Lorentz-invariant Newtonian mechanics for two particles

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Many ordinary Newtonian equations of motion in the center-of-mass frame can be made into Lorentz-invariant equations valid in all inertial frames. This is shown by outlining a construction of Poincaré-invariant Newtonian equations of motion, using global Lorentz transformations, starting with the assumption that at zero center-of-mass velocity the center-of-mass acceleration is zero, and the relative acceleration is specified as a function of the relative position and relative velocity, subject to some weak conditions. The only remarkable condition is a limit on repulsive forces, needed to keep the transformed center-of-mass velocity from being zero. This is generally satisfied up to energies comparable with the rest-mass energy. For Coulomb forces, it fails for higher energies, when the particles get closer than the classical electron radius and violate the uncertainty principle.

INTRODUCTION

There is an awakening interest in the description of relativistic particle interactions in terms of particle variables alone, without fields. This could be a useful complement to field theory, for example in handling bound states, which has always been difficult in field theory, as well as another area for investigating fundamental questions.

Here we consider particle interactions described in the most primitive form, Newtonian equations of motion, which specify the acceleration of each particle as a function of the positions and velocities of the particles at one time. This can be Lorentz-invariant, which means the equations of motion are the same for a moving reference frame, in terms of the variables of that frame at one time in that frame.

The conditions for Lorentz invariance of Newtonian equations of motion can be expressed, using infinitesimal Lorentz transformations, as differential equations for the acceleration functions or forces.[1-5] They are difficult to solve, because they are nonlinear and the accelerations of the different particles are coupled. Therefore, examples have not been known, and recent research has found it difficult to produce examples without physically artificial features.[6,7]

This problem has been posed in an equivalent manifestly invariant form.[8] Perturbation expansions have been developed for electrodynamics and other interactions.[9,3,10,11] The Hamiltonian form has been studied, first in the natural representation of the Poincaré group with canonical transformations[12,13] which was found to be inadequate,[14] and then using alternative approaches[15,16] including noncanonical representations equivalent to invariant Newtonian equations.[17]

The promise of many interesting Lorentz-invariant Newtonian equations of motion for two particles was evident in Ref. 1: Currie[1] showed that for parity-conserving forces for two identical particles in one-dimensional space, a solution of the Lorentz-invariance conditions can be constructed, using power series, with infinitesimal Lorentz transformations, starting from the specification of the accelerations as functions of the relative position and relative velocity when the center-of-mass velocity is zero. This has since been put on a mathematically rigorous basis.[18] The parity-conserving forces for two identical particles are a particular case of those considered in the present paper.

Here we will see that Lorentz-invariant Newtonian equations of motion for two particles in three-dimensional space can be constructed, by making global Lorentz transformations, from a specification of the relative acceleration as a function of the relative position and relative velocity at zero center-of-mass velocity, if the center-of-mass acceleration is assumed to be zero at zero center-of-mass velocity. We use the ordinary nonrelativistic definition of the center of mass. The relative acceleration in the center-of-mass frame can be specified as we choose, subject to only rather weak conditions. Thus many ordinary Newtonian equations of motion in the center-of-mass frame can be made into Lorentz-invariant Newtonian equations of motion valid in all inertial frames. For example, we will see that the conditions are satisfied by "nonrelativistic" repulsive Coulomb forces (up to a certain energy) in the center-of-mass frame.

The conditions to be met in specifying the relative acceleration in the center-of-mass frame are mostly rather obvious, such as rotation invariance, or the requirement that the particles not move faster than light. The one condition that was not anticipated is a limit on the strength of repulsive forces, which is required to keep the world lines from bending so sharply that a Lorentz transformation from the center-of-mass frame can give zero for the transformed center-of-mass velocity. This condition is generally satisfied if the energy of the interaction between the particles is not as large as their rest-mass energy. In the example of repulsive Coulomb forces it fails to be satisfied when the potential energy is larger than the rest-mass energy, which means the particles are closer than the classical electron radius and the uncertainty principle is violated.

This is a condition on those Lorentz-invariant Newtonian equations that can be constructed as outlined here, those for which it can be assumed that the center-of-mass acceleration is zero when the center-of-mass velocity is zero. This assumption, using the nonrelativistic center of mass, is surely not valid in general for all Lorentz-invariant Newtonian equations. However, it does hold, in particular, for parity-conserving forces for identical particles, to which we give some special attention. The method of constructing invariant equations of motion is also described in a more general form of wider applicability.

For comparison, we outline the analogous construction of Galilei-invariant equations of motion. Galilei transformations at time zero change only the center-of-mass velocity; they do not change the accelerations. Therefore, the accelerations are independent of the center-of-mass velocity; they are the same in any frame as in the center-of-mass frame. In contrast, the accelerations for the Lorentz-invariant equations of motion depend on the center-of-mass velocity in a complicated way.

In particular, the center-of-mass acceleration for the Galilei-invariant equations of motion is zero in every frame; the center of mass moves with constant velocity. For Lorentz-invariant equations of motion for two interacting particles, the center-of-mass velocity is not constant in every frame[2] (neither is the total relativistic kinematic particle momentum, nor the angular momentum, for two or three particles[4,19,20]).

The Lorentz invariance of Newtonian equations of motion is not a manifest invariance of four-vector or tensor forms for transformations at fixed space-time points. When we make a Lorentz transformation we go from variables at one time in the original frame to variables at one time in the new frame. The transformed center-of-mass velocity involves particle velocities at different times in the original frame. So it is not easy, as it is in the Galilei case, to find a Lorentz transformation to the center-of-mass frame.

Instead we simply start in the center-of-mass frame and see what we get by making Lorentz transformations. We may not get all positions and velocities that the particles can have, as we do with Galilei transformations, which give all center-of-mass velocities without changing the positions or relative velocity. If the particles can have positions and velocities that are not obtained by making Lorentz transformations from the center-of-mass frame, the accelerations for those positions and velocities are not related by Lorentz transformations to the accelerations in the center-of-mass frame; they must be specified independently.

For justification of ignoring Einstein causality in considering Newtonian equations of motion for a closed system of two particles, we refer to the very clear explanations given by Havas.[21,22]

CONSTRUCTION OF INVARIANT EQUATIONS OF MOTION

For each frame of reference of space-time coordinates $\vec{x}, t$, the world lines of the particles are determined by their positions $\vec{x}_1$ and $\vec{x}_2$ and velocities $\vec{v}_1$ and $\vec{v}_2$ (where $\vec{v}_n = d\vec{x}_n/dt$) at $t=0$. Consider the transformations of these initial positions and velocities resulting from the Poincaré group of transformations of space-time coordinates, that is, the group made from translations in space and time, rotations, and Lorentz transformations. For example, for a Lorentz transformation

$$x' = x\,,$$

$$y' = y\,,$$

$$z' = \frac{z - \beta t}{(1-\beta^2)^{1/2}}\,,$$

$$t' = \frac{t - \beta z}{(1-\beta^2)^{1/2}}\,,$$

with velocity $\vec{\beta}$ in the z direction (we use units such that $c=1$), we must have

$$t = \beta z$$

for $t'=0$; so for the positions and velocities in the new frame, at $t'=0$, we get

$$x_n' = x_n(t=\beta z_n)\,,$$

$$y_n' = y_n(t=\beta z_n)\,,$$

$$z_n' = (1-\beta^2)^{1/2} z_n(t=\beta z_n)\,,$$

$$v_{nx}' = (1-\beta^2)^{1/2}\frac{v_{nx}(t=\beta z_n)}{1-\beta v_{nz}(t=\beta z_n)}\,,$$

$$v_{ny}' = (1-\beta^2)^{1/2}\frac{v_{ny}(t=\beta z_n)}{1-\beta v_{nz}(t=\beta z_n)}\,,$$

$$v_{nz}' = \frac{v_{nz}(t=\beta z_n) - \beta}{1-\beta v_{nz}(t=\beta z_n)}$$

for $n=1,2$. The $\vec{x}_n(t=\beta z_n)$ and $\vec{v}_n(t=\beta z_n)$ are determined from $\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2$ at $t=0$, so we have a transformation, depending on β, from these positions and velocities at $t=0$ to the positions and velocities $\vec{x}_1', \vec{x}_2', \vec{v}_1', \vec{v}_2'$ at $t'=0$. When we want to simplify the notation we will let x stand for $\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2$, and Lx stand for the $\vec{x}_1', \vec{x}_2', \vec{v}_1', \vec{v}_2'$ thus obtained for the various transformations of the Poincaré group.

The equations of motion specify the accelerations as functions of the positions and velocities,

$$d\vec{v}_n/dt = \vec{f}_n(\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2)$$

for $n=1,2$. We calculate transformations of time-zero accelerations just as positions and velocities, but use the equations of motion to express the accelerations for the original frame as functions of positions and velocities. For our example Lorentz transformation we find that the accelerations in the new frame, at $t'=0$, are

$$dv_{nx}'/dt' = \frac{1-\beta^2}{(1-\beta v_{nz})^3}(f_{nx} - \beta v_{nz} f_{nx} + \beta v_{nx} f_{nz})\bigg|_{t=\beta z_n},$$

$$dv_{ny}'/dt' = \frac{1-\beta^2}{(1-\beta v_{nz})^3}(f_{ny} - \beta v_{nz} f_{ny} + \beta v_{ny} f_{nz})\bigg|_{t=\beta z_n},$$

$$dv_{nz}'/dt' = \frac{(1-\beta^2)^{3/2}}{(1-\beta v_{nz})^3} f_{nz}\bigg|_{t=\beta z_n}$$

for $n=1,2$, where the components of $\vec{v}_n$ shown and the $\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2$ in $\vec{f}_n$ are all at $t=\beta z_n$. Remembering that these are determined from the initial positions and velocities, we can think of this transformation of accelerations as defining functions $\vec{f}_n'$ of the $\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2$ at $t=0$ so that

$$d\vec{v}_n'/dt' = \vec{f}_n'(\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2)$$

for $n=1,2$. Our simplified notation will be $f(x)$ for $\vec{f}_1(\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2), \vec{f}_2(\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2)$ and $Lf(x)$ for $\vec{f}_1'(\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2), \vec{f}_2'(\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2)$.

Invariance of the equations of motion means that the acceleration is specified by the same function of the positions and velocities for every frame,

$$d\vec{v}_n'/dt' = f_n(\vec{x}_1', \vec{x}_2', \vec{v}_1', \vec{v}_2'),$$

or

$$\vec{f}_n'(\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2) = \vec{f}_n(\vec{x}_1', \vec{x}_2', \vec{v}_1', \vec{v}_2')$$

for $n=1,2$. In our simplified notation, invariant equations of motion are those for which

$$Lf(x) = f(Lx)$$

for all the transformations in the Poincaré group.

We construct invariant equations of motion by starting with equations of motion for the center-of-mass frame and making Lorentz transformations. Let

$$\vec{x} = \vec{x}_1 - \vec{x}_2,$$

$$\vec{v} = d\vec{x}/dt = \vec{v}_1 - \vec{v}_2,$$

$$d\vec{v}/dt = \vec{f} = \vec{f}_1 - \vec{f}_2,$$

$$\vec{X} = \frac{1}{m_1+m_2}(m_1\vec{x}_1 + m_2\vec{x}_2),$$

$$\vec{V} = d\vec{X}/dt = \frac{1}{m_1+m_2}(m_1\vec{v}_1 + m_2\vec{v}_2),$$

$$d\vec{V}/dt = \vec{F} = \frac{1}{m_1+m_2}(m_1\vec{f}_1 + m_2\vec{f}_2).$$

For $\vec{V}=0$ and a set S_0 of $\vec{x}, \vec{v}$ we let

$$\vec{F} = 0,$$

$$\vec{f} = \vec{f}_0(\vec{x}, \vec{v}),$$

specifying S_0 and $\vec{f}_0$ as we choose, subject to the following conditions.

If $\vec{V}=0$ at $t=0$, then $\vec{V}(t)=0$ for all t; so $\vec{X}$ is constant, and the motion is described by $\vec{x}(t)$ and $\vec{v}(t)$, which must be determined by the equation of motion

$$d\vec{v}/dt = \vec{f}_0(\vec{x}, \vec{v})$$

for any initial $\vec{x}, \vec{v}$ in S_0. We must have

$$|\vec{v}_1(t)| < 1$$

and

$$|\vec{v}_2(t)| < 1,$$

or

$$|\vec{v}(t)| < \frac{m_1+m_2}{m_2}$$

and

$$|\vec{v}(t)| < \frac{m_1+m_2}{m_1},$$

to have each world line cross $t=\vec{\beta}\cdot\vec{x}$ once, so the times $t=\vec{\beta}\cdot\vec{x}_n$ in our Lorentz transformations will be well defined for all $\vec{\beta}$ and all initial $\vec{x}, \vec{v}$ in S_0 and $\vec{X}$. This also prevents singularities occurring from the factors $1-\vec{\beta}\cdot\vec{v}_n$ in the denominators in the Lorentz transformations.

The set S_0 of $\vec{x}, \vec{v}$ being considered with $\vec{V}=0$ must be invariant for rotations and the changes of $\vec{x}$ and $\vec{v}$ in time. For example, if

$$\vec{f}_0 = -\nabla V(|\vec{x}|),$$

S_0 could be the set of $\vec{x}, \vec{v}$ for which

$$\tfrac{1}{2}\vec{v}^2 + V(|\vec{x}|)$$

is less than some specified value.

The equations of motion in the center-of-mass frame are invariant for time translations,

$t' = t - t_0$, because at $t' = 0$ we have $t = t_0$ and

$$\vec{x}' = \vec{x}(t = t_0),$$

$$\vec{v}' = \vec{v}(t = t_0),$$

$$d\vec{v}'/dt' = \vec{f}_0(\vec{x}(t = t_0), \vec{v}(t = t_0)),$$

so

$$d\vec{v}'/dt' = \vec{f}_0(\vec{x}', \vec{v}'),$$

and we have

$$\vec{V}' = \vec{V}(t = t_0) = 0$$

and

$$d\vec{V}'/dt' = 0.$$

The equations of motion in the center-of-mass frame are invariant for space translations, because velocities and accelerations are not changed by a space translation and $\vec{f}_0$ is not changed, because it depends only on $\vec{x}$ and $\vec{v}$ and not on $\vec{X}$. A rotation does not change $\vec{V} = 0$ and $d\vec{V}/dt = 0$. To make the equations of motion in the center-of-mass frame rotation-invariant, we choose $\vec{f}_0$ to be a rotational vector function of $\vec{x}$ and $\vec{v}$, so that it rotates when $\vec{x}$ and $\vec{v}$ are rotated; we require that

$$R\vec{f}_0(\vec{x}, \vec{v}) = \vec{f}_0(R\vec{x}, R\vec{v}),$$

so that

$$\begin{aligned} d\vec{v}'/dt &= dR\vec{v}/dt \\ &= R\vec{f}_0(\vec{x}, \vec{v}) \\ &= \vec{f}_0(R\vec{x}, R\vec{v}) \\ &= \vec{f}_0(\vec{x}', \vec{v}'), \end{aligned}$$

where R denotes the rotation of the vectors.

There will be one more condition on $\vec{f}_0$ that we will discuss later, putting a limit on the strength of repulsive accelerations, so that a Lorentz transformation from $\vec{V} = 0$ cannot yield $\vec{V}' = 0$.

Consider the set, which we call S, of all values of $\vec{x}, \vec{v}, \vec{X}, \vec{V}$ that we get from the Poincaré group of transformations, starting with $\vec{V} = 0$, all values of $\vec{x}, \vec{v}$ in S_0, and all $\vec{X}$. These are actually just the values that we get from Lorentz transformations, because every transformation in the Poincaré group is a product of space and time translations followed by a rotation followed by a Lorentz transformation,[23,24] and only the Lorentz transformation changes $\vec{V} = 0$. From our example Lorentz transformation, we can see how to obtain these transformed positions and velocities, and also the transformed accelerations. Our construction is simply to assign the transformed accelerations to the transformed positions and velocities: For each x in S there is an x_0 with $\vec{V} = 0$ and $\vec{x}, \vec{v}$ in S_0, so that

$$x = L_0 x_0$$

for some transformation in the Poincaré group; we let

$$f(x) = L_0 f_0(x_0),$$

where $f_0(x_0)$ is the $f(x_0)$ we have specified, namely, $\vec{F} = 0$ and $\vec{f} = \vec{f}_0$.

The set S of $\vec{x}, \vec{v}, \vec{X}, \vec{V}$ to which we have assigned accelerations is invariant for the Poincaré group of transformations. Any world lines determined from initial positions and velocities in S in one frame of reference are also determined from initial positions and velocities in S in all frames related by the Poincaré group. But we cannot be certain that S includes all values of $\vec{x}, \vec{v}, \vec{X}, \vec{V}$ that the two particles could have, even if S_0 includes all values of $\vec{x}, \vec{v}$ that the two particles could have with $\vec{V} = 0$. If there are $\vec{x}, \vec{v}, \vec{X}, \vec{V}$ that cannot be obtained with Lorentz transformations from $\vec{V} = 0$, the accelerations for them are not related by the Poincaré group to the accelerations for $\vec{V} = 0$; they must be specified independently.

It remains to show that our construction does produce well-defined, invariant equations of motion. First we show that the accelerations are specified uniquely as functions of the positions and velocities. Suppose x is obtained from both x_0 and x_{00}, two different $\vec{x}, \vec{v}$ in S_0 and $\vec{X}$ with $\vec{V} = 0$ for both, with two different transformations,

$$x = L_0 x_0$$

and

$$x = L_{00} x_{00}.$$

We have to show that

$$L_0 f_0(x_0) = L_{00} f_0(x_{00}).$$

This is equivalent to

$$f_0(x_0) = L_0^{-1} L_{00} f_0(x_{00}).$$

We have

$$x_0 = L_0^{-1} L_{00} x_{00},$$

so we just need to know that the equations of motion for $\vec{V} = 0$ are invariant under the transformation $L_0^{-1} L_{00}$. They are if this transformation is a product of space and time translations followed by a rotation but no Lorentz transformation. It is, because it preserves $\vec{V} = 0$ in taking x_{00} to x_0. We are assuming that the motion for $\vec{V} = 0$ is such that a Lorentz transformation never preserves $\vec{V} = 0$. This is the condition we have to look at later.

Finally we show that the equations of motion are invariant, that

$$f(Lx) = Lf(x) .$$

Let

$$x = L_0 x_0$$

for x_0 an $\vec{x}, \vec{v}$ in S_0 and $\vec{X}$ with $\vec{V} = 0$. Then

$$Lx = LL_0 x_0 ,$$

$$f(x) = L_0 f_0(x_0) ,$$

$$f(Lx) = LL_0 f_0(x_0) = Lf(x) .$$

COMPARISON WITH GALILEI INVARIANCE

If we use Galilei transformations in place of Lorentz transformations, our construction of invariant equations of motion is trivial. For a Galilei transformation,

$$\vec{x}' = \vec{x} - \vec{\beta} t ,$$

$$t' = t ,$$

at $t' = 0$ we have $t = 0$ and

$$\vec{x}_n' = \vec{x}_n ,$$

$$\vec{v}_n' = \vec{v}_n - \vec{\beta} ,$$

$$d\vec{v}_n'/dt' = d\vec{v}_n/dt$$

for $n = 1, 2$. After making a Galilei transformation, starting from $\vec{V} = 0$, we have

$$\vec{V} = -\vec{\beta} ,$$

but nothing else is changed, not $\vec{x}, \vec{v}, \vec{X}$ or the accelerations; we still have

$$d\vec{V}/dt = 0$$

and

$$d\vec{v}/dt = \vec{f}_0(\vec{x}, \vec{v}) ,$$

so these are the equations of motion for all $\vec{x}, \vec{v}, \vec{X}, \vec{V}$. Their Galilei invariance is evident; the accelerations do not depend on $\vec{V}$. The construction defines $\vec{F}$ and $\vec{f}$ for all real $\vec{x}, \vec{v}, \vec{X}, \vec{V}$.

The only condition on $\vec{f}_0$ is that of rotation invariance. A Galilei transformation cannot preserve $\vec{V} = 0$, and there can be no ambiguity about where the world lines cross $t' = 0$.

This construction does not produce all Galilei-invariant equations of motion, but only those for which $\vec{F} = 0$. Galilei invariance does not imply that $\vec{F} = 0$. It requires only that $\vec{F}$ and $\vec{f}$ be independent of $\vec{V}$.

PARITY-CONSERVING FORCES FOR IDENTICAL PARTICLES

Our construction does not produce all Lorentz-invariant equations of motion, just as it does not produce all Galilei-invariant equations of motion. It gives only those for which $\vec{F} = 0$ when $\vec{V} = 0$. The latter include all parity-conserving forces for two identical particles, which are what Currie[1] considered. For $\vec{V} = 0$, invariance under interchange of the two particles, for $m_1 = m_2$, implies

$$\vec{F}(-\vec{x}, -\vec{v}) = \vec{F}(\vec{x}, \vec{v}) ,$$

whereas invariance under space reflection implies

$$\vec{F}(-\vec{x}, -\vec{v}) = -\vec{F}(\vec{x}, \vec{v}) ,$$

so the only possibility is $\vec{F} = 0$ when $\vec{V} = 0$. Either identical-particle or space-reflection symmetry implies

$$\vec{f}(-\vec{x}, -\vec{v}) = -\vec{f}(\vec{x}, \vec{v})$$

for $\vec{V} = 0$.

Conversely, if we let $m_1 = m_2$, choose S_0 to be invariant under particle interchange or space reflection, and let

$$\vec{f}_0(-\vec{x}, -\vec{v}) = -\vec{f}_0(\vec{x}, \vec{v}) ,$$

our construction yields equations of motion that are invariant for both particle interchange and space reflection. We consider particle interchange first.

For each x we have

$$x = L_0 x_0 ,$$

with L_0 a pure Lorentz transformation and $\vec{V} = 0$ for x_0. Let x^i and x_0^i be obtained from x and x_0 by interchanging $\vec{x}_1$ with $\vec{x}_2$ and $\vec{v}_1$ with $\vec{v}_2$. From our example of Lorentz transformation of positions and velocities, we can see that

$$x^i = L_0 x_0^i ,$$

because for $\vec{V} = 0$ interchanging the initial positions and velocities interchanges the world lines, since the equations of motion are assumed invariant for $\vec{V} = 0$. From our example of Lorentz transformation of accelerations, we can see that

$$f(x^i) = L_0 f_0(x_0^i)$$

is obtained from

$$f(x) = L_0 f_0(x_0)$$

by interchanging the accelerations of the two particles, since we have assumed this for f_0.

The argument for space-reflection invariance is similar. For each x denoting $\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2$ and f denoting $\vec{f}_1, \vec{f}_2$, let $-x$ denote $-\vec{x}_1, -\vec{x}_2, -\vec{v}_1, -\vec{v}_2$ and $-f$ denote $-\vec{f}_1, -\vec{f}_2$. For each x we have

$$x = L_0 x_0 ,$$

with L_0 a pure Lorentz transformation and $\vec{V} = 0$ for x_0. From our example of Lorentz transformation of positions and velocities, we can see that

$$-x = L_0^{-1}(-x_0) ,$$

because for $\vec{V}=0$ reflection of the initial positions and velocities produces the reflection of the world lines, since the equations of motion are assumed invariant for $\vec{V}=0$. From our example of Lorentz transformation of accelerations, we can see that

$$L_0^{-1}f_0(-x_0) = -L_0 f_0(x_0) ,$$

because we have assumed that

$$f_0(-x_0) = -f_0(x_0) .$$

Thus we have

$$f(-x) = L_0^{-1}f_0(-x_0) = -L_0 f_0(x_0) = -f(x) .$$

GENERALIZATION

The method of constructing invariant equations of motion, being so simple, can easily be described in a more general form of wider applicability. All that is needed is the specification of a set of x_0, values of $\vec{x}_1, \vec{x}_2, \vec{v}_1, \vec{v}_2$, and accelerations $f_0(x_0)$ for each x_0, such that:

(i) The world lines are determined by f_0 for initial positions and velocities x_0. The particle speeds of these world lines never exceed the speed of light.

(ii) The set of x_0 is invariant for time translations, space translations, and rotations.

(iii) The equations of motion defined by f_0 for the x_0 are invariant for time translations, space translations, and rotations.

(iv) A Lorentz transform of an x_0 is not an x_0.

For example, one can use the relativistic free-particle momentum variables

$$\vec{u}_n = m_n \vec{v}_n / (1 - v_n^2)^{1/2} ,$$

let

$$\vec{U} = \vec{u}_1 + \vec{u}_2 ,$$

specify a set S_0 of $\vec{x}, \vec{v}$, define an x_0 to be an $\vec{x}, \vec{v}$ in S_0 with any $\vec{X}$ and $\vec{U}=0$, and define f_0 by

$$d\vec{U}/dt = 0$$

and

$$d\vec{v}/dt = \vec{f}_0(\vec{x}, \vec{v}) ,$$

with $\vec{f}_0$ a rotational vector function of $\vec{x}$ and $\vec{v}$. The remaining conditions are (i) that the particle speeds for $\vec{U}=0$ be less than the speed of light; (ii) that S_0 be invariant for rotations and the changes of $\vec{x}$ and $\vec{v}$ in time for $\vec{U}=0$; and (iv) that a Lorentz transformation from $\vec{U}=0$ never gives $\vec{U}'=0$.

We might get an idea of what is needed for a completely general construction of all Lorentz-invariant Newtonian equations by considering a more appropriate definition of the center of mass for a relativistic system of particles.[25] This appears to be difficult, because the center-of-mass position and the conserved total momentum cannot be defined in general independent of the interaction.[25,4]

LIMIT REQUIRED ON ACCELERATIONS

For the construction of invariant equations of motion, we have required that a Lorentz transformation never takes $\vec{V}=0$ to $\vec{V}'=0$. The fastest way to see what this involves is to look at Minkowski diagrams. Figure 1 shows world lines for two identical particles that have $\vec{V}=0$ at $t=0$ and $\vec{V}'=0$ at $t'=0$. At $t=0$ both particles have zero velocity in the unprimed frame, because both world lines have zero slope relative to $\vec{x}=0$. At $t'=0$ the particle on the left has a (negative) velocity in the primed frame, as indicated by the slope of its world line relative to $\vec{x}'=0$, and the particle on the right has an equal but opposite (positive) velocity in the primed frame. For this to happen, the world line on the right has to be bent from zero slope to about twice the slope of $\vec{x}'=0$, between $t=0$ and $t'=0$. By drawing diagrams like this, one can see that the only way one can get $\vec{V}=0$ at $t=0$ and $\vec{V}'=0$ at $t'=0$ is to have repulsive forces and a change of particle velocity Δv of magnitude

$$\Delta v = \beta c$$

in a time interval Δt of magnitude given by

$$c\Delta t = \beta r ,$$

where r is the distance between the particles and βc is the velocity of the Lorentz transformation. This implies a particle acceleration of magnitude

$$a = \Delta v / \Delta t = c^2/r .$$

Therefore, the particle accelerations for repulsive forces have to be limited in magnitude by

$$ar < c^2$$

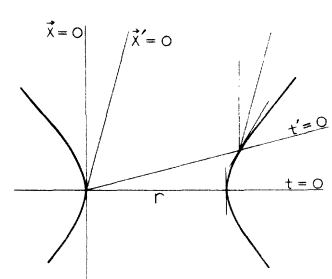

FIG. 1. World lines for which $\vec{V}=0$ at $t=0$ and $\vec{V}'=0$ at $t'=0$.

or

$$mar < mc^2\,,$$

where m is the mass of the particle. We can interpret this as meaning that the potential energy of the interaction between the two particles must not be as great as their rest-mass energy.

To bring some familiar magnitudes into play, we can look at "nonrelativistic" Coulomb forces and substitute

$$ma = e^2/r^2\,.$$

Then our condition is

$$e^2/mc^2 < r\,,$$

which means the particles must not get closer than the classical electron radius for their mass and charge, or

$$(e^2/\hbar c)\hbar < rmc\,,$$

which is satisfied by more than two orders of magnitude, with

$$e^2/\hbar c \approx \tfrac{1}{137}\,,$$

and $v < c$, if the uncertainty relation

$$\hbar < rmv$$

is satisfied.

It seems our requirement is satisfied up to a point where it is clear that quantum effects cannot be ignored. In the next section we will look at the Coulomb example in more detail, and see that when the particles get closer than the classical electron radius, there are Lorentz transformations that take $\vec{V}=0$ to $\vec{V}'=0$, and the construction of invariant equations of motion breaks down.

EXAMPLE OF TOO MUCH ACCELERATION

We again consider "nonrelativistic" Coulomb forces, specifying the relative acceleration

$$\vec{f}_0 = \frac{2e^2}{m|\vec{x}|^3}\vec{x}\,,$$

with $m_1 = m_2 = m$. For $\vec{V}=0$,

$$E = \tfrac{1}{4}mv^2 + \frac{e^2}{|\vec{x}|}$$

is constant. We get particle speeds less than c, or $v < 2c$, if we specify S_0 to include only $\vec{x}, \vec{v}$ for which $E < mc^2$. Then

$$|\vec{x}|_{\min} > \frac{e^2}{mc^2}\,.$$

From the last section, we know that if we limit S_0 to lower energies, we will not find a Lorentz transformation that takes $\vec{V}=0$ to $\vec{V}'=0$. Here we give an example to show that we will find such Lorentz transformations if we venture into the forbidden high-energy region.

We choose positions and velocities along the z axis, at some time, with $z>0$, $Z=0$, and $V=0$. Then this is the case for all times. The particles approach each other along the z axis to a minimum separation z_m and then move apart along the z axis, with particle 1 on the right and particle 2 on the left. At any time we have $z_1 = -z_2 = \frac{1}{2}z$ and $v_1 = -v_2 = \frac{1}{2}v$. Let $t=0$ be when the particles are closest to each other. Then $z(-t) = z(t)$ and $v(-t) = -v(t)$, with $v(t)>0$ for $t>0$. The time $t=\beta z_2/c$ is the negative of the time $t=\beta z_1/c$, so

$$v_1(t=\beta z_1/c) = v_2(t=\beta z_2/c),$$

and a Lorentz transformation in the z direction gives $v_1' = v_2'$. Thus $V' = v_1'$, which is zero if

$$v_1(t=\beta z_1/c) = \beta c$$

or

$$v(t=\beta z/2c) = 2\beta c.$$

If this last equation holds, our construction of invariant equations of motion breaks down, for then V' is zero at $t'=0$, but V' is in general not zero for $t'\neq 0$. (You can check that in fact d^2V'/dt'^2 is not zero at $t'=0$.) It remains to be seen how this can happen in our example.

From the energy equation

$$\tfrac{1}{4}mv^2 + \frac{e^2}{z} = \frac{e^2}{z_m}$$

we have

$$\frac{dz}{dt} = v = 2em^{-1/2}z^{-1/2}\left(\frac{z}{z_m}-1\right)^{1/2}.$$

The solution satisfying the boundary condition that $z=z_m$ at $t=0$ is

$$2em^{-1/2}z_m^{-3/2}t = \left[\frac{z}{z_m}\left(\frac{z}{z_m}-1\right)\right]^{1/2} + \ln\left[\left(\frac{z}{z_m}\right)^{1/2} + \left(\frac{z}{z_m}-1\right)^{1/2}\right].$$

We let $t=\beta z/2c$, and for β substitute the value obtained from $v=2\beta c$. Using the above equation for v in terms of z, we get

$$\left(\frac{e^2}{mc}-z_m\right)\frac{1}{z_m}\left[\left(\frac{z}{z_m}\right)\left(\frac{z}{z_m}-1\right)\right]^{1/2} = \ln\left[\left(\frac{z}{z_m}\right)^{1/2} + \left(\frac{z}{z_m}-1\right)^{1/2}\right]$$

as the condition for V' to be zero at $t'=0$. Here z means the value at $t=\beta z/2c$ or $t=\beta z_1/c$. We get

$V'=0$ only if

$$z_m < \frac{e^2}{mc^2}.$$

Then the energy equation implies that the particles move faster than light, $v>2c$, as $z\to\infty$. This does not mean that the particles are moving faster than light when $V'=0$. For example, we get

$$z(t=\beta z_1/c)=1.1z_m,$$

if $z_m=0.516e^2/mc^2$, and then

$$v_1(t=\beta z_1/c)=\beta c=0.42c.$$

It is easy to modify our example to get $V'=0$ with particles never moving faster than light. Just change $\vec{f}_0$ so that it drops to zero rather abruptly, and gives no further acceleration, when the distance between the particles becomes larger than that at which $V'=0$.

Alternatively, we can use the relativistic momentum to define $\vec{f}_0$ by letting

$$\frac{d}{dt}\frac{m\frac{1}{2}\vec{v}}{(1-\frac{1}{4}v^2/c^2)^{1/2}}=\frac{e^2}{|\vec{x}|^3}\vec{x}$$

for $\vec{V}=0$, with $m_1=m_2=m$. Then for $\vec{V}=0$ the energy

$$\frac{2mc^2}{(1-\frac{1}{4}v^2/c^2)^{1/2}}+\frac{e^2}{|\vec{x}|}$$

is constant. The particle speeds are always less than c, that is, $v<2c$, for any energy. If we rework the last paragraph, we get

$$\frac{1+s-s^2q}{s^2(sq+q-1)}[(2s+1)(q-1)(2sq+q-1)]^{1/2}=\ln\frac{(2s+1)q-s-1+[(2s+1)(q-1)(2sq+q-1)]^{1/2}}{s}$$

as the condition for V' to be zero at $t'=0$, where

$$s=\frac{2mc^2z_m}{e^2}$$

and

$$q=\frac{1}{z_m}z(t=\beta z_1/c).$$

For example, a solution is $q=1.1$ and $s=0.94$ which, for comparison with the previous case, means that

$$z(t=\beta z_1/c)=1.1z_m$$

if $z_m=0.47e^2/mc^2$, and then

$$v_1(t=\beta z_1/c)=\beta c=0.41c.$$

It seems there are solutions only for $s<1$, which would mean that the construction of invariant equations breaks down only if the particles get closer than $e^2/2mc^2$ and the potential energy (kinetic energy at infinite separation) is more than $2mc^2$.

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