PHYSICAL REVIEW D VOLUME 11, NUMBER 10 15 MAY 1975

Four-fermion interactions and scale invariance*

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(Received 15 January 1975)

Four-fermion interactions of the current-current type with U(n) symmetry, in one space and one time dimension, are investigated. It is shown that the equations of motion yield scale-invariant solutions only for two values of the coupling g_v of the SU(n) currents, namely $g_v = 0$ and $g_v = 4\pi/(n+1)$. This holds for any value of the coupling g_B of the U(1) currents. For the above two values of g_v and any g_B the theory is solved completely. Operator products of spinor fields are shown to be equal to c-number functions singular on the light cone times analytic bilocal operators expressed in terms of currents and free spinor fields. The currents are free for the above two values of g_v. The connection with the coupling as defined through four-point functions is discussed, and it turns out that the combination corresponding to SU(n) coupling is zero for both solutions. However, the solution for $g_v = 4\pi/(n+1)$ exhibits nontrivial four-point functions also for $g_B = 0$. It is shown, in an expansion around $g_v = 0$, that there is only one Callan-Symanzik function β which depends only on g_v and that $g_v = 0$ is relevant to the ultraviolet limit of the $g_v > 0$ theories. When mass terms are introduced, this still holds in an infinite interval for g_B, which is bounded below by a certain negative value and in which the mass term is soft.

I. INTRODUCTION

In this paper we treat four-fermion-coupling field theories in one space and one time dimension. We mainly consider the massless case. However, the effects of a mass term are discussed. The theories we treat have current-current interactions with U(n) symmetry. Thus there are two independent coupling constants in the Lagrangian,

$$L_I = -\tfrac{1}{2} g_B j_\mu j^\mu - \tfrac{1}{2} g_v j^a_\mu j^{a\mu} , \tag{1.1}$$

where j^a_μ, with $a = 1, 2, \ldots, n^2-1$, are the SU(n) currents and j_μ is the U(1) current. The spinor field is taken in the quark representation of SU(n). We are interested in finding those values of the coupling constants for which scale invariance is obtained.

Our main results are that the equations of motion yield a scale-invariant field theory only for two values of g_v, namely $g_v = 0$ and $g_v = 4\pi/(n+1)$. The value of g_B remains arbitrary. For any other values of g_v scale invariance is broken, through renormalization effects. In the scale-invariant cases, the dimension of the spinor field is anomalous and depends on the coupling constants. Moreover, the scalar and pseudoscalar densities have anomalous dimensions. The currents, however, are like free vector fields and have canonical dimensions.

The main ideas and an outline of the methods used to obtain these results were given by us earlier.[1] In this paper we also discuss effects of a mass term and calculate the renormalization-group parameters. The techniques we use are similar, to those employed[2] for solving the $n=1$ case, namely the case of the Thirring model.[3] We thus use normal ordering with respect to quanta of the currents[4] to define operator products. By this we avoid the necessity of defining the currents as regularized products of spinor fields at slightly different points. The latter method involves considerably more complications, making it harder to obtain correct results (see the discussion in Secs. III and IV for details). We also employ an energy-momentum tensor which is a function of the currents only,[5,6] thus giving the extra information replacing the explicit expressions of the currents in terms of the fields.

We obtain the values $g_v = 0$ and $g_v = 4\pi/(n+1)$ for scale invariance in the following way. We impose conformal invariance on the four-point function, and then use the equations of motion together with the requirement that the most singular short-distance behavior is like a product of two-point functions. This determines the above values of g_v. (We use the fact that for renormalizable theories scale invariance implies also conformal invariance[7].) We then show that for those values one indeed gets a field theory, by reconstructing the spinor fields. We also exhibit the structure of operator products of two and four spinor fields in terms of currents and free spinor fields multiplying c-number functions which are powers of coordinate differences and are singular on the light cone. The operator part is analytic in the coordinates. It is the "multilocal" operator; in the case of a product of two fields it is an example of a bilocal operator, as introduced for a general light-cone expansion.[8]

A short outline of this was given by us in Ref. 1.

Later, in Refs. 9 and 10, auxiliary scalar fields with an infrared cutoff[4] were used to construct the solutions for the two values of g_v. That method did not show that there are no other scale-invariant solutions. Also, in Ref. 10 it was argued that the solution with $g_v = 4\pi/(n+1)$ can be obtained formally from the $g_v = 0$ case with spin $s = -\frac{1}{2}$. We argue that one must have $s = \frac{1}{2}$ in order to be able to introduce mass terms (see the discussion in Secs. VI and VII). Since we view our model as a study of an asymptotic limit of a massive theory, we must have $s = \frac{1}{2}$ always. We also show that there is a region of coupling g_B for which the mass term is soft and asymptotic limits can be taken. The $g_v = 4\pi/(n+1)$ solution is, however, peculiar in the following way. When one calculates the four-point function corresponding to the perturbation-theory definition of the above coupling, one gets zero. However, insisting on $s = \frac{1}{2}$, not all four-point functions are trivial in this case even for $g_B = 0$. Thus the coupling obtained from the equations of motion does not simply correspond to the perturbation-theory definition.

Expanding perturbatively around $g_v = 0$ we show that there is only one Callan-Symanzik function[11] β, which is related to g_v and which depends only on g_v, and that the $g_v = 0$ solution is the asymptotic limit of the $g_v > 0$ case (up to calculable logarithmic corrections). The properties of the currents are determined from the conservation laws and the commutation relations. For the U(1) currents, conservation follows from phase and γ_5 invariance of the Lagrangian. The axial-vector SU(n) currents are in general not conserved. However, we insist on current algebra, which for the scale-invariant case also implies canonical dimensions. This in turn implies conservation of all currents. We also have finite Schwinger terms,[12] which for the U(1) currents j_μ serve as a normalization. For the SU(n) currents the Schwinger term has the free-field value for both cases of scale invariance.

A peculiarity of two-dimensional spinor theories is that the axial-vector currents are the dual of the vector ones. Conservation of both implies that they are free massless fields, which enables us to normal order with respect to their quanta. Another peculiarity is that j_μ and j_ν^a commute with each other at all times. The lack of these features in four dimensions may make it hard to generalize conclusions drawn from our model.

The spinor fields have anomalous dimensions[13] and the canonical commutation rules break down. Thus avoiding use of their regularized products in handling currents, as we do, is a great simplification. However, we do construct the scalar and pseudoscalar densities as products, properly regulated.

In the scale-invariant case, the Hamiltonian is a sum of two terms, one depending on $t+x$ and the other on $t-x$. The two commute with each other at all times. Thus no particle can reverse its direction of motion and there is no scattering (in a one-space-dimensional world). However, in the massless case we do not have asymptotic states. Our massless theory is a limit of a massive one when all momenta of Green's functions become large, such that all invariants become large too. Thus our theory is relevant for the highly off-shell behavior. When a mass term is introduced scattering will take place, and since the S matrix is defined for external particles on-shell, the limit of zero mass involves infrared problems. Thus the above-mentioned fact about the Hamiltonian in the massless case may not be relevant for the S matrix for any massive case.

The program of the paper is as follows. In Sec. II we discuss the implications of scale invariance and current algebra on the currents, and also give the commutation rules for the currents with the spinor field. In Sec. III we discuss the equations for the currents following from an energy-momentum tensor expressed in terms of currents only. The crucial role of the normal ordering is exhibited. In Sec. IV the equations for and the dimension of the spinor field are obtained. The equations include, besides the two obtained from the Dirac equation (for the two components of the spinor field), two extra ones which replace the information of having the currents in terms of products of the fields. The latter are identities for the free case. In Sec. V we deduce that scale invariance is maintained only for $g_v = 0$ and $g_v = 4\pi/(n+1)$ from considerations based on the four-point functions, as explained before. Operator products of two Fermi fields are then exhibited in terms of c-number functions and bilocal operators involving only U(1) currents and free spinor fields. In Sec. VI we explicitly construct the Fermi field. One way is to show that all conditions for reconstruction of the field[14] are obeyed. (Here we have a massless case, so that cluster decomposition involves a decrease only as a power of distance for large spacelike separations.) Another way is to introduce fermion fields which involve also some fixed space-time points, and then go to a limit of those points going to infinity. Another is to follow the construction of Ref. 2. In Sec. VII we discuss the question of coupling from the point of view of perturbation theory. We also compute the dimensions of the mass term and the function β. The latter is computed as an expansion around $g_v = 0$. We show that there is an interval in g_B which includes the origin, for which the mass term is soft, such that a massive theory with $g_v > 0$ tends to our $g_v=0$ solution in the asymptotic limit of large momenta.[15]

Finally, several demonstrations and calculations are performed in the Appendixes.

II. COMMUTATORS, DIVERGENCE EQUATIONS, AND SCALE INVARIANCE

We start with the Lagrangian (in one space and one time dimension)

$$L=\tfrac{1}{2}\bar{\psi}i\overleftrightarrow{\partial}\psi-\tfrac{1}{2}g_B{:}j_\mu j^\mu{:}-\tfrac{1}{2}g_v{:}j^{a\mu}j^a_\mu{:}\ , \tag{2.1}$$

where ψ is a Dirac spinor transforming as the fundamental (quark) representation of SU(n), and

$$j_\mu={:}\bar{\psi}\gamma_\mu\psi{:}\ , \tag{2.2a}$$

$$j^a_\mu={:}\bar{\psi}\gamma_\mu\tfrac{1}{2}\lambda^a\psi{:}\ , \tag{2.2b}$$

with $\overleftrightarrow{\partial}=\gamma^\mu(\overrightarrow{\partial}_\mu-\overleftarrow{\partial}_\mu)$ and λ^a the $n\times n$ traceless Hermitian matrices of the adjoint representation satisfying

$$[\tfrac{1}{2}\lambda^a,\tfrac{1}{2}\lambda^b]=if^{abc}(\tfrac{1}{2}\lambda^c)\ , \tag{2.3}$$

with f^{abc} the structure constants of SU(n). The double dots : : denote the usual normal ordering with respect to creation and annihilation operators of the fermion field. Later we shall use another normal ordering, with respect to the quanta of the currents.[2,4] We cannot use the former since we are going to solve the model exactly, and for the full spinor field there is no way of simple decomposition into creation and annihilation operators (the field contains both timelike and spacelike frequencies). The expressions Eqs. (2.1) and (2.2) are relevant for perturbation studies.

Formal derivations yield the equations of motion,

$$i\partial\!\!\!/\psi=g_B{:}j\!\!\!/\psi{:}+g_v{:}j\!\!\!/^a\tfrac{1}{2}\lambda^a\psi{:}\ . \tag{2.4}$$

The vector currents j_μ and j^a_μ are conserved, due to the invariance under U(n) symmetry, namely, invariance under phase transformations $\psi\rightarrow e^{i\alpha}\psi$ and under SU(n) transformations $\psi\rightarrow e^{i(1/2)\lambda^a\beta^a}\psi$ (α and β^a are numbers). The Lagrangian (2.1) is also invariant under axial phase transformations, $\psi\rightarrow e^{i\alpha\gamma_5}\psi$ (with $\gamma_5=\gamma_0\gamma_1$). This implies that

$$j^5_\mu={:}\bar{\psi}\gamma_\mu\gamma_5\psi{:} \tag{2.5}$$

is also conserved. In one space and one time dimension

$$\gamma_\mu\gamma_5=\epsilon_{\mu\nu}\gamma^\nu\ , \tag{2.6}$$

where $\epsilon_{\mu\nu}=-\epsilon_{\nu\mu}$ and $\epsilon_{10}=1$ (our metric is $g_{00}=-g_{11}=1$). Thus the vector current is divergence-free and curl-free. Defining

$$u=t+x\ ,\qquad v=t-x\ , \tag{2.7}$$

and

$$j_\pm=j_0\pm j_1\ , \tag{2.8}$$

we thus get that[2]

$$j_+\equiv j_+(u)\ ,\qquad j_-\equiv j_-(v)\ , \tag{2.9}$$

and since $\Box=4\partial_u\partial_v$, also

$$\Box j_\mu=0\ . \tag{2.10}$$

The equal-time commutation rules for the currents are

$$[j_0(xt),j_0(yt)]=0\ ,\quad [j_0(xt),j_1(yt)]=iC_0\delta'(x-y)\ ,\quad [j_1(xt),j_1(yt)]=0\ , \tag{2.11}$$

where C_0 is the Schwinger term,[12] a finite number in this case of two dimensions. For free fields $C_0=n/\pi$. In any case, C_0 serves as a normalization of j_μ.[2]

Combining Eqs. (2.9) and (2.11) one gets

$$[j_+(u),j_+(u')]=2iC_0\delta'(u-u')\ ,\quad [j_-(v),j_-(v')]=2iC_0\delta'(v-v')\ ,\quad [j_+(u),j_-(v)]=0\ . \tag{2.12}$$

The Klein-Gordon equation (2.10) and the commutation rules (2.12) imply the decomposition into creation and annihilation operators

$$j_+(u)=\left(\frac{C_0}{\pi}\right)^{1/2}\int_0^\infty dp\,[a_{(+)}(p)\,e^{-ipu}+a^\dagger_{(+)}(p)\,e^{ipu}]\ ,\quad j_-(v)=\left(\frac{C_0}{\pi}\right)^{1/2}\int_0^\infty dp\,[a_{(-)}(p)\,e^{-ipv}+a^\dagger_{(-)}(p)\,e^{ipv}]\ , \tag{2.13}$$

with

$$[a_{(+)}(p),a^\dagger_{(+)}(p')]=[a_{(-)}(p),a^\dagger_{(-)}(p')]=p\delta(p-p')\ , \tag{2.14}$$

the other commutators vanishing.

For the commutation law of the singlet currents with the Fermi field we take[16]

$$[j_+(u),\psi(u'v')]=-(a+\bar{a}\gamma_5)\,\psi(u'v')\,\delta(u-u')\ ,\quad [j_-(v),\psi(u'v')]=-(a-\bar{a}\gamma_5)\,\psi(u'v')\,\delta(v-v')\ , \tag{2.15}$$

where $a,\bar{a}$ are numbers (to be determined later).

Now, the Lagrangian (2.1) is not invariant under an axial SU(n) transformation, namely $\psi\rightarrow e^{i(1/2)\lambda^a\beta^a\gamma_5}\psi$. The axial-vector SU($n$) currents

$$j^{5a}_\mu={:}\bar{\psi}\gamma_\mu\gamma_5\tfrac{1}{2}\lambda^a\psi{:} \tag{2.16}$$

are therefore not conserved. The divergence equation, derived formally from our Lagrangian, is

$$\partial^\mu j^{5a}_\mu=g_vf^{abc}j^{b\lambda}j^{c5}_\lambda\ . \tag{2.17}$$

Since also for the SU(n) currents

$$j_\mu^{5a} = \epsilon_{\mu\nu} j^{a\nu} , \qquad (2.18)$$

it follows that the SU(n) vector current has a nonzero curl and therefore is not a free field.

At this stage we cannot proceed to solve the model, as was done in the case of the Thirring model ($g_v = 0$),[2] since the SU(n) currents are not free. Products of SU(n) currents and spinor fields cannot therefore be normal-ordered in the same manner as for the singlet currents, since only for the singlet currents does a decomposition into canonical creation and annihilation operators exist.

The equal-time commutation rules are

$$\begin{aligned}
&[j_0^a(xt), j_0^b(yt)] = if^{abc} j_0^c(xt)\,\delta(x-y) , \\
&[j_0^a(xt), j_1^b(yt)] = if^{abc} j_1^c(xt)\,\delta(x-y) \\
&\qquad\qquad + iC_1 \delta^{ab}\,\delta'(x-y) , \\
&[j_1^a(xt), j_1^b(yt)] = if^{abc} j_0^c(xt)\,\delta(x-y) .
\end{aligned} \qquad (2.19)$$

C_1 is the c-number Schwinger term. Here the currents are normalized by the charge algebra, and therefore C_1 is an independent constant (unlike the case of the singlet currents). Also,

$$[j_\mu^a(xt), j_\nu(yt)] = 0 . \qquad (2.20)$$

The latter is peculiar to one space dimension, where the space components of the vectors are time components of the axial-vectors and vice versa.

The Lagrangian (2.1) is formally invariant under scale transformations

$$\psi(x) \to \lambda^{1/2}\psi(\lambda x) . \qquad (2.21)$$

However, it is known that, owing to the renormalization procedure, scale invariance does not hold in general.[17] Scale invariance may hold for certain values of the coupling constants.[17] Even then, the fields in general acquire anomalous dimensions,[13] and the law, Eq. (2.21), is not valid (hence also the canonical anticommutator of Fermi fields breaks down). In the case of the usual Thirring model ($n=1$) scale invariance holds for any g_B, with anomalous dimensions that depend on the value of g_B.[2, 13, 18]

We will be interested in solutions to the generalized model that exhibit scale invariance. For those, the notion of dimensionality of operators is well defined. Current algebra, Eq. (2.19), then forces the currents to have the canonical value 1 for their dimension, namely

$$j_\mu^a(x) \to \lambda j_\mu^a(\lambda x) . \qquad (2.22)$$

We will now show that in a scale-invariant theory in one space dimension any vector operator with dimension 1 is conserved and curl-free. For consider a vector $A_\mu(x)$ with dimension 1. The two-point function of $A_+ = A_0 + A_1$ must then have the form

$$\langle 0 | A_+(uv) A_+(u'v') | 0 \rangle = \frac{C_A}{2\pi} \frac{1}{[i(u-u')+\epsilon]^2} . \qquad (2.23)$$

The reason is that since A_+ transforms under Lorentz transformations as $1/u$, the left-hand side of (2.23) can differ from the right-hand side only by a multiplicative factor which is an invariant function only. However, the latter must be a constant due to the fact that it has dimension zero. Thus $\partial_v A_+$ annihilates the vacuum, and is therefore zero.[19] Similarly, $\partial_u A_- = 0$. Hence $\partial_\mu A^\mu = 0$, $\epsilon_{\mu\nu}\partial^\mu A^\nu = 0$. (Note that this result does not involve the assumption of conformal invariance. In four dimensions conformal invariance is needed[20] to prove conservation of a current of dimension 3.)

We thus conclude that for those values of the coupling constants where we have scale invariance the right-hand side of Eq. (2.17) is zero and all currents are conserved. Similar to the case of the singlet currents we now deduce from the equal-time commutators (2.19) the commutation rules for any space-time points,

$$\begin{aligned}
&[j_+^a(u), j_+^b(u')] = 2if^{abc} j_+^c(u)\,\delta(u-u') \\
&\qquad\qquad + 2iC_1 \delta^{ab}\,\delta'(u-u') , \\
&[j_-^a(v), j_-^b(v')] = 2if^{abc} j_-^c(v)\,\delta(v-v') \\
&\qquad\qquad + 2iC_1 \delta^{ab}\,\delta'(v-v') , \\
&[j_+^a(u), j_-^b(v)] = 0 ,
\end{aligned} \qquad (2.24)$$

and from (2.20),

$$[j_\mu^a(x), j_\nu(y)] = 0 . \qquad (2.25)$$

In analogy with Eq. (2.13) we can now decompose also the currents $j_\pm^a$ into creation and annihilation parts, each one of massless excitations. We have

$$\begin{aligned}
&j_+^a(u) = j_+^{a(+)}(u) + j_+^{a(-)}(u) , \\
&j_-^b(v) = j_-^{b(+)}(v) + j_-^{b(-)}(v) ,
\end{aligned} \qquad (2.26)$$

where (+) is the creation part and (−) the annihilation part. Note that here two creation or two annihilation operators of different SU(n) indices do not commute.

In analogy with Eqs. (2.15) we write

$$\begin{aligned}
&[j_+^a(u), \psi(u'v')] = -b_+(1+\delta_+\gamma_5)\tfrac{1}{2}\lambda^a \psi(u'v') \\
&\qquad\qquad \times\delta(u-u') , \\
&[j_-^a(v), \psi(u'v')] = -b_-(1-\delta_-\gamma_5)\tfrac{1}{2}\lambda^a \psi(u'v') \\
&\qquad\qquad \times\delta(v-v') .
\end{aligned}$$

The Jacobi identities combined with Eqs. (2.24) yield $b_+ = b_- = 1$, $\delta_+ = \delta_- = \delta$, and $\delta^2 = 1$. Thus

$$[j^a_+(u), \psi(u'v')] = -(1+\delta\gamma_5)\tfrac{1}{2}\lambda^a\psi(u'v')\delta(u-u') , \tag{2.27}$$

$$[j^a_-(v), \psi(u'v')] = -(1-\delta\gamma_5)\tfrac{1}{2}\lambda^a\psi(u'v')\delta(v-v') .$$

Note that these commutation rules show invariance under parity (the fact that the same δ appears in both), a result of the parity invariance of Eqs. (2.24).

We can now also define normal-ordered products of currents and spinor fields at the same point. They are defined with respect to the quanta of the currents

$$\begin{aligned} :j_\pm\psi: &= j^{(+)}_\pm\psi + \psi j^{(-)}_\pm , \\ :j^a_\pm\psi: &= j^{(+)a}_\pm\psi + \psi j^{(-)a}_\pm . \end{aligned} \tag{2.28}$$

In Appendix A we show that Eq. (2.28) is the same as a limiting procedure, with products at slightly different points.

III. THE "WORLD" OF CURRENTS

In this section we treat the currents only. We solve for the model in a way of the "theory of currents,"[5,21] namely expressing the energy-momentum tensor $\theta_{\mu\nu}$ in terms of currents only. That same $\theta_{\mu\nu}$ will later be used also when we include spinors.

In a scale-invariant theory, $\theta_{\mu\nu}$ has a vanishing trace. Define now

$$\theta_\pm = \theta_{00} \pm \theta_{01} . \tag{3.1}$$

In a two-dimensional world, conservation of $\theta_{\mu\nu}$, its symmetry under $\mu \leftrightarrow \nu$ and its tracelessness imply that θ_+ is a function of u only and θ_- of v only. We now write $\theta_{\mu\nu}$ as

$$\theta_{\mu\nu} = \theta^B_{\mu\nu} + \theta^v_{\mu\nu} , \tag{3.2}$$

with

$$\begin{aligned} \theta^B_{\mu\nu} &= \frac{1}{2\tilde{C}_0}[2:j_\mu j_\nu: - g_{\mu\nu}:j_\lambda j^\lambda:] , \\ \theta^v_{\mu\nu} &= \frac{1}{2\tilde{C}_1}[2:j^a_\mu j^a_\nu: - g_{\mu\nu}:j^a_\lambda j^{a\lambda}:] . \end{aligned} \tag{3.3}$$

The relative coefficient between the two terms is determined from tracelessness. $\tilde{C}_0$ and $\tilde{C}_1$ are to be determined. Note that on the level of the currents only, the SU(n) part and the singlet part are completely decoupled. We have

$$\begin{aligned} \theta^B_+ &= \frac{1}{2\tilde{C}_0}:[j_+(u)]^2: , \\ \theta^B_- &= \frac{1}{2\tilde{C}_0}:[j_-(v)]^2: , \\ \theta^v_+ &= \frac{1}{2\tilde{C}_1}:[j^a_+(u)]^2: , \\ \theta^v_- &= \frac{1}{2\tilde{C}_1}:[j^a_-(v)]^2: . \end{aligned} \tag{3.4}$$

The u and v parts are thus decoupled too. Now, from Eq. (2.12)

$$[:j_+^2(u):, j_+(u')] = 4iC_0 j_+(u)\delta'(u-u') . \tag{3.5}$$

Hence,

$$\frac{i}{2}[H+P, j_+(u)] = \frac{C_0}{\tilde{C}_0}\partial_u j_+(u) ,$$

and thus necessarily

$$\tilde{C}_0 = C_0 . \tag{3.6}$$

This is the case discussed in Ref. 2, where it was shown that Lorentz invariance is maintained. For the commutator of energy-momentum densities we get

$$[\theta^B_+(u), \theta^B_+(u')] = 2i[\theta^B_+(u) + \theta^B_+(u')]\delta'(u-u') - \frac{i}{6\pi}\delta'''(u-u') . \tag{3.7}$$

In getting the c-number term we used the positive- and negative-frequency parts of the δ function,

$$\delta^{(+)}(x) = \delta^{(-)}(-x) = \frac{i}{2\pi}\frac{1}{x+i\epsilon} \tag{3.8}$$

and

$$[\delta^{(+)\prime}(x)]^2 - [\delta^{(-)\prime}(x)]^2 = -\frac{i}{12\pi}\delta'''(x) . \tag{3.9}$$

Note that to obtain the c-number term on the right-hand side of (3.7) the normal ordering in θ is essential. Without normal ordering we obtain only the first term. It is well known that the existence of such a c-number term follows from positivity and locality.[22] Note that by the normal ordering with respect to quanta of the currents, we got the c-number term easily without any complicated techniques of point separation, the latter necessary when expressing the currents in terms of the spinor fields. Also, the c-number term is independent of the value of the four Fermi coupling constants.

Let us now go to the SU(n) part. We start with

$$[\theta^v_+(u), j^b_+(u')] = \frac{2i}{\tilde{C}_1}\left(\frac{n}{2\pi} + C_1\right) j^b_+(u)\delta'(u-u') . \tag{3.10}$$

The calculation is performed in Appendix B. Note that the $n/2\pi$ term came from the δ part in Eq. (2.24) and the normal-ordering procedure. If we were not careful about the normal ordering, only the δ' part in the commutators (2.24) would contribute to the right-hand side of (3.10), and the latter would have only $2iC_1/\tilde{C}_1$ as its coefficient. The $n/2\pi$ part is easily missed when point-separation techniques of expressions involving spinor fields are employed, resulting in using an algebra of currents which is not symmetric between space and time components,[23] unlike the correct algebra Eqs. (2.24).

Translational invariance now requires

$$\tilde{C}_1 = \frac{n}{2\pi} + C_1 . \tag{3.11}$$

With this choice (for computation see Appendix B),

$$[\theta_+^v(u), \theta_+^v(u')] = 2i[\theta_+^v(u) + \theta_+^v(u')]\delta'(u-u') - \frac{i}{6\pi} S_v \delta'''(u-u') , \tag{3.12}$$

with

$$S_v = \frac{(n^2-1)C_1}{n/2\pi + C_1} = \frac{(n^2-1)(2\pi C_1)}{n+(2\pi C_1)} . \tag{3.13}$$

Again, the normal ordering was essential to obtain the c-number term. For the case of free spinor fields we have

$$C_1 = \frac{1}{2\pi} \text{ and } S_v = (n-1) .$$

With the relation (3.11) obeyed, the theory is Lorentz-invariant for any value of C_1. To fix C_1 we have to discuss the Fermi fields.

IV. EQUATIONS FOR THE SPINOR FIELD

Given the expressions for the energy-momentum tensor in terms of the currents, Eqs. (3.3), and the commutation rules of the currents with the spinor fields, Eqs. (2.15) and (2.27), we can now get the equations of motion for the spinor fields by calculating the commutation relations of the energy-momentum tensor and the spinor field. We get

$$[\theta_+(u), \psi(u'v')] = -\left[\frac{1+\delta\gamma_5}{\tilde{C}_1}\tfrac{1}{2}\lambda^b : j_+^b(u)\psi(uv): + \frac{a+\bar{a}\gamma_5}{C_0} : j_+(u)\psi(uv):\right]\delta(u-u') + \frac{i}{4\pi}\left[\frac{n^2-1}{n\tilde{C}_1}(1+\delta\gamma_5) + \frac{(a+\bar{a}\gamma_5)^2}{C_0}\right]\psi(u'v')\delta'(u-u') , \tag{4.1a}$$

$$[\theta_-(v), \psi(u'v')] = -\left[\frac{1-\delta\gamma_5}{\tilde{C}_1}\tfrac{1}{2}\lambda^b : j_-^b(v)\psi(uv): + \frac{a-\bar{a}\gamma_5}{C_0} : j_-(v)\psi(uv):\right]\delta(v-v') + \frac{i}{4\pi}\left[\frac{n^2-1}{n\tilde{C}_1}(1-\delta\gamma_5) + \frac{(a-\bar{a}\gamma_5)^2}{C_0}\right]\psi(u'v')\delta'(v-v') . \tag{4.1b}$$

The calculation is similar to the U(1) case discussed in Ref. 2. Note again the importance of normal ordering; here the δ' terms are not obtained without the introduction of a normal ordering between the currents and the spinor field as in Eq. (2.28). Without those δ' terms spinor fields cannot be incorporated. Since

$$\begin{aligned} H+P &= \int \theta_+(u)\,du , \\ H-P &= \int \theta_-(v)\,dv , \\ D+M &= \int u\theta_+(u)\,du , \\ D-M &= \int v\theta_-(v)\,dv , \end{aligned} \tag{4.2}$$

where D is the generator for dilatations and M for Lorentz transformations, we get

$$\partial_u \psi(uv) = -\frac{i}{2}\left[\frac{1+\delta\gamma_5}{\tilde{C}_1}\tfrac{1}{2}\lambda^b : j_+^b(u)\psi(uv): + \frac{a+\bar{a}\gamma_5}{C_0} : j_+(u)\psi(uv):\right] , \tag{4.3a}$$

$$\partial_v \psi(uv) = -\frac{i}{2}\left[\frac{1-\delta\gamma_5}{\tilde{C}_1}\tfrac{1}{2}\lambda^b : j_-^b(v)\psi(uv): + \frac{a-\bar{a}\gamma_5}{C_0} : j_-(v)\psi(uv):\right] , \tag{4.3b}$$

and

$$\frac{i}{2}[D+M, \psi(uv)] = u\partial_u\psi(uv) + \frac{1}{8\pi}\left[\frac{n^2-1}{n\tilde{C}_1}(1+\delta\gamma_5) + \frac{(a+\bar{a}\gamma_5)^2}{C_0}\right]\psi(uv) , \tag{4.4a}$$

$$\frac{i}{2}[D-M, \psi(uv)] = v\partial_v\psi(uv) + \frac{1}{8\pi}\left[\frac{n^2-1}{n\tilde{C}_1}(1-\delta\gamma_5) + \frac{(a-\bar{a}\gamma_5)^2}{C_0}\right]\psi(uv) . \tag{4.4b}$$

From Eqs. (4.4a) and (4.4b) we deduce that the spin s of ψ is

$$s=\tfrac{1}{2}\left[\frac{a\bar{a}}{\pi C_0}+\frac{(n^2-1)\delta}{(2\pi\tilde{C}_1)n}\right]\ , \tag{4.5}$$

and the dimension d of ψ is

$$d=\frac{1}{4\pi}\left(\frac{a^2+\bar{a}^2}{C_0}+\frac{n^2-1}{n\tilde{C}_1}\right)$$

$$=s+\frac{(a-\bar{a})^2}{4\pi C_0}+\frac{(n^2-1)(1-\delta)}{(4\pi\tilde{C}_1)n}\ . \tag{4.6}$$

Naturally we set $s=\frac{1}{2}$. In the case of two-dimensional massless spinor fields, the Dirac equation (2.4) actually admits any value of s. This is so since under a Lorentz transformation

$$\psi(x)\to e^{s\gamma_5\varphi}\psi(\Lambda^{-1}x)\ , \tag{4.7}$$

where Λ is the transformation matrix on the coordinates and φ is the boost parameter $\tanh\varphi=\beta$.

We still insist on $s=\frac{1}{2}$, since that is the only value consistent with Lorentz invariance when a mass term is introduced in Eq. (2.4). In our treatment of the massless case, which we can solve, we have in mind a massive theory which in certain limits behaves as the massless case (we shall discuss this in more detail in Sec. VII). Thus we must have $s=\frac{1}{2}$ (in Ref. 10 the case $s=-\frac{1}{2}$ was discussed; see Sec. VII for details regarding this).

Let us now return to Eqs. (4.3a) and (4.3b). Since ψ is a two-component spinor, we have here four equations. However, the original equations of motion (2.4) are only two equations. What happened is that the extra two equations are the new information which replaces the expressions of the currents in terms of singular products of spinor fields. This is similar to what happened in the case of one spinor field discussed in Ref. 2. The equations of motion (2.4) can be rewritten as

$$i\gamma_0[(1-\gamma_5)\partial_u+(1+\gamma_5)\partial_v]\psi(uv)=\tfrac{1}{2}g_B\gamma_0[(1-\gamma_5):j_+(u)\psi(uv):+(1+\gamma_5):j_-(v)\psi(uv):]$$
$$+\tfrac{1}{2}g_v\gamma_0[(1-\gamma_5):j_+^a(u)\tfrac{1}{2}\lambda^a\psi(uv):+(1+\gamma_5):j_-^a(v)\tfrac{1}{2}\lambda^a\psi(uv):]\ .$$

Defining

$$\psi_1(uv)=\tfrac{1}{2}(1+\gamma_5)\psi(uv),\quad \psi_2(uv)=\tfrac{1}{2}(1-\gamma_5)\psi(uv)\ , \tag{4.8}$$

we get

$$\begin{aligned}i\partial_v\psi_1(uv)&=\tfrac{1}{2}g_B:j_-(v)\psi_1(uv):+\tfrac{1}{2}g_v:j_-^a(v)\tfrac{1}{2}\lambda^a\psi_1(uv):\ ,\\ i\partial_u\psi_2(uv)&=\tfrac{1}{2}g_B:j_+(u)\psi_2(uv):+\tfrac{1}{2}g_v:j_+^a(u)\tfrac{1}{2}\lambda^a\psi_2(uv):\ .\end{aligned} \tag{4.9}$$

In a representation where γ_5 is diagonal, ψ_1 is the eigenfunction with $\gamma_5=1$ and ψ_2 with $\gamma_5=-1$. Both ψ_1 and ψ_2 are one component in spin space.

Comparing (4.3a) and (4.3b) with (4.9) we get

$$g_B=(a-\bar{a})/C_0\ , \tag{4.10}$$

$$g_v=(1-\delta)/\tilde{C}_1\ . \tag{4.11}$$

Besides (4.9), Eqs. (4.3a) and (4.3b) also include

$$\begin{aligned}i\partial_u\psi_1(uv)&=\frac{1}{2}\left[\frac{1+\delta}{\tilde{C}_1}\tfrac{1}{2}\lambda^b:j_+^b(u)\psi_1(uv):+\frac{a+\bar{a}}{C_0}:j_+(u)\psi_1(uv):\right],\\ i\partial_v\psi_2(uv)&=\frac{1}{2}\left[\frac{1+\delta}{\tilde{C}_1}\tfrac{1}{2}\lambda^b:j_-^b(v)\psi_2(uv):+\frac{a+\bar{a}}{C_0}:j_-(v)\psi_2(uv):\right],\end{aligned} \tag{4.12}$$

which are extra equations of motion.[24] In Appendix A we demonstrate that for the free case $\delta=1$, $a=\bar{a}=1$, $C_0=n/\pi$, $\tilde{C}_1=(n+1)/2\pi$, Eqs. (4.12) become identities.

V. SOLUTION FOR PRODUCTS OF FERMI FIELDS

We note that the equations for ψ_1 and ψ_2 are decoupled. From (4.9) and (4.12) it follows that ψ_2 is obtained from ψ_1 by $u\leftrightarrow v$, with $j_+(u)\leftrightarrow j_-(v)$ and $j_+^a(u)\to j_-^a(v)$.

We therefore start solving for ψ_1. We define an operator $M(uv;u'v')$ by

$$\psi_1(uv)\psi_1^\dagger(u'v')=f_0[i(u-u')+\epsilon]^{-(d+s)}[i(v-v')+\epsilon]^{-(d-s)}:\exp\left\{-\frac{i}{2C_0}\left[(a+\bar{a})\int_{u'}^{u}j_+(u'')\,du''+(a-\bar{a})\int_{v'}^{v}j_-(v'')\,dv''\right]\right\}$$
$$\times M(uu';vv')\ . \tag{5.1}$$

We normalize to f_0 such that

$$\langle 0|M^{ab}(uu';vv')|0\rangle=\delta^{ab} . \quad (5.2)$$

The form of the c-number function in Eq. (5.1) is determined from the two-point function,

$$\begin{aligned}\langle 0|\psi_1(uv)\psi_1^\dagger(u'v')|0\rangle &= f_0[i(u-u')+\epsilon]^{-(d+s)}[i(v-v')+\epsilon]^{-(d-s)}\\ &= f_0[i(u-u')+\epsilon]^{-2s}[-(x-x')^2+i\epsilon(x_0-x_0')]^{-(d-s)} .\end{aligned} \quad (5.3)$$

From Lorentz invariance it follows that the two-point function (for $s=\frac{1}{2}$) is $1/[i(u-u')+\epsilon]$ times a function of $(x-x')^2$ (up to $i\epsilon$ terms), and from dimensionality that it is $[\text{length}]^{-2d}$. The $i\epsilon$ structure is determined from the spectral conditions. Also, it follows from Eq. (5.3) that f_0 is real. Now, from Eqs. (2.2) we deduce that

$$\left[j_+(u_1), :\exp\left(-\frac{1}{2C_0}(a+\bar{a})\int_{u'}^{u} j_+(u'')\,du''\right):\right] = -(a+\bar{a})[\delta(u_1-u)-\delta(u_1-u')] :\exp\left(-\frac{i}{2C_0}(a+\bar{a})\int_{u'}^{u} j_+(u'')\,du''\right):, \quad (5.4a)$$

$$\left[j_-(v_1), :\exp\left(-\frac{i}{2C_0}(a-\bar{a})\int_{v'}^{v} j_-(v'')\,dv''\right):\right] = -(a-\bar{a})[\delta(v_1-v)-\delta(v_1-v')] :\exp\left(-\frac{i}{2C_0}(a-\bar{a})\int_{v'}^{v} j_-(v'')\,dv''\right). \quad (5.4b)$$

This entails

$$[j_+(u_1), M(uu';vv')]=0, \quad [j_-(v_1), M(uu';vv')]=0 . \quad (5.5)$$

Comparing with Ref. 2 we see that $M=1$ for $n=1$, the U(1) case. From Eq. (5.5) we see that M is independent of the U(1) currents.

In order to proceed from here we need the equations of motion. Now

$$\begin{aligned} i\partial_u[\psi_1(uv)\psi_1^\dagger(u'v')] &= \frac{1}{2}\,\frac{1+\delta}{\tilde{C}_1}\,\tfrac{1}{2}\lambda^b[j_+^{b(+)}(u)\psi_1(uv)+\psi_1(uv)j_+^{b(-)}(u)]\psi_1^\dagger(u'v')\\ &\quad+\frac{1}{2}\,\frac{a+\bar{a}}{C_0}[j_+^{(+)}(u)\psi_1(uv)+\psi_1(uv)j_+^{(-)}(u)]\psi_1^\dagger(u'v')\\ &= \frac{1+\delta}{2\tilde{C}_1}\,\tfrac{1}{2}\lambda^b:j_+^b(u)[\psi_1(uv)\psi_1^\dagger(u'v')]: + \frac{a+\bar{a}}{2C_0}:j_+(u)[\psi_1(uv)\psi_1^\dagger(u'v')]:\\ &\quad+\frac{(d+s)}{i(u-u')+\epsilon}[\psi_1(uv)\psi_1^\dagger(u'v')]\\ &\quad+\frac{1+\delta}{8\pi\tilde{C}_1}\,\frac{1}{i(u-u')+\epsilon}\left\{\lambda^b[\psi_1(uv)\psi_1^\dagger(u'v')]\lambda^b-\frac{2(n^2-1)}{n}[\psi_1(uv)\psi_1^\dagger(u'v')]\right\} .\end{aligned}$$

Comparing with Eq. (5.1) we get that

$$i\partial_u M(uu';vv') = \frac{1+\delta}{2\tilde{C}_1}\left\{\tfrac{1}{2}\lambda^b:j_+^b(u)M(uu';vv'): + \frac{1}{4\pi}\,\frac{1}{i(u-u')+\epsilon}\left[\lambda^b M(uu';vv')\lambda^b-\frac{2(n^2-1)}{n}M(uu'vv')\right]\right\} . \quad (5.6)$$

Similarly,

$$i\partial_v M(uu';vv') = \frac{1-\delta}{2\tilde{C}_1}\left\{\tfrac{1}{2}\lambda^b:j_-^b(v)M(uu';vv'): + \frac{1}{4\pi}\,\frac{1}{i(v-v')+\epsilon}\left[\lambda^b M(uu';vv')\lambda^b-\frac{2(n^2-1)}{n}M(uu';vv')\right]\right\} . \quad (5.7)$$

The equations for u' and v' are obtained by $\partial_u\to\partial_{u'}$ and $\partial_v\to\partial_{v'}$ on the left-hand side, $j_+^b(u)\to j_+^b(u')$ and $j_-^b(v)\to j_-^b(v')$ on the right-hand side.

Note that for $\delta=1$, M is a function of (u,u') only, and for $\delta=-1$ of (v,v') only.

The initial conditions are

$$M^{ab}_{(\delta=1)}(uu)=\delta^{ab}, \quad M^{ab}_{(\delta=-1)}(vv)=\delta^{ab} . \quad (5.8)$$

Equations (5.6)–(5.8) are to determine M.

It is more convenient, at this stage, to go to the four-point function. We write

$$\begin{aligned}\langle 0|\psi_{1a}(u_1 v_1)\psi^\dagger_{1a'}(u_1' v_1')\psi_{1b}(u_2 v_2)\psi^\dagger_{1b'}(u_2' v_2')|0\rangle = f_0{}^2\{[i(u_1-u_1')+\epsilon][i(u_2-u_2')+\epsilon]\}^{-(d+s)}\\ \times\{[i(v_1-v_1')+\epsilon][i(v_2-v_2')+\epsilon]\}^{-(d-s)}\\ \times\xi^{(d+s)}\eta^{(d-s)}G_{aa'\,bb'}(\xi,\eta)\ ,\end{aligned} \tag{5.9}$$

where

$$\begin{aligned}\xi&=\frac{[i(u_1-u_2)+\epsilon][i(u_1'-u_2')+\epsilon]}{[i(u_1-u_2')+\epsilon][i(u_1'-u_2)+\epsilon]}\ ,\\ \eta&=\frac{[i(v_1-v_2)+\epsilon][i(v_1'-v_2')+\epsilon]}{[i(v_1-v_2')+\epsilon][i(v_1'-v_2)+\epsilon]}\ .\end{aligned} \tag{5.10}$$

The fact that G as defined in Eq. (5.9) depends on the variables ξ, η only follows from conformal invariance; for under a conformal transformation characterized by a vector e_ν

$$x_\nu \to \frac{x_\nu - e_\nu x^2}{1-2e\cdot x+e^2x^2}\ . \tag{5.11}$$

In two dimensions, the u and v parts transform separately,

$$u\to u/(1-e_-u),\qquad v\to v/(1+e_+v)\ . \tag{5.12}$$

We also have

$$\begin{aligned}\psi_1(uv)&\to(1-e_-u)^{-(d+s)}(1-e_+v)^{-(d-s)}\psi_1\left(\frac{u}{1-e_-u},\ \frac{v}{1-e_+v}\right)\ ,\\ \psi_2(uv)&\to(1-e_-u)^{-(d-s)}(1-e_+v)^{-(d+s)}\psi_2\left(\frac{u}{1-e_-u},\ \frac{v}{1-e_+v}\right)\ ,\end{aligned} \tag{5.13}$$

and

$$j_+(u)\to(1-e_-u)^{-2}j_+\left(\frac{u}{1-e_-u}\right),\qquad j_-(v)\to(1-e_+v)^{-2}j_-\left(\frac{v}{1-e_+v}\right), \tag{5.14}$$

and the same transformation law for $j^a_+(u)$, $j^a_-(v)$. The quantities (ξ,η) are the only two independent conformal invariants one can form out of four points. Thus conformal invariance implies that G be a function of (ξ,η) only. By writing the factors $\xi^{(d+s)}\eta^{(d-s)}$ separately we ensure that G is regular whenever a point in a ψ coincides with a point in a $\psi^\dagger$. Also, $G=1$ for the case $n=1$ (see Ref. 2).

We now use the equations of motion for ψ_1, Eqs. (4.9) and (4.12), to obtain equations for G. We get

$$\begin{aligned}\frac{1+\delta}{8\pi\tilde{C}_1}\Bigg\{&\left[(\lambda^b)_{a\bar a}(\lambda^b)_{\bar a' a'}-\frac{2(n^2-1)}{n}\delta_{a\bar a}\delta_{\bar a' a'}\right]\frac{1}{i(u_1-u_1')+\epsilon}G_{\bar a\bar a'\,bb'}\\ &-\left[(\lambda^b)_{a\bar a}(\lambda^b)_{b\bar b}-\frac{2(n^2-1)}{n}\delta_{a\bar a}\delta_{b\bar b}\right]\frac{1}{i(u_1-u_2)+\epsilon}G_{\bar a a'\,\bar b b'}\\ &+\left[(\lambda^b)_{a\bar a}(\lambda^b)_{\bar b' b'}-\frac{2(n^2-1)}{n}\delta_{a\bar a}\delta_{\bar b' b'}\right]\frac{1}{i(u_1-u_2')+\epsilon}G_{\bar a a'\,b\bar b'}\Bigg\}=i\partial_{u_1}G_{aa'\,bb'}\ ,\end{aligned} \tag{5.15}$$

$$\begin{aligned}\frac{1-\delta}{8\pi\tilde{C}_1}&\left[(\lambda^b)_{a\bar a}(\lambda^b)_{\bar a' a'}-\frac{2(n^2-1)}{n}\delta_{a\bar a}\delta_{\bar a' a'}\right]\frac{1}{i(v_1-v_1')+\epsilon}G_{\bar a\bar a'\,bb'}\\ &-\left[(\lambda^b)_{a\bar a}(\lambda^b)_{b\bar b}-\frac{2(n^2-1)}{n}\delta_{a\bar a}\delta_{b\bar b}\right]\frac{1}{i(v_1-v_2)+\epsilon}G_{\bar a a'\,\bar b b'}\\ &+\left[(\lambda^b)_{a\bar a}(\lambda^b)_{\bar b' b'}-\frac{2(n^2-1)}{n}\delta_{a\bar a}\delta_{\bar b' b'}\right]\frac{1}{i(v_1-v_2')+\epsilon}G_{\bar a a'\,b\bar b'}=i\partial_{v_1}G_{aa'\,bb'}\ .\end{aligned} \tag{5.16}$$

Let us decompose G in terms of invariant functions,

$$G_{aa'\,bb'}=\delta_{aa'}\delta_{bb'}H_1+\delta_{ab'}\delta_{ba'}H_2\ . \tag{5.17}$$

Then, from (5.15),

$$\frac{1+\delta}{4\pi\tilde{C}_1}\left\{\left[\frac{1}{i(u_1-u_2)+\epsilon}-\frac{1}{i(u_1-u_{2'})+\epsilon}\right]nH_1+\left[\frac{1}{i(u_1-u_{1'})+\epsilon}-\frac{1}{i(u_1-u_2)+\epsilon}\right]H_2\right\}$$
$$=i\partial_{u_1}H_1=\left[\frac{1}{i(u_1-u_{2'})+\epsilon}-\frac{1}{i(u_1-u_2)+\epsilon}\right]\xi\partial_\xi H_1\,, \quad (5.18)$$

$$\frac{(1+\delta)}{4\pi\tilde{C}_1}\left\{-\left[\frac{1}{i(u_1-u_2)+\epsilon}-\frac{1}{i(u_1-u_{2'})+\epsilon}\right]H_1-\left[\frac{1}{i(u_1-u_{1'})+\epsilon}-\frac{1}{i(u_1-u_2)+\epsilon}\right]nH_2\right\}$$
$$=i\partial_{u_1}H_2=\left[\frac{1}{i(u_1-u_{2'})+\epsilon}-\frac{1}{i(u_1-u_2)+\epsilon}\right]\xi\partial_\xi H_2\,, \quad (5.19)$$

where we used

$$(\lambda^b)_{aa'}(\lambda^b)_{bb'}=2\delta_{ab'}\delta_{ba'}-\frac{2}{n}\delta_{aa'}\delta_{bb'}\,. \quad (5.20)$$

From (5.18) and (5.19) we get

$$\frac{1+\delta}{4\pi\tilde{C}_1}\left[\frac{1}{1-\xi}H_2-nH_1\right]=\xi\partial_\xi H_1\,,$$
$$\frac{1+\delta}{4\pi\tilde{C}_1}\left[-\frac{1}{1-\xi}nH_2+H_1\right]=\xi\partial_\xi H_2\,, \quad (5.21)$$

where we used

$$\frac{[i(u_1-u_{1'})+\epsilon][i(u_{2'}-u_2)+\epsilon]}{[i(u_{1'}-u_2)+\epsilon][i(u_1-u_{2'})+\epsilon]}=1-\xi\,. \quad (5.22)$$

Similarly, from Eq. (4.9) for $\partial_v\psi_1$,

$$\frac{1-\delta}{4\pi\tilde{C}_1}\left[\frac{1}{1-\eta}H_2-nH_1\right]=\eta\partial_\eta H_1\,,$$
$$\frac{1-\delta}{4\pi\tilde{C}}\left[-\frac{1}{1-\eta}nH_2+H_1\right]=\eta\partial_\eta H_2\,. \quad (5.23)$$

Note that for $\delta=1$ the functions H_1, H_2 depend on ξ only, and for $\delta=-1$ on η only. We also have

$$H_1(1)=1,\quad H_2(1)=0\,,$$
$$H_1(\infty)=0,\quad H_2(\infty)=1\,. \quad (5.24)$$

These follow from the fact that as $x_1\to x_1'$, $x_2\to x_2'$ the four-point function tends to $\delta_{aa'}\delta_{bb'}$ times a product of two-point functions in (x_1-x_1') and (x_2-x_2'), and as $x_1\to x_2'$, $x_2\to x_1'$ to $\delta_{ab'}\delta_{ba'}$ times a product of two-point functions in (x_1-x_2') and (x_2-x_1').

We thus have to solve

$$\frac{1}{2\pi\tilde{C}_1}\left[-nH_1(z)+\frac{1}{1-z}H_2(z)\right]=z\partial_z H_1(z)\,,$$
$$\frac{1}{2\pi\tilde{C}_1}\left[H_1(z)-\frac{1}{1-z}nH_2(z)\right]=z\partial_z H_2(z)\,, \quad (5.25)$$

with (5.24) as initial conditions. Note that Eqs. (5.25) are two first-order differential equations, and therefore two initial conditions fix the solution. However, Eqs. (5.24) constitute four initial conditions. Thus we have more conditions than needed, and this will turn out to be consistent only for one value of $\tilde{C}_1$.

Define

$$H=H_1+H_2\,. \quad (5.26)$$

From Eqs. (5.25) and conditions (5.24) we get

$$z(z-1)\partial_z^2H+\frac{1}{2\pi\tilde{C}_1}[(2\pi\tilde{C}_1+n)z-2n]\partial_z H-\frac{(n-1)}{z(2\pi\tilde{C}_1)^2}(n+1-2\pi\tilde{C}_1)H=0\,,$$
$$H(1)=H(\infty)=1\,. \quad (5.27)$$

Equation (5.27) has three regular singular points $z=0,1,\infty$. A solution regular at 1 and ∞ with an equal value must be a constant $H(z)\equiv1$.[25] (See Appendix C for details). For that to be a solution we must have (for $n\neq1$)

$$\tilde{C}_1=\frac{n+1}{2\pi}\,. \quad (5.28)$$

This implies, from Eq. (3.11), that $C_1=1/2\pi$, which is equal to the value for free fields. Thus for $\delta=1$, Eq. (5.6), with the condition $M^{ab}(uu)=\delta^{ab}$, is identical to the free-field case, with $j_+^b(u)$ obeying free-field commutation rules. Thus $M^{ab}(u,u')$ is identical to the operator in the case of free fields. For $\delta=-1$, we have $M^{ab}(v,v')$ identical to the former case, with $j_-^b(v)$ replacing $j_+^b(u)$ of the former case.

As is evident from Eq. (4.11), the case $\delta=1$ corresponds to $g_v=0$, and $\delta=-1$ to

$$g_v=\frac{4\pi}{n+1}\,. \quad (5.29)$$

We thus found a scale-invariant theory at a nonvanishing value of the coupling constant.

Adding Eqs. (5.25) we get

$$H_1+\frac{1}{1-z}H_2=0\,,$$

which together with $H_1+H_2=1$ gives

$$H_1(z)=\frac{1}{z}\,,\quad H_2(z)=1-\frac{1}{z}\,, \quad (5.30)$$

which completes the solution for the four-point function.

Let us now introduce a free spinor field $\varphi_1(u)$, $\varphi_2(v)$ such that

$$\{\varphi_1(u), \varphi_1^\dagger(u')\}_+ = \delta(u-u') ,$$

$$[j_+(u), \varphi_1(u')] = -2\left(\frac{\pi C_0}{n}\right)^{1/2} \varphi_1(u')\,\delta(u-u'), \quad (5.31)$$

$$[j_+^a(u), \varphi_1(u')] = -\lambda^a \varphi_1(u')\,\delta(u-u') ,$$

$$\{\varphi_2(v), \varphi_2^\dagger(v')\}_+ = \delta(v-v') ,$$

$$[j_-(v), \varphi_2(v')] = -2\left(\frac{\pi C_0}{n}\right)^{1/2} \varphi_2(v')\,\delta(v-v'), \quad (5.32)$$

$$[j_-^a(v), \varphi_2(v')] = -\lambda^a \varphi_2(v')\,\delta(v-v') .$$

Note that a natural choice for C_0 would be n/π, the free-field value, which would then yield $j_+(u)=2:\varphi_1^\dagger(u)\,\varphi_1(u):$, $j_-(v)=2:\varphi_2^\dagger(v)\,\varphi_2(v):$ (this time the ordering is with respect to the Fermi creation and annihilation operators). We define the commutation rules (5.31) and (5.32) for a normalization of $j_\pm(x)$ given by C_0. In such a case, for a free field $\varphi(x)$ we have

$$a = \bar{a} = \left(\frac{\pi C_0}{n}\right)^{1/2} ,$$

as is evident from Eq. (4.5) for $\delta = 1$ and $2\pi\tilde{C}_1 = n+1$ (with $s=\frac{1}{2}$, of course).

Thus, from Eq. (5.1) and the fact that M is the same operator as in the free case, we get

$$\varphi_1(u)\,\varphi_1^\dagger(u') = \frac{1}{2\pi}\,\frac{1}{i(u-u')+\epsilon} : \exp\left[-i\left(\frac{\pi}{nC_0}\right)^{1/2}\int_{u'}^{u} j_+(u'')\,du''\right] : M(u,u') , \quad (5.33)$$

$$\varphi_2(v)\,\varphi_2^\dagger(v') = \frac{1}{2\pi}\,\frac{1}{i(v-v')+\epsilon} : \exp\left[-i\left(\frac{\pi}{nC_0}\right)^{1/2}\int_{v'}^{v} j_-(v'')\,dv''\right] : M(v,v') . \quad (5.34)$$

The coefficient $1/2\pi$ comes from our normalization of the field φ through the canonical anticommutation relations.

Now, since M commutes with $j_\pm$, we have

$$M(u,u') = (2\pi)\,[i(u-u')+\epsilon]\exp i\left[\left(\frac{\pi}{nC_0}\right)^{1/2}\int_{u'}^{u} j_+^{(+)}(u'')\,du''\right]\varphi_1(u)\,\varphi_1^\dagger(u')\exp i\left[\left(\frac{\pi}{nC_0}\right)^{1/2}\int_{u'}^{u} j_+^{(-)}(u'')\,du''\right] , \quad (5.35)$$

$$M(v,v') = (2\pi)\,[i(v-v')+\epsilon]\exp i\left[\left(\frac{\pi}{nC_0}\right)^{1/2}\int_{v'}^{v} j_-^{(+)}(v'')\,dv''\right]\varphi_2(v)\,\varphi_2^\dagger(v')\exp i\left[\left(\frac{\pi}{nC_0}\right)^{1/2}\int_{v'}^{v} j_-^{(-)}(v'')\,dv''\right] . \quad (5.36)$$

In Appendix D we show that M can be expressed in terms of currents only, by solving Eqs. (5.6) and (5.7) as a power series in $(u'-u)$. For the products of Fermi fields we now have

$$\begin{aligned}\psi_1(uv)\,\psi_1^\dagger(u'v') = {}& f_0[i(u-u')+\epsilon]^{-(d+s)}[i(v-v')+\epsilon]^{-(d-s)}\\ &\times\exp\left\{-\frac{i}{2C_0}\left[(a+\bar{a})\int_{u'}^{u} j_+^{(+)}(u'')\,du''+(a-\bar{a})\int_{v'}^{v} j_-^{(+)}(v'')\,dv''\right]\right\}\left\{\frac{1+\delta}{2}M(u,u')+\frac{1-\delta}{2}M(v,v')\right\}\\ &\times\exp\left\{-\frac{i}{2C_0}\left[(a+\bar{a})\int_{u'}^{u} j_+^{(-)}(u'')\,du''+(a-\bar{a})\int_{v'}^{v} j_-^{(-)}(v'')\,dv''\right]\right\}\end{aligned} \quad (5.37)$$

and [by $u,u' \to v,v'$ and $j_+(u) \to j_-(v)$, $j_+^a(u) \to j_-^a(v)$]

$$\begin{aligned}\psi_2(uv)\,\psi_2^\dagger(u'v') = {}& f_0[i(u-u')+\epsilon]^{-(d-s)}[i(v-v')+\epsilon]^{-(d+s)}\\ &\times\exp\left\{-\frac{i}{2C_0}\left[(a-a)\int_{u'}^{u} j_+^{(+)}(u'')\,du''+(a+\bar{a})\int_{v'}^{v} j_-^{(+)}(v'')\,dv''\right]\right\}\left[\frac{1+\delta}{2}M(v,v')+\frac{1-\delta}{2}M(u,u')\right]\\ &\times\exp\left\{-\frac{i}{2C_0}\left[(a-\bar{a})\int_{u'}^{u} j_+^{(-)}(u'')\,du''+(a+\bar{a})\int_{v'}^{v} j_-^{(-)}(v'')\,dv''\right]\right\} .\end{aligned} \quad (5.38)$$

We have thus expressed the products $\psi_1\psi_1^\dagger$ and $\psi_2\psi_2^\dagger$ in terms of free-field operators and c-number functions.

Note that in Eqs. (5.37) and (5.38) the right-hand sides are a product of a c-number singular function times an analytic bilocal operator. Here the light-cone expansion consists of one term only, as in the case of the usual Thirring model.[2]

VI. CONSTRUCTION OF THE FERMI FIELDS

We first show how to calculate Green's functions involving Fermi fields. In order to do that we need to know how to change orders of two ψ's or two $\psi^\dagger$'s or a ψ with a $\psi^\dagger$, so as to bring any vacuum expectation value to one where only products of $(\psi_1\psi_1^\dagger)$ and $(\psi_2\psi_2^\dagger)$ factors appear. The vacuum expectation value can then be calculated by using expressions (5.37) and (5.38) and the commutation rules of the currents and the fields φ, which are the same as in the noninteracting case.

We start with the product $\psi_1^\dagger(u'v')\psi_1(uv)$. It is a general consequence of locality[8] that the bilocal operator remains the same as in Eq. (5.37), and $(uv)\to(u'v')$ in the c-number part. Also, this result can be obtained by solving directly for $\psi_1^\dagger\psi_1$ as we did for $\psi_1\psi_1^\dagger$. We thus obtain

$$\psi_1^\dagger(u'v')\psi_1(uv)=\left[\frac{i(u-u')+\epsilon}{i(u'-u)+\epsilon}\right]^{(d+s)}\left[\frac{i(v-v')+\epsilon}{i(v'-v)+\epsilon}\right]^{(d-s)}\times\psi_1(uv)\psi_1^\dagger(u'v')\ , \tag{6.1}$$

and similarly

$$\psi_2^\dagger(u'v')\psi_2(uv)=\left[\frac{i(u-u')+\epsilon}{i(u'-u)+\epsilon}\right]^{(d-s)}\left[\frac{i(v-v')+\epsilon}{i(v'-v)+\epsilon}\right]^{(d+s)}\times\psi_2(uv)\psi_2^\dagger(u'v')\ . \tag{6.2}$$

In order to determine how to change orders of two ψ_1's or a ψ_1 with a ψ_2 or a $\psi_2^\dagger$, we use the method of Ref. 2, namely first compute how to change the order of a ψ_1 with $\psi_1\psi_1^\dagger$ and $\psi_2\psi_2^\dagger$, and deduce the laws for the former changes from the latter. To determine the latter we need to know how to change orders of ψ_1 with $M(u,u')$ and $M(v,v')$. We thus write, for the change of ψ_1 with $M(u,u')$,

$$\psi_1(uv)M(u',u'')=M(u',u'')\psi_1(uv)\,C(uv;u'u'')\ , \tag{6.3}$$

and then determine $C(uv;u'u'')$ from the equations of motion. From the commutation rules it follows that C commutes with all currents, and is thus a c number. It then follows from translation invariance that it depends only on $(u-u')$ and $(u-u'')$. From Lorentz invariance it follows that only ratios of u's appear, and combined with conformal invariance it follows that only

$$\frac{i(u-u')+\epsilon}{i(u'-u)+\epsilon}\,,\ \frac{i(u-u'')+\epsilon}{i(u''-u)+\epsilon}$$

appear. Using an equation of motion (for either M or ψ_1) then determines G to be powers of the above arguments.

Another way to compute those changes of orders is to express products of four spinor fields in terms of c-number singular functions and analytic operators of the four points involved. This can be done by starting from Eqs. (5.37) and (5.38), then moving all the factors that involve creation operators for the currents to the left, all factors that involve annihilation operators for the currents to the right, and then using Wick's theorem for the products of the φ fields to rewrite them in terms of c-number singular functions and normal-ordered products. (The latter normal-ordered products are with respect to the free fermion creation and annihilation operators, and for free fields the normal-ordered products are analytic.) From the singular c-number factors one can read off the rules for changing orders, as was done for the product of two fields to arrive at Eqs. (6.1) and (6.2).

This way we can compute all n-point functions. One then has to show that they satisfy the properties required for the reconstruction of the spinor field, namely Lorentz invariance, cluster decomposition, and positivity.[14] Lorentz invariance is obvious from our previous rules. As far as cluster decomposition is concerned, the decrease of Green's functions minus the appropriate products of vacuum expectation values is like a power of the large separation. This is so since all our Green's functions are powers—the theory contains no mass. The only mass parameter is the linear dimension of the coefficient f_0, $f_0\sim\mu^{-2(d-1/2)}$, needed to ensure that ψ has linear dimension $\frac{1}{2}$, but there is no mass as a pole in any Green's function.

Let us now illustrate how we prove positivity. For the two-point function $\langle 0|\psi_1(uv)\psi_1^\dagger(u'v')|0\rangle$ positivity means that

$$\int du\,du'dv'dv' f(uv)f^*(u'v')\times[i(u-u')+\epsilon]^{-(d+s)}[i(v-v')+\epsilon]^{-(d-s)}$$

is positive-definite. But since

$$\int_{-\infty}^{\infty}du\,e^{ip_+u}[iu+\epsilon]^{-\lambda}=\frac{2\pi}{\Gamma(\lambda)}\,\theta(p_+)(p_+)^{(\lambda-1)}\ , \tag{6.4}$$

The above expression becomes proportional to

$$\int_0^\infty dp_+\int_0^\infty dp_-|\tilde f(p_+p_-)|^2\,\frac{(p_+)^{d+s-1}}{\Gamma(d+s)}\,\frac{(p_-)^{d-s-1}}{\Gamma(d-s)}\ ,$$

which is positive-definite.

As for the four-point function, it is obvious from Eq. (5.37) that

$$\int f(u_1v_1u_1'v_1')f^*(u_2v_2u_2'v_2')\psi_1(u_1v_1)\psi_1^\dagger(u_1'v_1')\times\psi_1(u_2'v_2')\psi_1^\dagger(u_2v_2)$$

is positive-definite, since the right-hand side of Eq. (5.37) guarantees $\psi_1(u_2'v_2')\psi_1^\dagger(u_2v_2)=[\psi_1(u_2v_2)\times\psi_1^\dagger(u_2'v_2')]^+$. The problem is to show positivity of

$$\int f(u_1v_1u_1'v_1')f^*(u_2v_2u_2'v_2')\psi_1(u_1v_1)\psi_1(u_1'v_1')\times\psi_1^\dagger(u_2'v_2')\psi_1^\dagger(u_2v_2)\,.$$

One can show that directly by manipulating with the operators involved. Instead, we are now going to give an explicit operator expression for auxiliary operators, which tend to the Fermi fields in a certain limit. This construction, in fact, means that we did not need to prove all the properties above for the Green's functions, since we have an explicit reconstruction of the Fermi field.

Define, for $\delta=1$,

$$\begin{aligned}\psi_1(uv\,|\,u_0u_0'v_0v_0';\delta=1)=(2\pi f_0)^{1/2}&\exp\left[-\frac{i}{2C_0}(a-\bar a)\int_{v_0}^{v}j_-^{(+)}(v'')dv''\right]\exp\left[-\frac{i}{2C_0}(a-\bar a)\int_{v_0'}^{v}j_-^{(-)}(v'')dv''\right]\\&\times\left(\exp\left\{-\frac{i}{2C_0}\left[a+\bar a-2\left(\frac{\pi C_0}{n}\right)^{1/2}\right]\int_{u_0}^{u}j_+^{(+)}(u'')du''\right\}\right)\varphi_1(u)\\&\times\exp\left\{-\frac{i}{2C_0}\left[a+\bar a-2\left(\frac{\pi C_0}{n}\right)^{1/2}\right]\int_{u_0'}^{u}j_+^{(-)}(u'')du''\right\}\\&\times[i(v-v_0)+\epsilon]^{-(a-\bar a)^2/4\pi C_0}[i(u-u_0)+\epsilon]^{-(a+a')[a+\bar a-2(\pi C_0/n)^{1/2}]/4\pi C_0}\\&\times[i(v_0'-v_0)+\epsilon]^{(a-\bar a)^2/8\pi C_0}[i(u_0'-u_0)+\epsilon]^{[a+\bar a-2(\pi C_0/n)^{1/2}]^2/8\pi C_0}\,,\end{aligned}\tag{6.5}$$

and for $\delta=-1$,

$$\begin{aligned}\psi_1(uv\,|\,u_0u_0'v_0v_0';\delta=-1)=(2\pi f_0)^{1/2}&\exp\left[-\frac{i}{2C_0}(a+\bar a)\int_{u_0}^{u}j_+^{(+)}(u'')du''\right]\exp\left[-\frac{i}{2C_0}(a+\bar a)\int_{u_0'}^{u'}j_+^{(-)}(u'')du''\right]\\&\times\left(\exp\left\{-\frac{i}{2C_0}\left[a-\bar a-2\left(\frac{\pi C_0}{n}\right)^{1/2}\right]\int_{v_0}^{v}j_-^{(+)}(v'')dv''\right\}\right)\varphi_2(v)\\&\times\exp\left\{-\frac{i}{2C_0}\left[a-\bar a-2\left(\frac{\pi C_0}{n}\right)^{1/2}\right]\int_{v_0'}^{v}j_-^{(-)}(v'')dv''\right\}\\&\times[i(u-u_0)+\epsilon]^{-(a+\bar a)^2/4\pi C_0}[i(v-v_0)+\epsilon]^{-(a-\bar a)[a-\bar a-2(\pi C_0/n)^{1/2}]/4\pi C_0}\\&\times[i(u_0'-u_0)+\epsilon]^{(a+\bar a)^2/8\pi C_0}[i(v_0'-v_0)+\epsilon]^{[a-\bar a-2(\pi C_0/n)^{1/2}]^2/8\pi C_0}\,.\end{aligned}\tag{6.6}$$

We then obtain that

$$\psi_1(uv\,|\,u_0u_0'\,v_0v_0')[\psi_1(u'v'\,|\,u_0'u_0v_0'\,v_0)]^\dagger$$

reproduces the product as in Eq. (5.37), for any choice of $(u_0u_0'\,v_0v_0')$. To arrive at this result, the following formulas are useful:

$$\begin{aligned}&\exp\left[-i\lambda\int_{v_0'}^{v}j_-^{(-)}(v'')dv''\right]\exp\left[-i\mu\int_{v'}^{v_0}j_-^{(+)}(v'')dv''\right]\\&\quad=\exp\left[-i\mu\int_{v'}^{v_0}j_-^{(+)}(v'')dv''\right]\exp\left[-i\lambda\int_{v_0'}^{v}j^{(-)}(v'')dv''\right]\left\{\frac{[i(v-v_0)+\epsilon][i(v_0'-v')+\epsilon]}{[i(v-v')+\epsilon][i(v_0'-v_0)+\epsilon]}\right\}^{\lambda\mu C_0/\pi}\,,\end{aligned}\tag{6.7}$$

$$\varphi_1(u)\exp\left[-i\lambda\int_{u'}^{u_0}j_+^{(+)}(u'')du''\right]=\exp\left[-i\lambda\int_{u'}^{u_0}j_+^{(+)}(u'')du''\right]\varphi_1(u)\left[\frac{i(u-u_0)+\epsilon}{i(u-u')+\epsilon}\right]^{\lambda(C_0/n\pi)^{1/2}}\,,\tag{6.8}$$

and the analogous formulas for u and v interchange. Also, from Eqs. (4.5) and (4.6)

$$\frac{(a-\delta\bar a)^2}{4\pi C_0}=(d-\delta s)\,,\quad\frac{(a+\delta\bar a)^2}{4\pi C_0}+\frac{n-1}{n}=(d+\delta s)\,,\tag{6.9}$$

which are also needed.

One can define

$$\psi_1(uv\,|\,f)=\int du_0du_0'\,dv_0dv_0'\,\psi_1(uv\,|\,u_0v_0u_0'v_0')f(u_0v_0';\,v_0v_0')$$

such that f is symmetric under $(u_0, v_0) \to (u_0', v_0')$, and $\int du_0\, du_0'\, dv_0\, dv_0' f(u_0 u_0'; v_0 v_0') = 1$. In this case $\psi_1(uv|f) \times \psi_1^\dagger(uv|f)$ tends to the expression on the right-hand side of (5.37) for f's, with support becoming narrower. We can imagine the support of f moving to infinity in its variables, thus giving, in the limit (for suitably chosen f), a local Fermi field. As for ψ_2, it is obtained from ψ_1 by $u \to v$, $j_+(u) \to j_-(v)$, and $\varphi_1(u) \to \varphi_2(v)$, as

$$\begin{aligned}
\psi_2(uv|\,u_0u_0'\,v_0v_0'\,;\,\delta=1) = (2\pi f_0)^{1/2} &\exp\left[-\frac{i}{2C_0}(a-\bar{a})\int_{u_0}^{u} j_+^{(+)}(u'')\,du''\right]\exp\left[-\frac{i}{2C_0}(a-\bar{a})\int_{u_0'}^{u} j_+^{(-)}(u'')\,du''\right] \\
&\times\left(\exp\left\{-\frac{i}{2C_0}\left[a+\bar{a}-2\left(\frac{\pi C_0}{n}\right)^{1/2}\right]\int_{v_0}^{v} j_-^{(+)}(v'')\,dv''\right\}\right)\varphi_2(v) \\
&\times\ \exp\left\{-\frac{i}{2C_0}\left[a+\bar{a}-2\left(\frac{\pi C_0}{n}\right)^{1/2}\right]\int_{v_0'}^{v} j_-^{(-)}(v'')\,dv''\right\} \\
&\times[i(u-u_0)+\epsilon]^{-(a-\bar{a})^2/4\pi C_0}[i(v-v_0)+\epsilon]^{-(a+\bar{a})[a+\bar{a}-2(\pi C_0/n)^{1/2}]/4\pi C_0} \\
&\times[i(u_0'-u_0)+\epsilon]^{(a-\bar{a})^2/8\pi C_0}[i(v_0'-v_0)+\epsilon]^{[a+a-2(\pi C_0)^{1/2}]^2/8\pi C_0}\,,
\end{aligned} \tag{6.10}$$

$$\begin{aligned}
\psi_2(uv|\,u_0u_0'\,v_0v_0'\,;\,\delta=-1) = (2\pi f_0)^{1/2} &\exp\left[-\frac{i}{2C_0}(a+\bar{a})\int_{v_0}^{v} j_-^{(+)}(v'')\right]\exp\left[-\frac{i}{2C_0}(a+\bar{a})\int_{v_0'}^{v'} j_-^{(-)}(v'')\,dv''\right] \\
&\times\left(\exp\left\{-\frac{i}{2C_0}\left[a-\bar{a}-2\left(\frac{\pi C_0}{n}\right)^{1/2}\right]\int_{u_0}^{u} j_+^{(+)}(u'')\,du''\right\}\right)\varphi_1(u) \\
&\times\ \exp\left\{-\frac{i}{2C_0}\left[a-\bar{a}-2\left(\frac{\pi C_0}{n}\right)^{1/2}\right]\int_{u_0'}^{u} j_+^{(-)}(u'')\,du''\right\} \\
&\times[i(v-v_0)+\epsilon]^{-(a+\bar{a})^2/4\pi C_0}[i(u-u_0)+\epsilon]^{-(a-\bar{a})[a-\bar{a}-2(\pi C_0/n)^{1/2}]/4\pi C_0} \\
&\times[i(v_0'-v_0)+\epsilon]^{(a+\bar{a})^2/8\pi C_0}[i(u_0'-u_0)+\epsilon]^{[a-\bar{a}-2(\pi C_0/n)^{1/2}]^2/8\pi C_0}\,.
\end{aligned} \tag{6.11}$$

Note that ψ_2 for $\delta = -1$ is obtained from ψ_1 of $\delta = 1$ by changing $\bar{a} \to -\bar{a}$. However, from Eq. (6.9) it follows that (by subtracting one from the other)

$$a\bar{a}/\pi C_0 = 2s - [(n-1)/n]\delta\,, \tag{6.12}$$

which means that for $s > 0$ we must have $(a\bar{a}) > 0$ for both $\delta = \pm 1$, and a change $\bar{a} \to -\bar{a}$ without changing a is not allowed. A change $\bar{a} \to -\bar{a}$ with $\delta \to -\delta$ entails also $s \to -s$. This in turn means that the solution for ψ_2 for $\delta = -1$ and $s = \frac{1}{2}$ is the same as for ψ_1 for $\delta = 1$ and $s = -\frac{1}{2}$. This does not mean, however, that the case $g_v = 4\pi/(n+1)$ and the case of $g_v = 0$ are the same, since we cannot possibly introduce a mass term for $s = -\frac{1}{2}$ (see Sec. IV). Thus the remarks in Ref. 10 regarding the equivalence of the two solutions are not correct. Also, starting from $g_v = 0$ and continuously varying it all the time with $s = \frac{1}{2}$ we should reach the other scale-invariant solution of the equations of motion at $g_v = 4\pi/(n+1)$, and we showed that this is the only other scale-invariant solution. For all other couplings scale invariance is broken. Also, by having $s = \frac{1}{2}$ everywhere, we can consider a theory with a mass term, and we show in the next section that for a certain range of parameters this mass term is "soft" (see the next section for more details).

It may appear useful, for the same purposes, to use Eq. (6.5) with arbitrary "a" and "$\bar{a}$" so as to include the case of $s = -\frac{1}{2}$ or ($s = \frac{1}{2}$, $\delta = -1$). However, this should be considered only as a mathematical device, in view of our discussion above.

Finally, we could have reconstructed the spinor field by the rigorous procedure of Ref. 2.

VII. COUPLINGS, RENORMALIZATION GROUP, AND MASS TERMS

We first construct the operator expression for $\psi_1\psi_1^\dagger\psi_2\psi_2^\dagger$. Using Eqs. (5.37) and (5.38) and commutation rules Eq. (6.7) together with the fact that the M's commute with the singlet current we get

$$\psi_{1a}(u_1v_1)\psi^\dagger_{1a'}(u_1'v_1')\psi_{2b}(u_2v_2)\psi^\dagger_{2b'}(u_2'v_2') = f_0{}^2[i(u_1-u_1')+\epsilon]^{-(d+s)}[i(v_1-v_1')+\epsilon]^{-(d-s)}$$
$$\times[i(u_2-u_2')+\epsilon]^{-(d-s)}[i(v_2-v_2')+\epsilon]^{-(d+s)}(\xi\eta)^{(a^2-\bar a^2)/4\pi C_0}$$
$$\times\!:\exp\left\{-\frac{i}{2C_0}\left[(a+\bar a)\int_{u_1'}^{u_1}j_+(u'')du''+(a-\bar a)\int_{u_2'}^{u_2}j_+(u'')du''\right.\right.$$
$$\left.\left.+(a-\bar a)\int_{v_1'}^{v_1}j_-(v'')dv''+(a+\bar a)\int_{v_2'}^{v_2}j_-(v'')dv''\right]\right\}$$
$$\times\left[\frac{1+\delta}{2}M_{aa'}(u_1u_1')M_{bb'}(v_2v_2')+\frac{1-\delta}{2}M_{aa'}(v_1v_1')M_{bb'}(u_2u_2')\right]. \tag{7.1}$$

To define the local operator corresponding to the regularized product $\psi_1^\dagger\psi_2$ we thus have to multiply the product by the $(a^2-\bar a^2)/2\pi C_0$ power of the point separation and then let the separation go to zero. Thus

$$\sigma_{12}=R[\psi_1^\dagger\psi_2]=\lim_{\substack{u\to u'\\ v\to v'}}\{[i(u'-u)+\epsilon][i(v'-v)+\epsilon]\}^{(a^2-\bar a^2)/4\pi C_0}\psi^\dagger_{1a}(u'v')\psi_{2a}(uv). \tag{7.2}$$

From here and Eq. (4.6),

$$d(\sigma_{12})=2d(\psi)+\frac{\bar a^2-a^2}{2\pi C_0}$$
$$=\frac{\bar a^2}{\pi C_0}+\frac{n-1}{n}$$
$$=\frac{\bar a}{a}\left(1-\delta\frac{n-1}{n}\right)+\frac{n-1}{n}. \tag{7.3}$$

Also, the operators (choosing $\gamma_0=\sigma_1$, $\gamma_5=\sigma_3$)

$$\sigma=R[\bar\psi\psi]=R[\psi_1^\dagger\psi_2]+R[\psi_2^\dagger\psi_1],$$
$$\pi=R[i\bar\psi\gamma_5\psi]=iR[\psi_2^\dagger\psi_1]-iR[\psi_1^\dagger\psi_2] \tag{7.4}$$

have the same dimensions as $d(\sigma_{12})$. We can express $\bar a/a$ in terms of $G=g_B(C_0/4\pi)^{1/2}$, δ, and n. Physical quantities will always depend on the combination G rather than g_B or C_0 separately, as is obvious from the equations of motion and already discussed in Ref. 2. Thus

$$d(\sigma)=d(\pi)=\left[\left(G^2+\left(1-\delta\frac{n-1}{n}\right)\right)^{1/2}-G\right]^2+\frac{n-1}{n},$$
$$G=g_B\left(\frac{C_0}{4\pi}\right)^{1/2}. \tag{7.5}$$

Before discussing "softness" of mass terms any further, we go on to discuss the question of couplings. Formally, from the Lagrangian,

$$L_I=-\tfrac12 g_B(\psi\gamma_\mu\psi)(\bar\psi\gamma^\mu\psi)$$
$$-\tfrac12 g_v(\bar\psi\gamma_\mu\tfrac12\lambda^a\psi)(\bar\psi\gamma^\mu\tfrac12\lambda^a\psi), \tag{7.6}$$

using Eq. (5.20), and

$$(\gamma_\mu)_{\alpha\alpha'}(\gamma^\mu)_{\beta\beta'}=\delta_{\alpha\beta'}\delta_{\beta\alpha'}-(\gamma_5)_{\alpha\beta'}(\gamma_5)_{\beta\alpha'}, \tag{7.7}$$

we get

$$L_I=-\tfrac12\left(g_B-\frac{1}{2n}g_v\right)(\bar\psi\gamma_\mu\psi)(\bar\psi\gamma^\mu\psi)$$
$$+\tfrac14 g_v[(\bar\psi\psi)(\bar\psi\psi)+(i\bar\psi\gamma_5\psi)(i\bar\psi\gamma_5\psi)]. \tag{7.8}$$

The change in sign of the second term is due to the fact that we commuted Fermi fields an odd number of times to arrive from Eq. (7.6) to Eq. (7.8). We can rewrite Eq. (7.8) also as

$$L_I=-2\left(g_B-\frac{1}{2n}g_v\right)(\psi_1^\dagger\psi_1)(\psi_2^\dagger\psi_2)+g_v(\psi_1^\dagger\psi_2)(\psi_2^\dagger\psi_1). \tag{7.9}$$

Now, from Eq. (7.1), we obtain

$$\langle 0|\psi_{1a}(u_1v_1)\psi^\dagger_{1a'}(u_1'v_1')\psi_{2b}(u_2v_2)\psi^\dagger_{2b'}(u_2'v_2')|0\rangle_C$$
$$=G_1(u_1-u_1',\,v_1-v_1')G_2(u_2-u_2',\,v_2-v_2')$$
$$\times[(\xi\eta)^{(a^2-\bar a^2)/4\pi C_0}-1]\delta_{aa'}\delta_{bb'}, \tag{7.10}$$

where $G_{1,2}$ are the two-point functions

$$\langle 0|\psi_{(1,2)a}(uv)\psi^\dagger_{(1,2)a'}(u'\,v')|0\rangle=G_{1,2}(u-u',\,v-v')\delta_{aa'}, \tag{7.11}$$

and the subscript C on the left-hand side of Eq. (7.10) means that the disconnected part has been subtracted. Now, comparing the SU(n) structure in Eq. (7.10) with the interactions in Eq. (7.9), we see that a structure such as $(\psi_1^\dagger\psi_2)(\psi_2^\dagger\psi_1)$ does not appear in the four-point function of Eq. (7.10). The latter structure corresponds to $\delta_{ab'}\delta_{ba'}$, while we have $\delta_{aa'}\delta_{bb'}$, corresponding to a $(\psi_1^\dagger\psi_1)(\psi_2^\dagger\psi_2)$ interaction only. Thus the case $\delta=-1$, corresponding to $g_v=4\pi/(n+1)$ in the equations of motion, still corresponds to $g_v=0$ from the point of view of defining the coupling via the four-point function of Eq. (7.10). Moreover, the case $\delta=-1$ and $g_B=0$ ($a=\bar a$) corresponds to a vanishing right-hand side for Eq. (7.10). However, not all connected four-point functions vanish in this case. For example, from Eq. (5.9), we get

$$\begin{aligned}&\langle 0|\psi_{1a}(u_1v_1)\psi^\dagger_{1a'}(u_1'v_1')\psi_{1b}(u_2v_2)\psi^\dagger_{1b'}(u_2'v_2')|0\rangle_C\\&\quad= G_1(u_1-u_1', v_1-v_1')G_1(u_2-u_2', v_2-v_2')\left\{\xi^{(d+s)}\eta^{(d-s)}\left[\frac{1+\delta}{2}\frac{1}{\xi}+\frac{1-\delta}{2}\frac{1}{\eta}\right]-1\right\}\delta_{aa'}\delta_{bb'}\\&\qquad+G_1(u_1-u_2', v_1-v_2')G_1(u_1'-u_2, v_1'-v_2)\left\{\left(1-\frac{1}{\xi}\right)^{-(d+s)}\left(1-\frac{1}{\eta}\right)^{-(d-s)}\left[\frac{1+\delta}{2}\left(1-\frac{1}{\xi}\right)+\frac{1-\delta}{2}\left(1-\frac{1}{\eta}\right)\right]-1\right\}\delta_{ab'}\delta_{ba'}.\end{aligned}\tag{7.12}$$

Since $\delta=-1$ and $a=\bar{a}$ correspond to $d-s=(n-1)/n$ and $d+s=a^2/\pi C_0$, we see that the right-hand side of Eq. (7.12) does not vanish in this case. Thus $g_B=0$ and $\delta=-1$ is not like the free case, even though the four-point function of Eq. (7.10) vanishes. Thus $\delta=-1$ cannot give the free case for any g_B (with $s=\frac{1}{2}$).

Note that the right-hand side of Eq. (7.10) also vanishes when $\delta=-1$ and $a=-\bar{a}$. We argued against this case since it implies a negative spin, which will not allow the introduction of a mass term. (From locality, $2s$ must be an integer, by arguments similar to those in Ref. 2. Also, $2s$ is odd for vanishing anticommutators of the Fermi fields at spacelike separations.) In any case, the choice of $a=-\bar{a}$ for $\delta=-1$, with $d=-s=\frac{1}{2}$, yields all connected four-point functions to be zero. This implies $a^2/\pi C_0=1/n$, and thus $g_B=2(\pi/C_0n)^{1/2}$ and $g_v=4\pi/(n+1)$. Thus a massless spin $(-\frac{1}{2})$ Fermi field with the above values for the coupling constants in the equations of motion has vanishing connected four-point functions.

Since, as explained above, the meaning of the coupling g_v for $\delta=-1$ is not clear, we are going to discuss the Callan-Symanzik[11] functions β and the "softness" of the mass term in the vicinity of the solution for $\delta=1$, namely small g_v but arbitrary g_B. It turns out (see below) that in the vicinity of $g_v=0$ we have only one β which depends only on g_v and which is negative. Our solution for $g_v=0$ and a given g_B is therefore relevant to the asymptotic limit of a theory with a nonvanishing $g_v>0$ and the same g_B. The asymptotic limit is that where all momenta tend to large values with no combination squared being finite. In this limit all Green's functions tend, up to logarithmic modifications,[26] to the $g_v=0$ theory with the same g_B. When a mass term is introduced, the above statements remain true provided the mass term is soft, which in our case means $d(\sigma)<2$. From Eq. (7.5) we get that this is so when (we deal with $\delta=1$)

$$\begin{aligned}&d(\sigma)<2\Rightarrow G>-\frac{1}{2[1+1/n)^{1/2}]}\quad(\delta=1),\\&g_B>-\left[\frac{\pi}{C_0(1+1/n)}\right]^{1/2}.\end{aligned}\tag{7.13}$$

We thus have a domain in g_B where the $g_v=0$ solution is relevant to the asymptotic limit.

We now go to the calculation of the functions β for small g_v. Our interaction Lagrangian is therefore

$$\begin{aligned}L_I^v&=-\tfrac{1}{2}g_v(\bar\psi\gamma_\mu\tfrac{1}{2}\lambda^a\psi)(\bar\psi\gamma^\mu\tfrac{1}{2}\lambda^a\psi)\\&=-\tfrac{1}{2}g_vJ_+^a(u)J_-^a(v).\end{aligned}\tag{7.14}$$

Now, from Eq. (7.1) and the known commutation relations of the currents and the fields

$$\begin{aligned}&[\langle 0|T\{J_+^c(u)J_-^c(v)\psi_{1a}(u_1v_1)\psi^\dagger_{1a'}(u_1'v_1')\psi_{2b}(u_2v_2)\psi^\dagger_{2b'}(u_2'v_2')\}|0\rangle]_C\\&\quad=G_1(u_1-u_1', v_1-v_1')G_2(u_2-u_2', v_2-v_2')(\xi\eta)^{(a^2-\bar a^2)/4\pi C_0}\\&\qquad\times\left\langle 0\left|T\left\{J_+^c(u)J_-^c(v)\left[G_0^{-1}(u_1-u_1')G_0^{-1}(v_2-v_2')\frac{1+\delta}{2}\varphi_{1a}(u_1)\varphi^\dagger_{1a'}(u_1')\varphi_{2b}(v_2)\varphi^\dagger_{2b'}(v_2')\right.\right.\right.\right.\\&\qquad\left.\left.\left.\left.+G_0^{-1}(v_1-v_1')G_0^{-1}(u_2-u_2')\frac{1-\delta}{2}\varphi_{2a}(v_1)\varphi^\dagger_{2a'}(v_1')\varphi_{1b}(u_2)\varphi^\dagger_{1b'}(u_2')\right]\right\}\right|0\right\rangle\\&\qquad-G_1G_2\langle 0|T\{\ \}|0\rangle_{\rm Disc}\,,\end{aligned}\tag{7.15}$$

where

$$G_0(x)=\frac{1}{2\pi}\frac{1}{ix+\epsilon},\tag{7.16}$$

and we assumed, for simplicity, that the time sequence is $t_1>t_1'>t_2>t_2'$. The subscript "Disc" in the second term in Eq. (7.15) means that only the disconnected part should be considered in that term, and the arguments of G_1, G_2 and in the time-ordered product are the same as in the first term. We will take the $\delta=1$ case only, as discussed before.

Now,

$$i\int L_I^v d^2x = -\frac{i}{4}g_v\int du\,dv J_+^a(u)J_-^a(v), \tag{7.17}$$

so that $-\frac{1}{4}ig_v\int du\,dv$ of Eq. (7.15) gives the first-order correction due to the g_v coupling. Also,

$$\langle 0|\,T\{J_+^a(u)J_-^a(v)\psi_{1a}(u_1v_1)\psi_{1a'}^\dagger(u_1'v')\}\,|0\rangle=0. \tag{7.18}$$

Thus the propagator has no first-order corrections in g_v.

The Callan-Symanzik equation for a connected n-point function is[11]

$$\left[\mu\frac{\partial}{\partial\mu}+\beta(G,g_v)\frac{\partial}{\partial g_v}+\tilde\beta(G,g_v)\frac{\partial}{\partial G}+n\gamma(G,g_v)\right]\Gamma^{(n)}=0, \tag{7.19}$$

where γ is the anomalous dimension parameter for the spinor field. Also,

$$\mu\frac{\partial}{\partial\mu}f_0=(1-2d)f_0, \tag{7.20}$$

with d given in Eq. (4.6). It is thus obvious that to first order in g_v we have $\beta=\tilde\beta=0$ and $\gamma=d-\frac{1}{2}$. From $n=2$ in Eq. (7.19) it follows that γ and $\tilde\beta$ have no first-order contributions in g_v, and then the $n=4$ case [taking the combination $\delta_{ab'}\delta_{ba'}$ in Eq. (7.15), which is connected to the g_v coupling and does not appear in zeroth order] yields that also β is zero to first order in g_v.

We now go over to second order in g_v. Again, similar to arriving at Eq. (7.15), we get

$$\begin{aligned}
&[\langle 0|\,T\{J_+^c(u)J_-^c(v)J_+^{c'}(u')J_-^{c'}(v')\psi_{1a}(u_1v_1)\psi_{1a'}^\dagger(u_1'v_1')\psi_{2b}(u_2v_2)\psi_{2b'}^\dagger(u_2v_2)\}|0\rangle]_{C_1}\\
&\quad=G_1(u_1-u_1',\,v_1-v_1')G_2(u_2-u_2',\,v_2-v_2')(\xi\eta)^{(a^2-\bar a^2)/4\pi C_0}\\
&\quad\times\Big\langle 0\Big|\,T\Big\{J_+^c(u)J_-^c(v)J_+^{c'}(u')J_-^{c'}(v')\Big[G_0^{-1}(u_1-u_1')G_0^{-1}(v_2-v_2')\frac{1+\delta}{2}\varphi_{1a}(u_1)\varphi_{1a'}^\dagger(u_1')\varphi_{2b}(v_2)\varphi_{2b'}^\dagger(v_2')\\
&\qquad+G_0^{-1}(v_1-v_1')G_0^{-1}(u_2-u_2')\frac{1-\delta}{2}\varphi_{2a}(v_1)\varphi_{2a'}^\dagger(v_1')\varphi_{1b}(u_2)\varphi_{1b'}^\dagger(u_2')\Big]\Big\}\Big|\,0\Big\rangle_{C_1},
\end{aligned} \tag{7.21}$$

where the subscript C_1 means that only the part $\delta_{ab'}\delta_{ba'}$ is taken, which is relevant to g_v (this also excludes the disconnected part, which is proportional to $\delta_{aa'}\delta_{bb'}$). Now, when integrating $\int du\,dv\int du'\,dv'$, we are going to get a logarithmic divergence. Thus, as is well known, a subtraction is needed. We are going to use the Callan-Symanzik equation with $\Gamma^{(n)}$ in x space, since this is more convenient in our case. We also make the subtraction in x space in such a way that the second-order contribution to $\Gamma^{(4)}$ is (for $\delta=1$)

$$\begin{aligned}
[\Gamma^{(4)}]_{(\text{second order})C_1}&=-\tfrac{1}{16}g_v^2G_1(u_1-u_1',\,v_1-v_1')G_2(u_2-u_2',\,v_2-v_2')(\xi\eta)^{(a^2-\bar a^2)/4\pi C_0}G_0^{-1}(u_1-u_1')G_0^{-1}(v_2-v_2')\\
&\quad\times\int du\,dv\,du'\,dv'\,[\langle 0|\,T\{J_+^c(u)J_-^c(v)J_+^{c'}(u')J_-^{c'}(v')\,\varphi_{1a}(u_1)\varphi_{1a'}^\dagger(u_1')\varphi_{2b}(v_2)\varphi_{2b'}^\dagger(v_2')\}|0\rangle_{C_1}\\
&\quad-\langle 0|\,T\{J_+^c(u)J_-^c(v)J_+^{c'}(u')J_-^{c'}(v')\varphi_{1a}(\bar u_1)\varphi_{1a'}^\dagger(\bar u_1')\varphi_{2b}(\bar v_2)\varphi_{2b'}^\dagger(\bar v_2')\}|0\rangle_{C_1}],
\end{aligned} \tag{7.22}$$

where $(\bar u_1\bar u_1'\bar v_2\bar v_2')$ is a set with fixed differences $\propto 1/\mu$. Returning to Eq. (7.19) and using the $[\Gamma^{(4)}]_{C_1}$ to second order, we immediately see that $\tilde\beta=0$ and β has a second-order term in g_v which is the same as for the $G=0$ case. In order to get γ to second order in g_v we consider also $\Gamma^{(2)}$ to second order. By methods familiar by now we get

$$\begin{aligned}
[\Gamma_1^{(2)}]_{\text{second order}}&=-\tfrac{1}{16}g_v^2G_1(u_1-u_1',\,v_1-v_1')G_0^{-1}(u_1-u_1')\\
&\quad\times\int du\,dv\int du'\,dv'\langle 0|\,T\{J_+^c(u)J_-^c(v)J_+^{c'}(u')J_-^{c'}(v')[\varphi_1(u_1)\varphi_1^\dagger(u_1')-\varphi_1(\bar u_1)\varphi_1^\dagger(\bar u_1')]\}|0\rangle.
\end{aligned} \tag{7.23}$$

Taking $\Gamma^{(2)}$ to second order in Eq. (7.19) we now get that γ equals $(d-\frac{1}{2})$ + second-order contribution in g_v equal to the $G=0$ case.

Our results are that $\tilde{\beta}=0$ and β is a function of g_v only, equal to the one for $G=0$, and that γ is a sum of two terms, one dependent only on G and the same as for $g_v=0$ and the other only on g_v and the same as for $G=0$. These results are an immediate consequence from the fact that our $\Gamma^{(n)}$ have the structure of a product of two terms, one a factor dependent on G only and the same as for $g_v=0$ and the other dependent on g_v only and the same as for $G=0$. We demonstrated this to second order in g_v explicitly. By similar arguments one can show this to all orders in g_v, starting from Eq. (7.1) and using the fact that the perturbation is a product of the SU(n) currents.

To get β it is thus sufficient to consider the interaction in Eq. (7.8) with $g_B=0$. γ will then be given by the $g_B=0$ result plus a term which is G^2. Computing $\Gamma^{(4)}$ graphically it is obvious that to second order in g_v, β has a term linear in n and a term independent of n. The linear term in n is obtained from the contributions of one-fermion-loop diagrams. The first term in L_I of Eq. (7.8) does not contribute to β, since the one-fermion loop of a vector current is finite here (see, for example, Ref. 27). Thus only the scalar and pseudoscalar terms in Eq. (7.8) contribute to the one-loop divergence, each an equal amount. A straightforward calculation then gives $-g_v^2 n/2\pi$ for the term linear in n.[28] Since $\beta=0$ for the case of $n=1$ [because in this case we have the U(1) Thirring model],[18] we thus get[29]

$$\beta=-\frac{g_v^{\,2}}{2\pi}(n-1)+(\text{higher orders in } g_v). \qquad (7.24)$$

Thus the theory is asymptotically stable for $g_v>0$, namely the high-momentum limit is given, up to logarithmic modifications, by the $g_v=0$ case, provided that the mass term is soft, namely Eq. (7.13) holds.

Note that for $g_v<0$ it is the infrared limit of momenta tending to zero in which the $g_v=0$ theory is relevant. However, in this case the mass term can be ignored when $d(\sigma)>2$.

We would like to add that when discussing perturbation theory around the $\delta=-1$ solution, we get formally for β the same answer as in Eq. (7.24). However, as we discussed before, it is not clear that g_v in Eq. (7.24) can be viewed as the difference between the coupling constant for the coupling of the SU(n) currents and $4\pi/(n+1)$.

ACKNOWLEDGMENT

One of us (Y. F.) is happy to thank Dr. Carl Kaysen and the Institute for Advanced Study, Princeton, New Jersey, for their hospitality.

APPENDIX A

In this appendix we discuss the relation between our normal ordering (with respect to current quanta) and a limiting procedure, and also show that Eqs. (4.12) become identities for the case of free spinors.

Denote by J one of the currents j_μ or j_ν^a. Then

$$:J(x)\psi(x):=J^{(+)}(x)\psi(x)+\psi(x)J^{(-)}(x)\,. \qquad (A1)$$

Now [with $l=(l_0,l_1)$],

$$\begin{aligned}\tfrac{1}{2}[J(x+l)\psi(x)+\psi(x)J(x-l)] &= \tfrac{1}{2}[J^{(+)}(x+l)\psi(x)+\psi(x)J^{(-)}(x-l)]\\ &+\tfrac{1}{2}[J^{(-)}(x+l),\psi(x)]+\tfrac{1}{2}[\psi(x),J^{(+)}(x-l)]\\ &+\tfrac{1}{2}[\psi(x)J^{(-)}(x+l)+J^{(+)}(x-l)\psi(x)]\,.\end{aligned}$$

But

$$[J^{(+)}(x-l)-J^{(-)}(x+l),\psi(x)]=0\,, \qquad (A2)$$

as follows from the commutation rules of the currents with the fields, Eqs. (2.15) and (2.27), together with the fact that [Eq. (3.8)],

$$\delta^{(+)}(-l)=\delta^{(-)}(l)\,.$$

Thus

$$\begin{aligned}\tfrac{1}{2}[J(x+l)\psi(x)+\psi(x)J(x-l)] &= \tfrac{1}{2}[J^{(+)}(x+l)+J^{(+)}(x-l)]\psi(x)\\ &+\tfrac{1}{2}\psi(x)[J^{(-)}(x-l)+J^{(-)}(x-l)]\,.\end{aligned}$$

From this we get the connection between the normal ordering (with respect to current quanta) and the limiting procedure,

$$:J(x)\psi(x):=\lim\tfrac{1}{2}[J(x+l)\psi(x)+\psi(x)J(x-l)]\,. \qquad (A3)$$

For the case of the U(1) current this was stated in Ref. 2 (footnote 17 there).

We now want to show that Eqs. (4.12) become identities for the case of free spinor fields. From Eqs. (4.9) with $g_B=0$, $g_v=0$ it follows then that ψ_1 depends on u only and ψ_2 on v only. The first of Eqs. (4.12) then reads

$$\frac{i}{\pi}\partial_u\psi_1(u)=\frac{1}{n+1}\lambda^b:j_+^b(u)\psi_1(u):+\frac{1}{n}:j_+(u)\psi_1(u):\,.$$

The proof of the second of Eqs. (4.12) will be identical to the proof of this one.

Using (A3), the right-hand side of the last equation is the limit of $k\to 0$ of

$$\frac{1}{n+1}\,\tfrac{1}{2}\lambda^b[j_+^b(u+k)\psi_1(u)+\psi_1(u)j_+^b(u-k)]+\frac{1}{2n}\,[j_+(u+k)\psi_1(u)+\psi_1(u)j_+(j-k)]\,.$$

With free Fermi fields,

$$j_+(u)=2N[\psi_1^\dagger(u)\psi_1(u)]\,,\quad j_+^a(u)=2N[\psi_1^\dagger(u)\tfrac{1}{2}\lambda^a\psi_1(u)]\,, \tag{A4}$$

where $N[\;]$ here denotes normal ordering with respect to the fermion creation and annihilation operators. Now, using Wick's theorem [with Γ a matrix in SU(n) indices],

$$N[\psi_1^\dagger(u+k)\Gamma\psi_1(u+k)]\psi_1(u)+\psi_1(u)N[\psi_1^\dagger(u-k)\Gamma\psi_1(u-k)]$$
$$=N[(\psi_1^\dagger(u+k)\Gamma\psi_1(u+k))\psi_1(u)+(\psi_1^\dagger(u-k)\Gamma\psi_1(u-k))\psi_1(u)]+\frac{i}{2\pi}\frac{1}{k-i\epsilon}[\Gamma\psi_1(u+k)-\Gamma\psi_1(u-k)]\,.$$

We also used

$$\langle 0|\psi_{1a}^\dagger(u)\psi_{1b}(0)|0\rangle=\langle 0|\psi_{1a}(u)\psi_{1b}^\dagger(0)|0\rangle=\frac{1}{2\pi}\frac{1}{iu+\epsilon}\delta_{ab}\,. \tag{A5}$$

Thus,

$$\frac{1}{n+1}\lambda^b:j_+^b(u)\psi_1(u):+\frac{1}{n}:j_+(u)\psi_1(u):=\frac{1}{n+1}N\{[\psi_1^\dagger(u)\lambda^b\psi_1(u)](\lambda^b\psi_1(u))\}+\frac{2}{n}N\{[\psi_1^\dagger(u)\psi_1(u)]\psi_1(u)\}+\frac{i}{\pi}\partial_u\psi_1(u)\,, \tag{A6}$$

where we used the fact that normal-ordered products of free fields at the same point are regular. Now, using (5.20),

$$N\{[\psi_1^\dagger(u)\lambda^b\psi_1(u)]\lambda^b\psi_1(u)\}=-2N\{[\psi_1^\dagger(u)\psi_1(u)]\psi_1(u)\}-\frac{2}{n}N\{[\psi_1^\dagger(u)\psi_1(u)]\psi_1(u)\}$$
$$=-2\frac{n+1}{n}N\{[\psi_1^\dagger(u)\psi_1(u)]\psi_1(n)\}\,. \tag{A7}$$

Thus the sum of normal-ordered products in Eq. (A6) is zero, and we proved that

$$\frac{1}{n+1}\lambda^b:j_+^b(u)\psi_1(u):+\frac{1}{n}:j_+(u)\psi_1(u):=\frac{i}{\pi}\partial_u\psi_1(u)\,, \tag{A8}$$

which is what we wanted.

APPENDIX B

We want to prove Eqs. (3.10) and (3.12). Now, using the commutation rules Eq. (2.24),

$$\begin{aligned}
[:[j_+^a(u)]^2:,j_+^b(u')]&=[[j_+^{a(+)}(u)]^2+2j_+^{a(+)}(u)j_+^{a(-)}(u)+[j_-^{a(-)}(u)]^2,j_+^b(u')]\\
&=2if^{abc}\delta^{(+)}(u-u')[j_+^c(u')j_+^{a(+)}(u)+j_+^{a(+)}(u)j_+^c(u')]\\
&\quad+4if^{abc}[\delta^{(+)}(u-u')j_+^c(u')j_+^{a(-)}(u)+\delta^{(-)}(u-u')j_+^{a(+)}(u)j_+^c(u')]\\
&\quad+2if^{abc}\delta^{(-)}(u-u')[j_+^c(u')j_-^{a(-)}(u)+j_-^{a(-)}(u)j_+^c(u')]\\
&\quad+2iC_1\delta^{ab}[2\delta'^{(+)}(u-u')j_+^{a(+)}(u)+2\delta'^{(+)}(u-u')j_-^{a(-)}(u)\\
&\qquad+2\delta'^{(-)}(u-u')j_+^{a(+)}(u)+2\delta'^{(-)}(u-u')j_-^{a(-)}(u)]\\
&=4if^{abc}\delta^{(+)}(u-u')j_+^{a(+)}(u)j_+^c(u')-2if^{abc}[\delta^{(+)}(u-u')]^2\,2if^{acd}j_+^d(u')\\
&\quad+4if^{abc}\delta^{(+)}(u-u')j_+^c(u')j_+^{a(-)}(u)+4if^{abc}\delta^{(-)}(u-u')j_+^{a(+)}(u)j_+^c(u')\\
&\quad+4if^{abc}\delta^{(-)}(u-u')j_+^c(u')j_-^{a(-)}(u)+2if^{abc}[\delta^{(-)}(u-u')]^2\,2if^{acd}j_+^d(u')+4iC_1\delta'(u-u')j_+^b(u)\\
&=4if^{abc}\delta(u-u')j_+^{a(+)}(u)j_+^c(u)+4if^{abc}\delta(u-u')j_+^c(u)j_+^{a(-)}(u)\\
&\quad+4f^{acb}f^{acd}\{[\delta^{(-)}(u-u')]^2-[\delta^{(+)}(u-u')]^2\}j_+^d(u')+4iC_1\delta'(u-u')j_+^b(u)\,.
\end{aligned}$$

Now,

$$f^{acb}f^{acd}=n\delta^{bd}\,, \tag{B1}$$

and using also Eq. (3.8) we get

$$[:[j_+^a(u)]^2:, j_+^b(u')] = 2if^{abc}\delta(u-u')\{[j_+^{a(+)}(u), j_+^{c(+)}(u)] - [j_+^{a(-)}(u), j_+^{c(-)}(u)]\}$$
$$+ i\frac{2n}{\pi}\delta'(u-u')j_+^b(u') + 4iC_1\delta^{ab}\delta'(u-u')j_+^a(u).$$

Let us now calculate the commutators on the right-hand side. From Eqs. (2.24)

$$[j_+^{a(+)}(u), j_+^b(u')] = 2if^{abc}j_+^c(u')\delta^{(+)}(u-u') + 2iC_1\delta^{ab}\delta'^{(+)}(u-u').$$

Now, the positive-frequency part of an operator $A(u)$ is given by

$$A^{(+)}(u) = \frac{1}{2\pi}\int_0^\infty dp\, e^{ipu}\int_{-\infty}^\infty d\bar{u}\, e^{-ip\bar{u}}A(\bar{u})$$
$$= \int_{-\infty}^\infty d\bar{u}\,\delta^{(+)}(u-\bar{u})A(\bar{u}). \tag{B2}$$

Thus

$$[j_+^{a(+)}(u), j_+^{b(+)}(u')] = \int_{-\infty}^\infty d\bar{u}'\,\delta^{(+)}(u'-\bar{u}')[j_+^{a(+)}(u), j_+^b(\bar{u}')]$$
$$= 2if^{abc}\int_{-\infty}^\infty d\bar{u}'\,\delta^{(+)}(u-\bar{u}')\delta^{(+)}(u'-\bar{u}')j_+^c(\bar{u}') + 2iC_1\delta^{ab}\int_{-\infty}^\infty d\bar{u}'\,\delta'^{(+)}(u-\bar{u}')\delta^{(+)}(u'-\bar{u}').$$

Using Eq. (3.8) we get that the second term is zero and that

$$\delta^{(+)}(u-\bar{u}')\delta^{(+)}(u'-\bar{u}') = \frac{1}{2\pi i}\frac{1}{u-u'}[\delta^{(+)}(u-\bar{u}') - \delta^{(+)}(u'-\bar{u}')]. \tag{B3}$$

Thus

$$[j_+^{a(+)}(u), j_+^{b(+)}(u')] = \frac{1}{\pi}f^{abc}\frac{1}{u-u'}[j_+^{c(+)}(u) - j_+^{c(+)}(u')], \tag{B4}$$

and hence

$$[j_+^{a(+)}(u), j_+^{b(+)}(u)] = \frac{1}{\pi}f^{abc}\frac{\partial}{\partial u}j_+^{c(+)}(u), \tag{B5}$$

and similarly [or by taking the Hermitian conjugate of (B5)],

$$[j_+^{a(-)}(u), j_+^{b(-)}(u)] = -\frac{1}{\pi}f^{abc}\frac{\partial}{\partial u}j_+^{c(-)}(u). \tag{B6}$$

Thus

$$[\theta_+^v(u), j_+^b(u')] = \frac{1}{2\bar{C}_1}\left\{2if^{abc}\delta(u-u')\frac{1}{\pi}f^{acd}\frac{\partial}{\partial u}j_+^d(u) + i\frac{2n}{\pi}\delta'(u-u')j_+^b(u') + 4iC_1\delta'(u-u')j_+^b(u)\right\}$$
$$= \frac{i}{\bar{C}_1}\left\{\frac{n}{\pi}\left[\delta'(u-u')j_+^b(u') - \delta(u-u')\frac{\partial}{\partial u}j_+^b(u)\right] + 2C_1\delta'(u-u')j_+^b(u)\right\}.$$

Hence

$$[\theta_+^v(u), j_+^b(u')] = \frac{2i}{\bar{C}_1}\left(\frac{n}{2\pi} + C_1\right)\delta'(u-u')j_+^b(u). \tag{B7}$$

We thus proved Eq. (3.10). As for Eq. (3.12), we proceed from Eq. (B7),

$$[\theta_+^v(u), \theta_+^v(u')]$$
$$= \frac{-i}{\bar{C}_1^{\,2}}\left(\frac{n}{2\pi} + C_1\right)[\delta'^{(+)}(u'-u)j_+^{b(+)}(u')j_+^b(u) + \delta'^{(+)}(u'-u)j_+^b(u)j_+^{b(+)}(u') + 2\delta'^{(+)}(u'-u)j_+^b(u)j_+^{b(-)}(u')$$
$$+ 2\delta'^{(-)}(u'-u)j_+^{b(+)}(u')j_+^b(u) + \delta'^{(-)}(u'-u)j_+^{b(-)}(u')j_+^b(u) + \delta'^{(-)}(u'-u)j_+^b(u)j_+^{b(-)}(u')]$$
$$= -\frac{i}{\bar{C}_1^{\,2}}\left(\frac{n}{2\pi} + C_1\right)\{2\delta'(u'-u):j_+^b(u')j_+^b(u): + \delta'^{(+)}(u'-u)[j_+^b(u), j_+^{b(+)}(u')] + \delta'^{(-)}(u'-u)[j_+^{b(-)}(u'), j_+^b(u)]\}$$
$$= -\frac{i}{\bar{C}_1^{\,2}}\left(\frac{n}{2\pi} + C_1\right)\{\delta'(u'-u)[:j_+^b(u)j_+^b(u): + :j_+^b(u')j_+^b(u'):] + 2iC_1(n^2-1)[(\delta'^{(-)}(u'-u))^2 - (\delta'^{(+)}(u'-u))^2]\}.$$

Thus [with Eq. (3.9)]

$$[\theta_+^v(u),\theta_+^v(u')]=\frac{2i}{\tilde{C}_1}\left(\frac{n}{2\pi}+C_1\right)[\theta_+^v(u)+\theta_+^v(u')]\delta'(u-u')-\frac{i}{6\pi}\frac{C_1}{\tilde{C}_1^2}(n^2-1)\left(\frac{n}{2\pi}+C_1\right)\delta'''(u-u').$$

With $\tilde{C}_1=n/2\pi+C_1$ [Eq. (3.11)], we finally get

$$[\theta_+^v(u),\theta_+^v(u')]=2i[\theta_+^v(u)+\theta_+^v(u')]\delta'(u-u')-\frac{i}{6\pi}\frac{(n^2-1)(2\pi C_1)}{n+2\pi C_1}\delta'''(u-u'),\tag{B8}$$

which is Eq. (3.12).

APPENDIX C

We want to solve Eq. (5.27),

$$z(z-1)\partial_z{}^2H+\frac{1}{2\pi\tilde{C}_1}[(2\pi\tilde{C}_1+n)z-2n]\partial_zH-\frac{(n-1)}{z(2\pi\tilde{C}_1)^2}(n+1-2\pi\tilde{C}_1)H=0,\tag{C1}$$

with $H(1)=H(\infty)=1$. We first transform the point $z=\infty$ to $z=0$ by going over to

$$\tilde{H}(z)=H\left(\frac{1}{z}\right).\tag{C2}$$

Thus, the equation for $\tilde{H}$ is

$$z(1-z)\partial_z{}^2\tilde{H}+\left(1-\frac{n}{2\pi\tilde{C}_1}\right)(1-2z)\partial_z\tilde{H}-\frac{(n-1)}{(2\pi\tilde{C}_1)^2}(n+1-2\pi\tilde{C}_1)\tilde{H}=0,\tag{C3}$$

$\tilde{H}(0)=\tilde{H}(1)=1$.

Let us rewrite Eq. (C3) as[25]

$$z(1-z)\partial_z{}^2\tilde{H}+[\gamma-(\alpha+\beta+1)z]\partial_z\tilde{H}-\alpha\beta\tilde{H}=0.\tag{C4}$$

Then

$$\alpha=1-\frac{(n+1)}{2\pi\tilde{C}_1},\quad \beta=-\frac{(n-1)}{2\pi\tilde{C}_1},\quad \gamma=1-\frac{n}{2\pi\tilde{C}_1}.\tag{C5}$$

From Eq. (3.11), $2\pi\tilde{C}_1=n+2\pi C_1>n$, it follows that $\gamma>0$, $0>\beta>-1$, $1>\alpha>-1$. The general solution of Eq. (C4) is[25]

$$\tilde{H}(z)=AF(\alpha,\beta;\gamma;z)+Bz^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1;2-\gamma;z),\tag{C6}$$

with

$$F(\alpha,\beta;\gamma;x)=1+\frac{\alpha\beta}{1!\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\gamma(\gamma+1)}x^2+\cdots\tag{C7}$$

From Eq. (5.9) we see that the functions $H_1(z)$ and $H_2(z)$ should be real in the vicinity of the points 1 and ∞, since we can approach these points by $x_1'\to x_1$ and $x_2'\to x_1$ with all separations spacelike on the left-hand side of Eq. (5.9). Thus we must have $B=0$ in Eq. (C6). Also, in order not to have a cut near $z=1$ coming from the first term in Eq. (C6), the series in Eq. (C7) must terminate. Since $0>\alpha>-1$, the only way that the series can terminate is for $\alpha=0$. Then, with $A=1$, we have $H(z)\equiv1$. Thus only $2\pi\tilde{C}_1=n+1$ is possible.

APPENDIX D

We want to show here that M can be expressed in terms of SU(n) currents only. We start from the differential equations (5.6) and (5.7), which have the form

$$i\partial_uM(u,u')=\frac{1}{\tilde{C}_1}\left\{\tfrac{1}{2}\lambda^b:j_+^b(u)M(u,u'):+\frac{1}{4\pi}\frac{1}{i(u-u')+\epsilon}\left[\lambda^bM(uu')\lambda^b-\frac{2(n^2-1)}{n}M(uu')\right]\right\}.\tag{D1}$$

Equation (D1) is for the case of $\delta=1$. For $\delta=-1$ we just change $(u,u')\to(v,v')$ and $j_+(u)\to j_-(v)$. Also,

$$M^{ab}(u,u)=\delta^{ab}.\tag{D2}$$

Let us now write

$$M(u,u')=I+\sum_{r=1}^{\infty}(u-u')^rM_r(u).\tag{D3}$$

Thus

$$(r+1)M_{r+1}(u) + \frac{i}{4\pi\bar{C}_1}\left[\lambda^b M_{r+1}(u)\lambda^b - \frac{2(n^2-1)}{n}M_{r+1}(u)\right] = \frac{1}{2\bar{C}_1}\lambda^b : j_+^b(u)M_r(u): - i\partial_u M_r(u)\,, \quad \text{(D4)}$$

with $M_0(u)=I$. Equation (D4) enables us to get M_{r+1} in terms of M_r. Writing

$$M_r(u) = M_r^{(0)}(u)\,I + M_r^{(c)}(u)\lambda^c\,, \quad \text{(D5)}$$

we have

$$\lambda^b M_{r+1}(u)\lambda^b - \frac{2(n^2-1)}{n}M_{r+1}(u) = -2nM_{r+1}^{(c)}(u)\lambda^c\,, \quad \text{(D6)}$$

which is useful for the left-hand side of Eq. (D4) [we used $\lambda^b\lambda^a\lambda^b = -(2/n)\lambda^a$ to arrive at Eq. (D6)]. As for the right-hand side of Eq. (D4), we need

$$\lambda^b\lambda^c = \frac{2}{n}I + \sum_a B_a^{bc}\lambda^a\,, \quad \text{(D7)}$$

where

$$B_a^{bc} = \tfrac{1}{2}\,\mathrm{Tr}(\lambda^a\lambda^b\lambda^c)\,.$$

Using Eqs. (D6) and (D7) in Eq. (D4) we can solve for all M_r's recursively.

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†On leave from the Weizmann Institute, Rehovot, Israel.

*Research sponsored by the National Science Foundation under Grant No. NSF GP-40768X.

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