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Dynamical symmetries and the nonconservative classical system

Lawrence H. Buch and Harry H. Denman
Department of Physics, Wayne State University, Detroit, Michigan 48202
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All 3-dimensional, time-dependent classical Hamiltonian systems are shown to possess the dynamical symmetries SU_3 and O_4. As an example, the Newtonian, linearly damped, isotropic harmonic oscillator in a time-dependent external field is treated.

I. INTRODUCTION

The group algebraic properties of certain classical and quantum-mechanical systems have recently received considerable attention,[1-3] as one wishes to exploit the connections between dynamical symmetries and quantum-mechanical degeneracies and energy spectra. Dynamical symmetries of classical systems are usually expressed via the Poisson-bracket algebra developed among certain constants of the motion, in the expectation of finding corresponding relations for the commutators among quantum operators. A classic example is the O_4 algebra among the angular momentum and Runge-Lenz vector components for the 3-dimensional Kepler particle.

For all classical conservative central-force problems, Fradkin[2] has explicitly constructed constants of the motion exhibiting O_4 and SU_3 algebras, for both Newtonian and relativistic particles. Mukunda[3] has shown in a very general way that all single-particle, 3-dimensional classical systems characterized by time-independent Hamiltonians possess O_4 and SU_3 algebras. No such generalization appears possible, however, for the corresponding quantum systems.[2,3]

One question which remains is whether such algebraic structures are characteristic only of conservative systems. We show that all 3-dimensional classical Hamiltonian systems share the O_4 and SU_3 symmetries. (These results should readily generalize to n dimensions.) As an example, we treat explicitly the Newtonian, linearly damped, three-dimensional isotropic harmonic oscillator (DHO) in an arbitrarily time-dependent external field.

II. GENERAL TIME-DEPENDENT SYSTEM

Consider a time-dependent classical mechanical system with three degrees of freedom. Let the (relativistic or nonrelativistic) Hamiltonian be given by

$$H(r_i, p_i, t) , \tag{1}$$

where r_i and p_i are canonical coordinates and their conjugate momenta, respectively. A realization of the invariance of the above system under the O_4 and SU_3 Lie algebras is afforded by a finite canonical transformation and proceeds as follows:

Let us choose, for example, the class of canonical transformations $(r_i, p_i) \rightarrow (R_i, P_i)$ whose generators are of the form[4]

$$F_2(\vec{r}, \vec{P}, t) . \tag{2}$$

We then make a canonical transformation to a frame of reference in which the new Hamiltonian $\overline{H}$ is time-independent, i.e.,

$$H\left(r_i, \frac{\partial F_2}{\partial r_i}, t\right) + \frac{\partial F_2}{\partial t} = \overline{H}\left(\frac{\partial F_2}{\partial P_i}, P_i\right) . \tag{3}$$

The above partial differential equation can, in principle, be solved for $F_2(r_i, P_i, t)$ if the form of $\overline{H}$ is prescribed. The problem has thus been reduced to that for the conservative system $\overline{H}(R_i, P_i)$ for which the generators of the O_4 and SU_3 algebras can be obtained by the prescription of Mukunda.[3] These generators, when expressed in terms of the original canonical variables and time (r_i, p_i, t), provide us with O_4 and SU_3 algebras for the time-dependent system (1). We shall now construct these algebras for a forced, linearly damped, isotropic harmonic oscillator.

III. FORCED DHO

The equation of motion for a forced DHO is given by

$$\frac{d^2\vec{r}}{dt^2} + 2\gamma \frac{d\vec{r}}{dt} + \omega_0^2 \vec{r} = \frac{1}{m} \vec{f}(t) , \tag{4}$$

which can be obtained from the Hamiltonian

$$H = \frac{p^2}{2m} e^{-2\gamma t} + \tfrac{1}{2} m\omega_0^2 r^2 e^{2\gamma t} - \vec{f}\cdot\vec{r}\, e^{2\gamma t} , \tag{5}$$

where

$$\vec{p} = m \frac{d\vec{r}}{dt} e^{2\gamma t} . \tag{6}$$

Construction of SU_3 and O_4 algebras for the above can be accomplished by letting

$$\bar{H}(R_i, P_i) = \frac{P^2}{2m} + \tfrac{1}{2} m\omega^2 R^2, \quad \omega^2 = \omega_0^2 - \gamma^2 \tag{7}$$

so that (3) becomes

$$\frac{1}{2m} e^{-2\gamma t}\left(\frac{\partial F_2}{\partial r_i}\right)^2 + \tfrac{1}{2} m\omega_0^2 r^2 e^{2\gamma t} - \vec{f}\cdot\vec{r}\, e^{2\gamma t} + \frac{\partial F_2}{\partial t} = \frac{1}{2m} P^2 + \tfrac{1}{2} m\omega^2 \left(\frac{\partial F_2}{\partial P_i}\right)^2 . \tag{8}$$

Rather than solve the above nonlinear partial differential equation for $F_2(\vec{r}, \vec{P}, t)$, we can perform a series of canonical transformations, as shown in the Appendix, which is equivalent to a single canonical transformation generated by

$$F_2(\vec{r}, \vec{P}, t) = (\vec{r} - \vec{\xi})\cdot\vec{P}\, e^{\gamma t} + \vec{p}_\xi \cdot \vec{r} - \tfrac{1}{2} m\gamma e^{2\gamma t}(\vec{r} - \vec{\xi})^2 - \int L_0\, dt , \tag{9}$$

where

$$L_0 = \tfrac{1}{2} m\, e^{2\gamma t}(\dot{\xi}^2 - \omega_0^2 \xi^2) , \tag{10}$$

$$\vec{p}_\xi = m \frac{d\vec{\xi}}{dt} e^{2\gamma t} , \tag{11}$$

and $\vec{\xi}$ is a particular solution of (4).

Let us now construct SU_3 and O_4 algebras for the harmonic oscillator (7).[2]

SU_3 algebra

Consider the conserved canonical angular momentum

$$\vec{L} = \vec{R} \times \vec{P} , \tag{12}$$

and the symmetric tensor

$$A_{ij} = \frac{1}{m\omega} P_i P_j + m\omega R_i R_j . \tag{13}$$

We have the following Poisson-bracket relations (note that $\bar{H} = \tfrac{1}{2}\omega A_{ii}$):

$$\{A_{ij}, \bar{H}\} = \{L_i, \bar{H}\} = 0 , \tag{14}$$

$$\{L_i, L_j\} = \epsilon_{ijk} L_k , \tag{15}$$

$$\{L_i, A_{jk}\} = \epsilon_{ijn} A_{nk} + \epsilon_{ikn} A_{jn} , \tag{16}$$

$$\{A_{ij}, A_{kl}\} = (\delta_{ik}\epsilon_{jln} + \delta_{il}\epsilon_{jkn} + \delta_{jk}\epsilon_{iln} + \delta_{jl}\epsilon_{ikn}) L_n . \tag{17}$$

The SU_3 algebra can be constructed from $\vec{L}$ and the five traceless components of $\underline{A}$.

O_4 algebra

Consider the (conserved) Runge-Lenz vector[5]

$$\vec{A} = (A/2m\omega)^{1/2}(L^2 U^2 - m\bar{H} + m\omega A)^{-1/2} \times [(-m\bar{H} + m\omega A)(\vec{R}/R) + U\vec{P}\times\vec{L}] , \tag{18}$$

where

$$U = R^{-1}, \quad A = (\bar{H}^2 - \omega^2 L^2)^{1/2}/\omega . \tag{19}$$

It follows that

$$\{A_i, \bar{H}\} = \{L_i, \bar{H}\} = 0 , \tag{20}$$

$$\{L_i, L_j\} = \epsilon_{ijk} L_k , \tag{21}$$

$$\{L_i, A_j\} = \epsilon_{ijk} A_k , \tag{22}$$

$$\{A_i, A_j\} = \epsilon_{ijk} L_k . \tag{23}$$

The six quantities $\vec{A}$ and $\vec{L}$ form a closed algebra of O_4 structure.

To obtain the algebras for the system (5), we express all the above algebraic elements in terms of the original variables (r_i, p_i, t) via (9), i.e.,

$$\vec{R} = (\vec{r} - \vec{\xi})\, e^{\gamma t} , \tag{24}$$

$$\vec{P} = (\vec{p} - \vec{p}_\xi)\, e^{-\gamma t} + m\gamma e^{\gamma t}(\vec{r} - \vec{\xi}) . \tag{25}$$

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APPENDIX

Let us perform a series of canonical transformations equivalent to (9). First, we remove the effect of the force $\vec{f}(t)$ via the transformation $(r_i, p_i) \to (r_i', p_i')$, for which the new Hamiltonian is

$$H' = \frac{p'^2}{2m} e^{-2\gamma t} + \tfrac{1}{2} m\omega_0^2 r'^2 e^{2\gamma t} . \tag{A1}$$

The linearity of (4) suggests setting

$$\vec{r}' = \vec{r} - \vec{\xi} , \tag{A2}$$

where $\vec{\xi}$ is a particular solution of (4). From (5), (A1), and (A2), it follows that

$$\vec{p}' = \vec{p} - \vec{p}_\xi , \tag{A3}$$

where

$$\vec{p}_\xi = m \frac{d\vec{\xi}}{dt} e^{2\gamma t} .$$

The generating function $F_2(\vec{r}, \vec{p}', t)$ is then obtained from the integration of the relations

$$\vec{p} = \frac{\partial F_2}{\partial \vec{r}}, \quad \vec{r}' = \frac{\partial F_2}{\partial \vec{p}'}$$

and the condition

$$\bar{H} = H + \frac{\partial F_2}{\partial t} ,$$

and it is found to be

$$F_2 = \vec{r}\cdot(\vec{p}' + \vec{p}_\xi) - \vec{\xi}\cdot\vec{p}' - \int L_0\, dt , \tag{A4}$$

where L_0 is given by (10).

The transformation of the DHO (A1) to an undamped harmonic oscillator proceeds as follows: The Lagrangian corresponding to (A1) is

$$L' = \tfrac{1}{2} m e^{2\gamma t}(\dot{r}'^2 - \omega_0^2 r'^2) . \qquad (A5)$$

We consider the invariance of L' under the combined time translations and simultaneous coordinate scale transformations

$$t \to t + \delta, \; r_i' \to (1+\alpha) r_i', \quad \alpha \neq -1 .$$

If $1 + \alpha = e^{-\gamma\delta}$, L' is invariant. Since

$$r_i'' = r_i' e^{\gamma t}$$

are invariant under this transformation, and

$$\dot{r}_i'' = e^{\gamma t}(\dot{r}_i' + \gamma r_i')$$

are first differential invariants, the transformation to these variables is suggested. This yields the time-independent Lagrangian

$$L''(\dot{r}_i'', r_i'') = L'(\dot{r}_i', r_i', t) = \tfrac{1}{2} m (\dot{r}_i'' - \gamma r_i'')^2 - \tfrac{1}{2} m \omega_0^2 r_i''^2 , \qquad (A6)$$

and the corresponding Hamiltonian

$$H'' = \frac{1}{2m}(p_i'' + m\gamma r_i'')^2 + \tfrac{1}{2} m \omega^2 r_i''^2 , \quad \omega^2 = \omega_0^2 - \gamma^2 . \qquad (A7)$$

The above coordinate transformation corresponds to the canonical transformation $(r_i', p_i') \to (r_i'', p_i'')$ generated by

$$F_3(\vec{r}'', \vec{p}', t) = - e^{-\gamma t} \vec{r}'' \cdot \vec{p}' , \qquad (A8)$$

which can be obtained in a manner similar to that leading to (A4).

A final canonical transformation $(r_i'', p_i'') \to (R_i, P_i)$ with generator

$$\overline{F}_2(\vec{r}'', \vec{P}) = \vec{P} \cdot \vec{r}'' - \tfrac{1}{2} m \gamma r''^2 \qquad (A9)$$

transforms (A7) to

$$\overline{H} = \frac{1}{2m} P^2 + \tfrac{1}{2} m \omega^2 R^2 .$$

We note that the generator (9) is the sum of the generators (A4), (A8), and (A9), expressed in terms of the initial and final variables.

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[1]D. M. Fradkin, Am. J. Phys. 33, 207 (1965); H. Bacry, H. Ruegg, and J. M. Souriau, Commun. Math. Phys. 3, 323 (1966); V. B. Serebrennikov and A. E. Shabad, P. M. Lebedev Physical Institute Report No. N38, 1972 (unpublished).

[2]D. M. Fradkin, Prog. Theor. Phys. 37, 798 (1967).

[3]N. Mukunda, Phys. Rev. 155, 1383 (1967).

[4]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, Mass., 1950), Chap. 8.

[5]The Runge-Lenz vector (18) for the isotropic harmonic oscillator is only piecewise conserved. It is constant between two successive apogees pointing toward one perigee, and discontinuously reverses its direction each time the particle passes an apogee. However, the orbit geometry and the dynamics are not affected by this discontinuity. In a forthcoming note (in preparation), we show that this behavior of the Runge vector follows from the symmetry of the orbit. On the other hand, Serebrennikov and Shabad (Ref. 1) define a *smoothly varying* Runge vector (of constant magnitude), resulting in a "broken" O_4 symmetry for the harmonic oscillator (and also for more general central-force problems with nonclosed orbits).