Hence we conclude that in the case where the pure vector-meson proper vertex functions have Goldstone poles, their dominant contributions to the mass generation vanish because of the symmetry. (The functions P_2 and P_3 may still contribute to the mass generation via the infrared region. But this is outside the scope of this investigation.)

The above facts give us the hope that the presence of P_2 and P_3 in the Bethe-Salpeter equation for $B(p^2)$ does not change the asymptotic behavior of $B(p^2)$. To argue that this is indeed the case, consider the symmetry-breaking part of the fermion self-energy $\Sigma^V(p)$ which obeys the Dyson-Schwinger gap equation

$$\Sigma^V(p) = \mathrm{Tr} \int \frac{d^4k}{(2\pi)^4}\, g_u \gamma^\mu T D_{\mu\nu}(k-p)\, \Gamma^\nu(k,p) S(k) \times \Sigma^V(k) S(k) \ , \qquad (5.10)$$

where $g_u \gamma^\mu T$ is the bare vertex (T is a matrix depending on the representations). Defining the renormalized coupling constant $g_R = Z_3 X^{-1} g_u$, Eq. (5.10) can be written in renormalized quantities [where $D = D_R Z_3$, $S = S_R Z_2$, $\Sigma_R^V(p) = Z_2 \Sigma^V(p)$, $\Gamma = \Gamma_R Z_2^{-1} Z_3$], with the exception of an extra factor of the renormalization constant $Z = Z_2 X$ on the right-hand side,

$$\Sigma_R^V(p) = \mathrm{Tr} \int \frac{d^4k}{(2\pi)^4} (g_R Z) \gamma^\mu T D_{R\mu\nu}(k-p) \times \Gamma_R^\nu(k,p) S_R(k) \Sigma_R^V(k) S_R(k) \ . \qquad (5.11)$$

In order to express Eq. (5.11) in terms of renormalized quantities, we use Fig. 8(a), which can be written in the following form:

$$\Gamma_R^\mu(q,p) = Z g_R \gamma^\mu T + \mathrm{Tr} \int S_R(k)\, \Gamma_R^\mu(k) S_R(k) K_R(k,p) \frac{d^4k}{(2\pi)^4} + \cdots \ . \qquad (5.12)$$

[We note that the term $Z g_R \gamma^\mu T$ drops out in Eq. (4.1).] To the one-loop approximation, $Z g_R \gamma^\mu \Gamma$ in Eq. (5.11) is replaced by $\Gamma_R^\mu(q,p)$, and the resulting equation takes the same form as the Bethe-Salpeter equation (4.3) we discussed in Sec. IV. It then follows $\Sigma_R^V(p) \sim (\ln p^2)^{-a}$ is a solution of Eq. (5.11). Multiloop contributions to $g_R Z$ in Eq. (5.11) do not change the asymptotic form of $\Sigma_R^V(p)$, as can be seen following the arguments in Sec. IV and the Appendix. The WT identity (3.1) then tells us that $B(p)$ also must have an asymptotic form $(\ln p^2)^{-A}$ as we have obtained earlier. The $\Lambda(p)$ in Eq. (3.1) vanishes as $p^2 \to \infty$ can be shown following the discussion in Sec. IV. Since the power a is independent of the absence or presence of P_2 and P_3 and it also fixes the power A via the WT identity (3.1), we conclude that the power A obtained in Sec. IV is not changed in the presence of P_2 and P_3. This means that, even in the case of unequal vector-meson masses, the behavior of $B(p^2)$ obtained in Sec. IV remains true.

VI. CONCLUSION

The present investigation demonstrates that dynamically induced spontaneous symmetry breaking can generate masses for Yang-Mills mesons. Despite the formal complexities of non-Abelian gauge theories, the acquisition of vector-meson masses via symmetry-breaking fermion mass terms is essentially the same as in the Abelian case. One should investigate further in asymptotic free theories the possibility of an effective computational method bypassing ordinary perturbation theory. This should make model building basing on a dynamical symmetry breaking very attractive.

In strong interactions one can take hadrons to be bound states of fermion quarks mediated by "color" vector gluons so that form factors and scattering amplitudes may be computable. In weak interactions one can build unified theories of weak and electromagnetic interactions. Since the Lagrangians do not have scalar fields and hence have fewer parameters, one has a good chance of calculating quantities like the mass ratios (e.g., m_μ/m_e). Such models probably have asymptotic freedom for weak interactions, and possibly for QED as well. In fact, asymptotic freedom for QED can be checked experimentally, at least in principle, by going to very large transverse momenta in high-energy scattering processes, because experimentally, strong interactions have very sharp transverse momentum cutoffs.

On the theoretical side, an immediate problem is to understand better the gauge dependence of the function $B(p^2)$, and the gauge invariance of the mass of a vector meson in terms of the symmetry-breaking part of the fermion masses should be demonstrated. We note that the gauge dependence of dynamical symmetry breaking has been discussed by Cornwall,[24] who emphasizes the importance of the Landau gauge.

After the completion of an earlier version of this work, we received two reports on the same subject. Cornwall[24] used a phenomenological nonlocal, nonpolynomial Lagrangian. Feinberg and Eichten[11] studied the weak-coupling limit.

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We thank our colleagues at M.I.T. and Harvard for many useful discussions. In particular, we would like to thank Tom Appelquist, Mike Cornwall, Tony Duncan, Frank Feinberg, Roman Jackiw, Kenneth Johnson, and Peter Schattner. We also thank Sidney Coleman for pointing out to us that the illustrative example given in Sec. III of an earlier version of this work is not valid.

APPENDIX: HIGHER-LOOP CONTRIBUTION TO THE BETHE-SALPETER KERNEL

In this appendix we will explicitly work out a one-loop contribution to the Bethe-Salpeter kernel and we will show that it is down by one power of an inverse logarithm in momentum compared to the one-vector-boson exchange contribution.[26] Then we will give a general argument to extend this particular result to all loops and therefore confirm that, asymptotically, the kernel has the simple form (in the Landau gauge)

$$K(k,p) \underset{k\gg p}{\longrightarrow} \frac{1}{k^2 \ln k^2}\left[1+O\left(\frac{1}{\ln k^2}\right)\right] \quad .$$

The one-loop part of the kernel is shown in Fig. 12.

It is given by (group indices suppressed)

$$K_1(k,p)=\int \frac{d^4 s}{(2\pi)^4}\,\Gamma_\alpha(k,s)S(s)\,\Gamma_\mu(s,p)$$
$$\times D_{\mu\nu}(p-s)\,\Gamma_\nu(k-s+p,k)$$
$$\times S(k-s+p)\,\Gamma_\beta(k-s+p,p)\,D_{\beta\alpha}(k-s)\ . \tag{A1}$$

We assume the same lower cutoff λ^2 as in the one-meson exchange graph and divide the rest of the integral (A1) into three regions for both $K>p$ and $K<p$:

(1) $s>k>p$, (4) $p>k>s>\lambda$,

(2) $k>s>p$, (5) $p>s>k$,

(3) $k>p>s>\lambda$, (6) $s>p>k$.

For the first case, in the Landau gauge, using Eq. (4.25) to (4.28), we obtain

$$K_1(k,p) \underset{s>k>p}{\sim} \int_{k^2}^{\infty} s^2 ds^2 (g_s)^4 \frac{1}{s^6}$$
$$\sim \frac{1}{k^2}(\ln k^2)^{-2}\ .$$

Similarly for case (2)

$$K_1(k,p) \underset{k>s>p}{\sim} (\ln k^2)^{-c_0/b_0}\frac{g_k{}^3}{k^3}\int_{p^2}^{k^2} ds(\ln s^2)^{c_0/b_0} g_s$$
$$\sim \frac{1}{k^2}(\ln k^2)^{-2}\ .$$

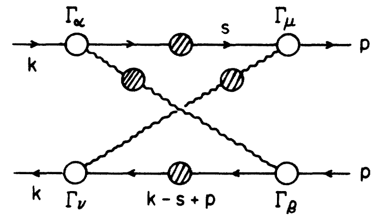

FIG. 12. One-loop approximation to the Bethe-Salpeter kernel discussed in Eq. (4.1).

The remaining four cases contribute at most as

$$K_1(k,p) \underset{k>p>s>\lambda}{\sim} \frac{p}{k^3}(\ln k^2)^{-c_0/b_0-3/2}(\ln p^2)^{c_0/b_0-1/2}\ ,$$

$$K_1(k,p) \underset{p>k>s>\lambda}{\sim} \frac{k}{p^3}(\ln k^2)^{c_0/b_0-1/2}(\ln p^2)^{-c_0/b_0-3/2}\ ,$$

$$K_1(k,p) \underset{p>s>k}{\sim} \frac{1}{k^2}(\ln k^2)^{-2}\ ,$$

$$K_1(k,p) \underset{s>p>k}{\sim} \frac{1}{p^2}(\ln p^2)^{-2}\ .$$

Therefore, the one-loop contribution to the kernel is down by either a power or a factor of a logarithm. We notice that this follows from

$$\int^x \frac{dy}{y^n}(\ln y)^m \sim \frac{(\ln x)^m}{x^n},\quad n\neq 1$$
$$\sim (\ln x)^{m+1},\quad n=1$$

where the power of the logarithm increases by one only when $n=1$. One can convince oneself that in general, for a multiloop graph of the kernel, there is at least one loop integral of the above form with $n\neq 1$. Since the extra coupling-constant factors $g_k{}^2$ introduce extra powers of $(\ln k^2)^{-1}$, the graph is therefore down by at least a factor of a logarithm in comparison with the one-meson exchange graph discussed in the text.

Actually for our solution to be valid, all that is required is that the multiloop graphs of the kernel contribute at most to the same order as the one-meson exchange graph.

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[15]See Ref. 5 for a complete discussion. See also a recent report by Ng Wing-Chiu and K. Young, Stony Brook Univ. Report No. ITP-SB-73-55 (unpublished).

[16]If the fermion fields in the Lagrangian have mass terms, then the renormalization-group equation is true only for asymptotic Euclidean momenta.

[17]We cannot prove that the leading term of $\Gamma_{\psi\bar\psi B}$ in the operator-product expansion must have a pole in q. If, indeed, $\tilde\Gamma^{(0)}(q)$ does not have a pole but some other nonleading terms have, then the only change is that the constraint $A>\frac{1}{2}$ will relax further the constraint on group representations.

[18]The normalization condition for the bound-state Bethe-Salpeter wave function has been discussed extensively in the literature. See, for example, C. H. Llewellyn Smith, Nuovo Cimento 60A, 348 (1969), where a list of earlier references can be found.

[19]S. Weinberg, Phys. Rev. D 8, 3497 (1973). See in particular Sec. VI.

[20]A more careful treatment requires a study of Eqs. (4.12) and (4.13) as a coupled set of differential equations. It will be interesting to calculate the two-loop graph contributions to $\beta(g,t,\alpha)$ and $\eta(g,t,\alpha)$.

[21]When $b>0$ and we start with $\alpha<0$, the effective gauge parameter $\bar\alpha\to-\infty$ asymptotically. Then the $A>\frac{1}{2}$ constraint is trivially satisfied. In this case, the finite part of the integral in the Bethe-Salpeter equation is probably important.

[22]We note A becomes gauge independent when $C_2(G)=2C_2(R)$, as is obvious from Eq. (4.36).

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[25]In analyzing a color gluon-quark bound-state model for the pion, T. W. Appelquist and E. C. Poggio [Phys. Rev. D 10, 3280 (1974)] have also found another asymptotic solution to the Bethe-Salpeter equation by a treatment analogous to the one in the present work.

[26]For a proof that, for $k^2\gg p^2$, multiloop contributions to the Bethe-Salpeter kernel for an asymptotically free theory are nonleading by powers of logs, see the Appendix of the reference in footnote 25.