Quantum cosmology: Exact solution for the Gowdy T^3 model

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A complete set of orthonormal wave functions valid at all times including that of the classical singularity is obtained in closed form for each mode of the quantized Gowdy T^3 universe. These wave functions are superposed to yield that wave function for each mode which reduces to a given initial state near the classical singularity. The expected number of quanta in each mode at times far from the singularity is obtained and depends only on the constants which characterize the initial state. All expectation values agree with those obtained classically and semiclassically except for a smearing out of the essentially classical picture due to quantum fluctuations. The precise description of the initial state at the singularity yields a model with the size and shape parameter of the universe satisfying an initial free-particle–like equation and later captured in the N-quantum state of a rising harmonic-oscillator potential. This contrasts with the graviton creation from vacuum fluctuations description of an earlier treatment by Berger. Misner has shown that the Gowdy T^3 model universe may be described as a scattering process in minisuperspace. He obtains a Klein-Gordon equation for the wave function of the universe which is separable in Fourier components of the wave part of the gravitational field. It is shown here that exact solutions exist for the Klein-Gordon equation which reduces for each mode to the Schrödinger equation for a time-dependent-frequency harmonic oscillator. The methods of Salusti and Zirilli are used to obtain wave functions characterized by the quantum number N with harmonic-oscillator spatial (in superspace) dependence and time-dependent coefficients. These N-quantum wave functions are fixed uniquely by requiring agreement with the known large-time-limit wave functions. Initial states are constructed for each mode near the time of the classical singularity which are wave packets characterized by an initial position and an initial momentum. These states form an overcomplete family of states. Their expectation values follow the classical equations of motion.

I. INTRODUCTION

The Gowdy T^3 universe[1] is the simplest example of a closed, anisotropic, spatially inhomogeneous empty cosmology. As such, it has been studied[2-5] as a useful model for classical and quantum-mechanical processes in early universes.

The phenomenon of pair creation in strong gravitational fields has been studied by Parker[6] and Zel'dovich and Starobinsky[7] by considering the behavior of a quantized field in a given classical background cosmology. The Gowdy T^3 universe is especially useful as an example of this process since the wave (dynamical) part of the gravitational field itself is quantized so that the known solution of Einstein's equations can be used to fully account for the back reaction of created quanta on the metric.[8]

The quantum-mechanical treatment of this universe as given by the author[2,3] followed the lines suggested by Zel'dovich.[9] Pairs of gravitons were produced from vacuum fluctuations. These gravitons were characterized (at all times) by a momentum (wave number) which was conserved and a time-dependent energy. The number of gravitons produced in each mode (momentum value) was proportional to the ratio of the wavelength of the mode to the horizon size at an initial time. In this calculation an arbitrary initial time must be selected and spacetime assumed flat prior to it so that the initial N-quantum states may be constructed. This initial-time choice then appears to control the further evolution of the system.[2,3]

Misner has argued[4] that the definition of initial quantum number requiring an arbitrary construction at an initial time is not appropriate for the Gowdy T^3 universe. The classical model has an initial singularity near which the cosmology is a (different) Kasner universe in each mode. The Kasner behavior of the universe corresponds to a trajectory in superspace following a free particle rather than a harmonic-oscillator-type equation of motion. Far from the singularity, the universe appears to consist of a spatially homogeneous background filled with gravitational waves.[2,3] Misner shows[4] that evolution of the Gowdy T^3 universe may be interpreted as a scattering process in (mini) superspace. The initial states in the quantized model would then be the in-states for the scattering process and would reflect the local Kasner appearance near the singularity.

In this paper we discuss the quantized Gowdy T^3 model from the point of view of a scattering in superspace. As before,[2-5] the appropriate Klein-Gordon equation is separable, yielding a Schrödinger equation for each mode. This equation in fact has exact solutions. Thus the conclusions drawn from this analysis of the quantized Gowdy T^3 universe reflect only the approximation involved in neglecting, in the quantized model, degrees of freedom removed by symmetry in the classical problem and the arbitrariness in the method of quantization[10] and factor ordering.[4]

The Gowdy T^3 universe is obtained by generalizing the closed Kasner universe to allow spatial dependence in one direction. It is described by the metric[4,3]

$$ds^2 = e^{-\tau-\lambda/2}(-e^{4\tau}dt^2 + d\theta^2) + e^{2\tau}(e^{\beta}d\sigma^2 + e^{-\beta}d\delta^2)\,, \tag{1}$$

where τ, λ, β, are functions of θ, t only and the space variables θ, σ, δ are closed to give a 3-torus topology. Misner has shown[4] that the scattering in superspace can be described by the Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial\lambda_0\partial\tau} + \sum_{n=-\infty}^{\infty}\frac{1}{2}\frac{\partial^2}{\partial q_n{}^2} - \tfrac{1}{2}e^{4\tau}\sum_{n=-\infty}^{\infty} n^2 q_n{}^2\right)\psi = 0\,. \tag{2}$$

The wave function ψ represents the amplitude for the universe being at a given point [i.e., specified q_n $(-\infty \leq n \leq +\infty)$, λ_0, τ] in superspace where λ_0 is the constant Fourier component of λ.

The wave function is separable in λ_0 and in the q_n with the wave function for each mode, ψ_n, satisfying the Schrödinger equation[4]

$$ip_\lambda\frac{\partial\psi_n}{\partial\tau} = \left(-\frac{1}{2}\frac{\partial^2}{\partial q_n{}^2} + \tfrac{1}{2}n^2 e^{4\tau}q_n{}^2\right)\psi_n\,. \tag{3}$$

(p_λ is the momentum conjugate to λ_0 and is a constant.)

Thus the Schrödinger equation for each mode of the Gowdy T^3 universe is that for a harmonic oscillator with time-dependent frequency

$$\omega_n = |n|e^{2\tau}\,.$$

The method of Salusti and Zirilli[11] is used to find exact solutions. This method assumes the usual harmonic-oscillator spatial dependence to set up equations for the time-dependent coefficients.

We obtain a complete orthonormal set of solutions to the time-dependent Schrödinger equation of the form (for mode n)

$$\psi_{n,N} = \mathfrak{N}_N(\tau)P_N(f(\tau)q_n)\exp[-\alpha(\tau)q_n{}^2]\,, \tag{4}$$

where $f \equiv (2\,\mathrm{Re}\,\alpha)^{1/2}$. P_N is a Hermite polynomial of order N in q_n with time (τ) dependent coefficients. $\mathfrak{N}_N$ is an N-dependent time-dependent normalization. The coefficient $\alpha(\tau)$ is found uniquely by requiring

$$\lim_{\tau\to+\infty}\psi_{n,N} = \phi_N(q_n;\omega_n) \times \exp\left[-ip_\lambda{}^{-1}(N+\tfrac{1}{2})\int^{\tau} d\tau'\,\omega_n(\tau')\right], \tag{5}$$

where $\tau \to +\infty$ is far from the singularity, ϕ_N is the usual harmonic-oscillator N-quantum wave function for frequency $\omega_n(\tau)$, and the time dependence of ω_n may be neglected. The large-time or radiation limit of the Gowdy T^3 universe is discussed in Refs. 2–4.

The wave functions $\psi_{n,N}$ are solutions of the Schrödinger equation at all times including the time $\tau = -\infty$ of the initial singularity. The total wave function for mode n is thus

$$\psi_n = \sum_N a_N\psi_{n,N}\,, \tag{6}$$

where the a_N are constants. Near the initial singularity, the time-dependent potential term $\frac{1}{2}\omega_n{}^2 q_n{}^2$ in the Schrödinger equation drops out (since $\lim_{\tau\to-\infty}\omega_n = 0$) so that the appropriate states are those for a free particle.

We choose as the initial states wave packets $|p_n^0 q_n^0\rangle$ which satisfy the free-particle Schrödinger equation and of which the expectation values $\langle q_n\rangle, \langle -i\partial/\partial q_n\rangle \equiv \langle p_n\rangle$ satisfy the classical (Kasner) equations of motion[3,4]

$$q_n = q_n^0 + p_n^0\tau/p_\lambda\,, \tag{7a}$$

$$p_n = p_n^0\,, \tag{7b}$$

where p_n is conjugate to q_n and p_n^0, q_n^0 are constants. The work of Klauder[12] on continuous-representation theory is used to show that the $|p_n^0 q_n^0\rangle$ for all p_n^0, q_n^0 form an overcomplete family of states (OFS) and thus span the Hilbert space of solutions of the free-particle Schrödinger equation.

For an initial state $|p_n^0 q_n^0\rangle$, the a_N of Eq. (6) are found by expanding $|p_n^0 q_n^0\rangle$ in the complete set $\psi_{n,N}$ evaluated near $\tau = -\infty$ (when both the $\psi_{n,N}$ and $|p_n^0 q_n^0\rangle$ are solutions of the same Schrödinger equation). Since the a_N are constants, the solution ψ_n such that

$$\lim_{\tau\to-\infty}\psi_n = |p_n^0 q_n^0\rangle \tag{8}$$

is now known for all times. Since all the in-states, $|p_n^0 q_n^0\rangle$, have been found and all the out-states, $\lim_{\tau\to\infty}\psi_{n,N}$, are known, the S matrix for the scattering can be determined. It is easily found to have the matrix elements

$$S^n_{N;p_n^0 q_n^0} = a_N(p_n^0, q_n^0)\,. \tag{9}$$

Thus the problem has been solved.

It becomes trivial to calculate any expectation values of interest. For example, the expected number of quanta in mode n, $\langle N_n\rangle$, is

$$\langle N_n \rangle = \frac{\pi}{8p_\lambda}(p_n^0)^2 + \frac{2p_\lambda}{\pi}(q_n^0)^2 \tag{10}$$

and is a conserved quantity because the $\psi_{n,N}$ have been constructed as eigenfunctions of the number operator. With this construction, $\langle N_n \rangle$ represents the number of quanta present as $\tau \to +\infty$ for an initial wave packet $|p_n^0 q_n^0\rangle$. In contrast to Refs. 2 and 3, no unphysical initial time must be specified. Initial conditions p_n^0, q_n^0 are imposed at $\tau = -\infty$, the time of the initial singularity.

To compare Eq. (10) directly with $\langle N_n \rangle$ as found in Refs. 2 and 3, we use as an initial state the lowest-energy ($N=0$) state of the wave equation (3) for the frequency $ne^{2\tau}$ evaluated at $\tau = \tau_0$ and assumed fixed. (This procedure is equivalent to stopping the expansion of the universe prior to τ_0.) This initial state $\Psi_0(\tau_0)$ is then expanded in the $\psi_{n,N}$ wave functions and $\langle N_n \rangle$ is calculated. The result obtained (for $p_\lambda = 1$ and multiplying by 2 to include the mode $-n$) is precisely Eq. (110) of Ref. 3 for $N_n(\tau)$ in the limit of $\tau \to \infty$. Thus the choice of $\Psi_0(\tau_0)$ as an initial state is equivalent to a diagonalization of the Hamiltonian quantization.[7,9]

All other expectation values in addition to Eq. (10) can be obtained from a semiclassical treatment, so that the full quantum treatment here yields no new information. Fluctuations Δq_n, ΔN_n, etc., can be calculated but serve merely to slightly smear out the classical picture.

In Sec. II we summarize the classical picture using the notation of Misner and the results of Refs. 2–5. Section II is a discussion of the quantization including the complete wave functions $\psi_{n,N}$, which are derived in Appendixes A and B. Section IV is a discussion of the initial states $|p_n^0 q_n^0\rangle$, which are shown to be an OFS in Appendix C. Section V contains a derivation of the S matrix $S_{N;p_n^0 q_n^0}$, and Sec. VI gives the results for the calculations of $\langle N_n \rangle$, $\langle q_n \rangle$, Δq_n, and ΔN_n. The results are discussed in Sec. VII.

II. THE CLASSICAL GOWDY T^3 MODEL

The Gowdy T^3 universe is described by the metric[4,13]

$$\begin{aligned} ds^2 = {} & e^{-\tau-\lambda/2}(-e^{4\tau}dt^2 + d\theta^2) \\ & + e^{2\tau}(e^{\beta}d\sigma^2 + e^{-\beta}d\delta^2), \end{aligned} \tag{11}$$

where the metric variables τ, λ, and β are functions of t and θ only. The 3-torus topology of the spacelike hypersurfaces is achieved by requiring $0 \le \theta \le 2\pi$ and

$$\oint \oint d\sigma\, d\delta = 8 . \tag{12}$$

The coordinate choice

$$\tau' = 0 = p_\lambda' \tag{13}$$

(which is preserved in time) leads to the integrated Hamiltonian constraint[4] ($G = 1 = c$)

$$\mathfrak{H} \equiv p_\lambda p_* + (2\pi)^{-1} \oint d\theta (\tfrac{1}{2}p^2 + \tfrac{1}{2}e^{4\tau}\beta'^2) = 0 , \tag{14}$$

where p_λ and p are the momenta canonically conjugate to λ and β,

$$p_* \equiv (2\pi)^{-1} \int d\theta\, p_\tau \tag{15}$$

is the constant Fourier component of p_τ, the momentum conjugate to τ, and the prime denotes $\partial/\partial\theta$. The dynamics of this universe is found from the super-Hamiltonian (14).

The spatially varying part of the metric variable λ which we call λ_+ is nondynamical and is found by identically solving the momentum constraint to yield[4]

$$\lambda_+ = -(p_\lambda)^{-1}\left[\int_0^\theta p\beta' d\theta - (2\pi)^{-1} \oint d\theta' \int_0^{\theta'} d\theta\, p\, d\beta\right]. \tag{16}$$

The constant Fourier component

$$\lambda_0 \equiv (2\pi)^{-1} \oint d\theta\, \lambda \tag{17}$$

is the dynamical variable conjugate to p_λ and is cyclical in the super-Hamiltonian (14).

To easily study Eq. (14), p and β are rewritten in Fourier-decomposed form[4]

$$\beta = q_0 + \sqrt{2} \sum_{n=1}^{\infty} (q_n \cos n\theta + q_{-n} \sin n\theta) , \tag{18a}$$

$$p = p_0 + \sqrt{2} \sum_{n=1}^{\infty} (p_n \cos n\theta + p_{-n} \sin n\theta) , \tag{18b}$$

and the θ integration in Eq. (14) is performed. We obtain[4]

$$\mathfrak{H} = p_\lambda p_* + \sum_{n=-\infty}^{\infty} \tfrac{1}{2}(p_n{}^2 + n^2 e^{4\tau} q_n{}^2) = 0 . \tag{19}$$

The 3-torus topology requires the total θ momentum $\mathfrak{P}$ to be zero[4]:

$$\mathfrak{P} = -(2\pi)^{-1/2} \oint p\beta' d\theta = 0 . \tag{20}$$

Equation (20) acts to restrict the solutions obtained by varying $\mathfrak{H}$. Upon Fourier transformation, we find[4]

$$\mathfrak{P} = -\sum_{n=1}^{\infty} n(p_n q_{-n} - p_{-n} q_n) = 0 . \tag{21}$$

Varying $\mathfrak{H}$ with respect to p_* and λ_0 yields

$$d\tau/dt = p_\lambda \text{ and } p_\lambda = \text{const}\,, \tag{22}$$

which allows us to use τ as the time variable.

We obtain the equations of motion[4] (the overdot denotes $\partial/\partial\tau$)

$$\dot{\lambda}_0 = p_*/p_\lambda\,, \tag{23a}$$

$$\dot{p}_* = -\frac{2}{p_\lambda} e^{4\tau} \sum_n n^2 q_n{}^2\,, \tag{23b}$$

$$\dot{q}_n = p_n/p_\lambda\,, \quad \dot{p}_n = -\frac{n^2}{p_\lambda} e^{4\tau} q_n{}^2\,. \tag{23c}$$

From Eqs. (23c) and (21), we see that $\mathfrak{P}=0$ is satisfied if the ratio of the odd-parity amplitude q_{-n} to the even-parity amplitude q_{+n} is constant in time τ for each mode n.

Equations (23c) combine to yield the second-order equation

$$p_\lambda{}^2 \ddot{q}_n + n^2 e^{4\tau} q_n = 0\,. \tag{24}$$

The dynamical equations for the field amplitude q_n can be found by varying the Arnowitt-Deser-Misner (ADM) Hamiltonian

$$H_{\text{ADM}} = (p_\lambda)^{-1} \sum_{n=-\infty}^{\infty} \tfrac{1}{2}(p_n{}^2 + n^2 e^{4\tau} q_n{}^2)\,, \tag{25}$$

which is obtained as $-p_*$, i.e., the negative of the momentum conjugate to τ. It has been shown that in the limit of $\tau \to -\infty$, the Gowdy T^3 universe reduces to a Kasner universe in each mode n with a true singularity at $\tau = -\infty$.[3]

Near the singularity,

$$H_{\text{ADM}} = (p_\lambda)^{-1} \sum_{n=-\infty}^{\infty} \tfrac{1}{2} p_n{}^2\,, \tag{26}$$

or

$$q_n = q_n^0 + p_n^0 \tilde{\tau}/p_\lambda\,, \tag{27}$$

where $2\tilde{\tau} \equiv 2\tau + \gamma + \ln(|n|/4p_\lambda)$ and γ is Euler's constant. (The use of $\tilde{\tau}$ as the time variable simplifies the form of the appropriate solution of Eq. (24) as given in Eq. (28). Equivalently, we could have taken $q_n^c \equiv q_n^0 + [\gamma/2 + \frac{1}{2}\ln(|n|/4p_\lambda)]p_n^0/p_\lambda$ as the coefficient of J_0 in Eq. (28). This would yield q_n^c in place of q_n^0 in the final results.[4]) The solution of Eq. (24) which reduces to Eq. (27) in the limit $\tau \to -\infty$ is

$$q_n = \frac{\pi}{4}\frac{p_n^0}{p_\lambda} N_0\left(\frac{|n|}{2p_\lambda} e^{2\tau}\right) + q_n^0 J_0\left(\frac{|n|}{2p_\lambda} e^{2\tau}\right), \tag{28}$$

where J_ν and N_ν are respectively the νth-order regular and irregular Bessel functions.[14] (Note: For the mode $n=0$, the Kasner relation $q_0 = q_0^0 + p_0^0\tau/p_\lambda$ is retained for all time.)

In the limit as $\tau \to +\infty$, we have

$$\lim_{\tau\to\infty} q_n = p_\lambda{}^{1/2}\omega_n{}^{-1/2}\left[\left(\frac{\pi}{8}\right)^{1/2} p_\lambda{}^{-1} p_n^0 \sin\left(p_\lambda{}^{-1}\int^\tau d\tau'\omega_n - \frac{\pi}{4}\right) + q_n^0\left(\frac{2}{\pi}\right)^{1/2} \cos\left(p_\lambda{}^{-1}\int^\tau d\tau'\omega_n - \frac{\pi}{4}\right)\right], \tag{29}$$

where

$$\omega_n \equiv |n| e^{2\tau}\,. \tag{30}$$

In this limit the time dependence of ω_n can be neglected compared to all other time derivatives. The ADM Hamiltonian can thus be expressed as the "energy" due to the sum of N_n harmonic oscillators of frequency ω_n for each mode n ($\neq 0$) divided by p_λ. For mode n, the component of the equivalent harmonic-oscillator Hamiltonian $p_\lambda H_n$ is

$$p_\lambda H_n = \omega_n N_n = \omega_n\left[\frac{\pi}{8p_\lambda}(p_n^0)^2 + \frac{2}{\pi} p_\lambda (q_n^0)^2\right] \tag{31}$$

or[4]

$$N_n = \frac{\pi}{8p_\lambda}(p_n^0)^2 + \frac{2p_\lambda}{\pi}(q_n^0)^2\,. \tag{32}$$

The interpretation of these oscillators as gravitons is described in Ref. 3. Thus, classically, this model behaves as a system of disjoint Kasner universes near the singularity and as a universe containing gravitational radiation far from the singularity.

III. QUANTIZING THE MODEL

To quantize this model, we follow Misner[4] and require the wave function for the universe Ψ to obey the (mini) superspace equation

$$\mathfrak{H}\Psi = 0\,, \tag{33}$$

where $\mathfrak{H}$ is given by Eq. (19). By imposing the usual commutation relations on the dynamical variables ($\hbar = 1$),

$$[p_\lambda, \lambda_0] = -i\,, \tag{34a}$$

$$[p_*, \tau] = -i\,, \tag{34b}$$

$$[p_n, q_n] = -i\,, \text{ etc.}\,, \tag{34c}$$

we can write Eq. (33) as the Klein-Gordon equation[4]

$$\left(\frac{\partial^2}{\partial\lambda_0\partial\tau} + \sum_{n=-\infty}^{\infty}\frac{1}{2}\frac{\partial^2}{\partial q_n{}^2} - \tfrac{1}{2}e^{4\tau}\sum_{n=-\infty}^{\infty} n^2 q_n{}^2\right)\Psi = 0\,. \tag{35}$$

This equation is easily seen to be separable in λ and for each n—i.e., we may write[4]

$$\Psi = (2\pi)^{-1} \int \frac{dp_\lambda}{p_\lambda} f(p_\lambda) e^{ip_\lambda \lambda_0} \prod_{n=-\infty}^{\infty} \psi_n(q_n, \tau; p_\lambda), \tag{36}$$

where f is an arbitrary function. The separated wave functions ψ_n each satisfy the Schrödinger equation[4]

$$ip_\lambda \frac{\partial \psi_n}{\partial \tau} = \left(-\frac{1}{2}\frac{\partial^2}{\partial q_n^2} + \tfrac{1}{2} n^2 e^{4\tau} q_n^2\right)\psi_n . \tag{37}$$

From Ref. 3 or Eq. (29) or by direct substitution, we find that in the limit of $\tau \to +\infty$ (i.e., far from the classical singularity)

$$\lim_{\tau\to+\infty} \psi_n = \sum_n a_N \exp\left[-\frac{i}{p_\lambda}(N+\tfrac{1}{2})\int^\tau d\tau' \omega_n\right] \times \phi_n(q_n; \omega_n), \tag{38}$$

where $\omega_n = |n| e^{2\tau}$, a_N is constant for all N, and ϕ_N is the usual N-quantum harmonic-oscillator wave function for a unit-mass oscillator with frequency ω_n. The Schrödinger equation (37) becomes in this limit

$$(N+\tfrac{1}{2})\omega_n \phi_N = \left(-\frac{1}{2}\frac{\partial^2}{\partial q_n^2} + \tfrac{1}{2}\omega_n^2 q_n^2\right)\phi_N . \tag{39}$$

Equations (38) and (39) are valid in the (adiabatic or WKB) regime for which $\omega_n^2 \gg p_\lambda d\omega_n/d\tau$ or $\omega_n \equiv |n| e^{2\tau} \approx p_\lambda$.[4]

Using the methods of Salusti and Zirilli[11] for finding wave functions for time-dependent-frequency harmonic oscillators, we obtain a complete set of normalized wave functions $\psi_{n,N}(q_n, \tau)$ which are solutions of Eq. (37) for all τ. These wave functions are completely determined by requiring agreement with Eqs. (38) and (39) in the limit of $\tau \to +\infty$. These solutions are constructed in detail in Appendixes A and B.

We find that the set of solutions to Eq. (37) has the form

$$\psi_{n,N} = \mathfrak{N}_N(\tau) P_N(q_n, \tau) \exp[-\alpha(\tau) q_n^2], \tag{40a}$$

where $\mathfrak{N}_N$ is a time-dependent normalization factor, P_N is an Nth-order polynomial in q_n with time-dependent coefficients, and α is a complex time-dependent function. We find (Appendix B) that

$$\mathfrak{N}_N = \left(\frac{2}{\pi}\right)^{1/2} e^{-iN\pi/4} e^{-i\pi/8} 2^{-N/2} p_\lambda^{1/4} (+i)^N (N!)^{-1/2} \times \left[H_0^{(2)}\left(\frac{|n|}{2p_\lambda} e^{2\tau}\right)\right]^{N/2} \left[H_0^{(1)}\left(\frac{|n|}{2p_\lambda} e^{2\tau}\right)\right]^{-(N+1)/2} \tag{40b}$$

(where $H_\nu^{(a)}$ is a νth-order Hankel function of the ath kind),

$$P_N = \mathcal{H}_N\left(2(p_\lambda/\pi)^{1/2} q_n \Big/ \left|H_0^{(1)}\left(\frac{|n|}{2p_\lambda} e^{2\tau}\right)\right|\right) = \mathcal{H}_N((2\,\mathrm{Re}\alpha)^{1/2} q_n), \tag{40c}$$

where $\mathcal{H}_N$ is the Nth-order Hermite polynomial, and

$$\alpha = \frac{i}{2} |n| e^{2\tau} H_1^{(1)}\left(\frac{|n|}{2p_\lambda} e^{2\tau}\right) q_n^2 \Big/ H_0^{(1)}\left(\frac{|n|}{2p_\lambda} e^{2\tau}\right). \tag{40d}$$

The properties of the Hermite polynomials are used to construct the generating function

$$F(s, q_n, \tau) = \exp[-s^2 + 2(2\,\mathrm{Re}\alpha)^{1/2} s q_n] = \sum_N P_N(q_n, \tau) s^N / N! \tag{41}$$

to show that the $\psi_{n,N}$ form a complete, orthogonal, normalized set of solutions of the Schrödinger equation (37). By using the large-argument ($\tau \to +\infty$) form of the Hankel functions, we find

$$\lim_{\tau\to\infty} \psi_{n,N} = \exp\left[-ip_\lambda^{-1}(N+\tfrac{1}{2})\int^\tau \omega_n d\tau'\right] \phi_N(q_n; \omega_n), \tag{42}$$

as required.

The $\psi_{n,N}$ can be constructed from the ground state

$$\psi_{n,0} = \left(\frac{2}{\pi}\right)^{1/2} p_\lambda^{1/4} e^{-i\pi/8} \left[H_0^{(1)}\left(\frac{|n|}{2p_\lambda} e^{2\tau}\right)\right]^{-1/2} \times \exp(-\alpha q_n^2) \tag{43}$$

by a rule analogous to that for the harmonic oscillator,

$$\psi_{n,N} = (N!)^{-1/2} (A_n^\dagger)^N \psi_{n,0}, \tag{44}$$

where

$$A_n^\dagger = e^{-i\pi/4} (\pi/8p_\lambda)^{1/2} \left[|n| e^{2\tau} H_1^{(2)}\left(\frac{|n|}{2p} e^{2\tau}\right) \hat{q}_n - iH_0^{(2)}\left(\frac{|n|}{2p_\lambda} e^{2\tau}\right)\frac{\partial}{\partial q_n}\right] \tag{45}$$

and the caret denotes an operator. The operator A_n constructed as the Hermitian conjugate of $A_n^\dagger$ yields

$$A_n \psi_{n,0} = 0 \tag{46}$$

for all times. We also obtain the time-independent commutation relation

$$[A_n, A_n^\dagger] = 1 . \tag{47}$$

To satisfy Eqs. (38) and (39), we completely determine A_n and $A_n^\dagger$ by requiring

$$\lim_{\tau \to +\infty} A_n^\dagger = a_n^\dagger = (2\omega_n)^{1/2}(\hat{p}_n + i\omega_n \hat{q}_n) \times \exp\left[-i(p_\lambda)^{-1}\int^{\tau'} \omega(\tau')d\tau'\right], \tag{48}$$

where ω_n is given by Eq. (30) and $a_n^\dagger$ is the usual harmonic-oscillator creation operator.

From Eq. (44) we easily show that

$$\psi_{n,N+1} = (N+1)^{-1/2} A_n^\dagger \psi_{n,N} \tag{49a}$$

and

$$\psi_{n,N-1} = (N)^{-1/2} A_n \psi_{n,N}. \tag{49b}$$

Thus we can construct the conserved number operator

$$\hat{N}_n \equiv A_n^\dagger A_n \tag{50}$$

such that

$$\hat{N}_n \psi_{n,N} = N\psi_{n,N}. \tag{51}$$

Thus the wave functions $\psi_{n,N}$ are eigenfunctions of the number operator. From Eq. (48) and its Hermitian conjugate, we conclude that $\hat{N}_n$ as defined in Eq. (50) is just the usual harmonic-oscillator number operator in the limit of $\tau \to +\infty$. Since it is a conserved operator, $\hat{N}_n$ must at all times produce, when acting on the wave function, the number of gravitons which will be present in the adiabatic regime for a given initial state.

IV. THE INITIAL-STATE WAVE FUNCTIONS

The total wave function for mode n must be a superposition of the N-quantum wave functions (40a)—i.e.,

$$\psi_n = \sum_{N=0}^{\infty} a_N \psi_{n,N}. \tag{52}$$

To determine the appropriate a_N, we study the solutions of Eq. (37) in the vicinity of the classical singularity $\tau = -\infty$.

In this limit, Eq. (37) becomes a "free-particle" Schrödinger equation

$$ip_\lambda \frac{\partial \psi_{-\infty}}{\partial \tau} = -\frac{1}{2}\frac{\partial^2}{\partial q_n^{\,2}}\psi_{-\infty} \tag{53}$$

(since the potential term goes to zero), where $\psi_{-\infty}$ is the wave function near the singularity. (For convenience we shall again use $\bar{\tau} \equiv \tau + \frac{1}{2}[\gamma + \ln(|n|/4p_\lambda)]$ as the time variable.)

It is clear that we must have

$$\psi_{-\infty}(q_n, \tau) = \int_{-\infty}^{\infty} dk\, F(k)\exp(ikq_n - \tfrac{1}{2}ik^2\bar{\tau}/p_\lambda), \tag{54}$$

where $F(k)$ is chosen so that $\psi_{-\infty}$ is normalized. For

$$F(k) = (4\pi)^{-1/2}\exp[-\tfrac{1}{8}\pi(p_n^0 - k)^2 + iq_n^0(p_n^0 - k)], \tag{55}$$

we obtain the normalized wave packet

$$\psi_{-\infty}(q_n, \tau) = p_\lambda^{-1/4}\left(\frac{\pi}{2p_\lambda} + 2i\bar{\tau}/p_\lambda\right)^{-1/2}\exp\left[ip_n^0 q_n - i\frac{(p_n^0)^2}{2}\frac{\bar{\tau}}{p_\lambda} - \frac{(q_n - q_n^0 - p_n^0\bar{\tau}/p_\lambda)^2}{\pi/2p_\lambda + 2i\bar{\tau}/p_\lambda}\right]. \tag{56}$$

In addition to being a solution of Eq. (53), $\psi_{-\infty}$ from Eq. (56) has the following properties:

$$\langle q_n\rangle \equiv \int dq_n\, \psi_{-\infty}^* q_n \psi_{-\infty} = q_n^0 + p_n^0\bar{\tau}/p_\lambda, \tag{57a}$$

$$\langle p_n\rangle \equiv \int dq_n\, \psi_{-\infty}^*(-i\partial/\partial q_n)\psi_{-\infty} = p_n^0, \tag{57b}$$

and

$$\psi_{-\infty}(q_n, \tau)\big|_{q_n^0, p_n^0 = 0} = \lim_{\tau \to -\infty} \psi_{n,0}(q_n, \tau), \tag{57c}$$

where $\psi_{n,0}$ is given by Eq. (43). Properties (57a) and (57b) imply that our initial state $\psi_{-\infty}$ is a wave packet whose expectation values follow in this limit the classical equation of motion (24). The spread of the wave packet is fixed by Eq. (57c) to coincide with that for the ground-state wave function $\psi_{n,0}$ which behaves as a wave packet in the limit $\tau \to -\infty$. In fact, the form of Eqs. (55) and (56) was suggested by the fact that

$$\lim_{\tau \to -\infty} \psi_{n,0} = p_\lambda^{-1/4}\left(\frac{\pi}{2p_\lambda} + 2i\bar{\tau}/p_\lambda\right)^{-1/2} \times \exp\left(\frac{-q_n^{\,2}}{\pi/2p_\lambda + 2i\bar{\tau}/p_\lambda}\right) \tag{58}$$

is a spreading minimum-uncertainty wave packet.[15]

Thus for each set of initial conditions p_n^0, q_n^0 a wave packet with the classical expectation values can be constructed. Thus let us rewrite the initial wave function as $\psi_{-\infty}(q_n, \tau; p_n^0, q_n^0) \equiv |p_n^0 q_n^0\rangle$. In Appendix C we show that the set $|p_n^0 q_n^0\rangle$ for all p_n^0, q_n^0 can be put into 1 to 1 correspondence with the overcomplete family of states (OFS) for phase space, as discussed by Klauder.[12] Thus the OFS $|p_n^0 q_n^0\rangle$ for all p_n^0, q_n^0 spans the space of solutions of the Schrödinger equation (53). Therefore, the wave functions (56) for all p_n^0, q_n^0 constitute all the initial states for mode n of this model.

V. THE SCATTERING MATRIX

The Klein-Gordon equation (35) is that for a particle (or, in this case, universe) scattering in a (mini) superspace with $\infty + 2$ dimensions $\lambda_0, \tau, q_0, q_1, \ldots, q_n, \ldots$.[4] The in-states for this scattering process are the $|p_n^0 q_n^0\rangle$. The out-states are the $\psi_{n,N}$ in the limit of $\tau \to +\infty$. The S matrix relating the in- and out-states has the elements[16]

$$S_{rs} = \langle \psi_r | U(+\infty, -\infty) | \psi_s \rangle , \tag{59}$$

where ψ_s is an in-state, ψ_r is an out-state, and $U(t,t')$ is the time-evolution operator—i.e.,

$$U(t,t')\psi(t') = \psi(t) . \tag{60}$$

Since $U^\dagger = U^{-1}$, Eq. (59) may also be written

$$S_{rs} = \langle U(-\infty, +\infty)\psi_r | \psi_s \rangle . \tag{61}$$

But

$$U(-\infty, \infty) \lim_{\tau\to\infty} \psi_{n,N} = \lim_{\tau\to-\infty} \psi_{n,N} , \tag{62}$$

where, from Eqs. (40), $\psi_{n,N}$ is known for all τ. Thus we define the S-matrix elements

$$S^n_{N;p_n^0 q_n^0} = \int_{-\infty}^{\infty} dq_n (\lim_{\tau\to-\infty} \psi^*_{n,N})\psi_{-\infty} . \tag{63}$$

We now reconsider the state of the system for mode n,

$$\psi_n = \sum_N a_N \psi_{n,N} . \tag{64}$$

For the system initially in the state $|p_n^0 q_n^0\rangle$, we have

$$\psi_n = U(\tau, -\infty) | p_n^0 q_n^0 \rangle , \tag{65}$$

i.e., the state of the system is the future evolved initial state. Comparing Eq. (63) with Eq. (65) substituted in Eq. (64), we find

$$a_N = S^n_{N;p_n^0 q_n^0} . \tag{66}$$

Using Eqs. (40) and (56), the integration (63) is easily performed to yield

$$a_N = e^{iN\pi/4} e^{i\pi/8} (N!)^{-1/2} (-i)^N \left(\frac{\pi}{8p_\lambda}\right)^{N/2} \left(ip_n^0 + \frac{4p_\lambda}{\pi} q_n^0\right)^N \exp\left[-\frac{\pi}{16p_\lambda}(p_n^0)^2 + \frac{i}{2} q_n^0 p_n^0 - \frac{p_\lambda}{\pi}(q_n^0)^2\right] . \tag{67}$$

Thus given the initial wave packet characterized by p_n^0, q_n^0 we have obtained the wave function which is a solution of Eq. (37) at all times for the mode n. (We note here that the mode $n = 0$ retains its $|p_n^0 q_n^0\rangle$ form for all τ. This is the quantized Kasner behavior.[17])

VI. THE STATE OF THE SYSTEM

To obtain the full solution (36), we must form the product of the wave functions ψ_n over the n modes. This product is restricted by the condition (21) on the total θ momentum. In the quantized model, Eq. (21) becomes

$$\langle \mathfrak{P} \rangle = 0 . \tag{68}$$

From Eq. (21), we see that the modes $\pm n$ must satisfy

$$\langle p_n q_{-n} - p_{-n} q_n \rangle \equiv \int\int dq_n dq_{-n} \psi_n^* \psi_{-n}^* (\hat{p}_n \hat{q}_{-n} - \hat{p}_{-n}\hat{q}_n)\psi_n \psi_{-n} \cdot$$
$$= 0 . \tag{69}$$

Using Eq. (52) for $\psi_{\pm n}$ with a_N from Eq. (67), we find that Eq. (69) is equivalent to

$$p_n^0 q_{-n}^0 = p_{-n}^0 q_n^0 , \tag{70}$$

as in the classical treatment.

Thus, subject to the restriction of Eq. (70), the complete wave function (36) may be written down with ψ_n given by (52) and a_N from Eq. (67). The distribution $f(p_\lambda)$ is, of course, an arbitrary initial condition.

Calculations of various expectation values in the state ψ_n defined by

$$\lim_{\tau\to-\infty} \psi_n = | p_n^0 q_n^0 \rangle \tag{71}$$

for some p_n^0, q_n^0 (i.e., the initial state is a basis state) are completely straightforward using Eq. (52) for ψ_n with a_N from Eq. (67).

The expected quantum number in the mode n, N_n is [using Eq. (50) for $\hat{N}_n$]

$$\langle \hat{N}_n \rangle = \int_{-\infty}^{\infty} dq_n \psi_n^* A_n^\dagger A_n \psi_n$$
$$= \sum_{N,M} \int dq_n a_M^* N a_N \psi^*_{n,M} \psi_{n,N}$$
$$= \sum_N |a_N|^2 N . \tag{72}$$

Using Eq. (67) for a_N, the summation is easily performed to yield

$$\langle \hat{N}_n \rangle = \frac{\pi}{8p_\lambda}(p_n^0)^2 + \frac{2p_\lambda}{\pi}(q_n^0)^2 . \tag{73}$$

A comparison with Eq. (32) for the (semi)classical N_n shows that the quantum treatment yields the same result.

We can also calculate a purely quantum-mechanical quantity—the fluctuation in quantum number

$$\Delta N_n \equiv \pm(\langle N_n{}^2 \rangle - \langle N_n \rangle^2)^{1/2} . \tag{74}$$

Straightforward use of a procedure as in Eq. (72) for

$$\hat{N}^2 = A_n^\dagger A_n A_n^\dagger A_n \tag{75}$$

yields

$$\langle N_n{}^2\rangle = \langle N_n\rangle^2 + \langle N_n\rangle \tag{76}$$

or

$$\Delta N_n = \pm (\langle N_n\rangle)^{1/2}\,, \tag{77}$$

where $\langle N_n\rangle$ is given by Eq. (73). Thus the fluctuations are at the usual statistical level. We emphasize here that both $\langle N_n\rangle$ and ΔN_n are conserved quantities in this treatment but describe the number of particle-like quanta (gravitons) only at large times. The value of $\langle N_n\rangle$ valid for all times in the quantum treatment agrees at large times with the semiclassical value obtained by recognizing the quantum-like (or radiation) character of the system in the adiabatic regime.

Other calculations are also easily performed. The expected value of the field amplitude $\tilde{q}_n$ is

$$\langle q_n\rangle = \frac{\pi}{4p_\lambda} p_n^0 N_0\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right) + q_n^0 J_0\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right). \tag{78}$$

Comparing Eq. (78) with the classical q_n [Eq. (28)] shows that the expected field amplitude satisfies the classical equations of motion at all times.

The field fluctuations

$$\Delta q_n \equiv \pm (\langle q_n{}^2\rangle - \langle q_n\rangle^2)^{1/2} = \pm\left(\frac{\pi}{8}\right)^{1/2}\left[J_0{}^2\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right) + N_0{}^2\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\right]^{1/2} \tag{79}$$

are independent of initial conditions p_n^0, q_n^0 and vanish at large times. The fluctuation Δq_n is infinite at the classical singularity.

VII. DISCUSSION

A comparison of the classical results of Sec. II with the quantum results of Secs. III–VI shows surprisingly little effect of quantization. The only significant difference appears to be the smearing out of the quantum picture due to fluctuations. The expectation values all agree with the classical quantities so that the singularity structure is unchanged. Even the Planck length, $L_p \equiv (G\hbar/c^3)^{1/2}$, which was introduced through the commutation relations,[2,3] appears to play no significant role since the number of Planck lengths, times, etc. in any relevant quantity can be controlled by adjusting the free parameter p_λ.[14]

We emphasize here that the $\psi_{n,N}(q,\tau)$ of Eq. (40) constitute a complete orthonormal set of solutions of the Schrödinger equation (37) for all times $-\infty \leq \tau \leq \infty$. Once the $\psi_{n,N}$ have been obtained, only questions of interpretation remain—i.e., which superpositions of the $\psi_{n,N}$ are physically interesting.

The $|p_n^0 q_n^0\rangle$ initial states [Eq. (56)], constructed as minimum-uncertainty wave packets, are those which most nearly reproduce the classical results for this model universe. They are readily interpretable and well defined at all times including that of the initial singularity, and thus enable us to avoid the problems usually associated with particle-number definition in strong gravitational field regimes.[6]

By contrast, we reproduce the results of Ref. 3 by requiring $\omega_n = \omega_n^0 \equiv ne^{2\tau_0}$ for all $\tau \leq \tau_0$, where τ_0 is arbitrary, and choosing as a vacuum state the lowest-energy eigenstate of the Hamiltonian $\frac{1}{2}p_n{}^2 + \frac{1}{2}\omega_n{}^2 q_n{}^2$. If ω_n is not in the adiabatic regime for mode n, gravitons are created. This creation regime is, however, just that where particle number is not well defined,[6] so that the creation is more a consequence of the inappropriateness of the particle description of the initial state than an actual physical process.

Thus the choices of initial states such as the $|p_n^0 q_n^0\rangle$ which are appropriate near the classical singularity are seen to follow the known classical behavior of the system. As such, they should (to within the limitations of the model due to the absence of degrees of freedom absent classically by symmetry) eliminate artificial creation associated with poorly defined graviton number and should describe interesting physical processes.

This use of plane-wave initial states to describe a Kasner-like singularity has been generalized to the case of a quantized massive scalar field in a given classical background spacetime which is spatially homogeneous with a 3-torus topology.[18] Very similar results are obtained.

The conjectures which have been made[19] as to the importance of quantization of gravity do not appear to be valid in this model since the quantization essentially reproduces the classical results. It is possible that the quantization of a more complex cosmology with interacting modes would yield results absent from classical universes.[20]

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APPENDIX A: THE GROUND-STATE WAVE FUNCTION

We use the method of Salusti and Zirilli[11] to find solutions to Eq. (37) of the form

$$\psi_n = \sum_N a_N \psi_{n,N}(q_n, \tau; p_\lambda) . \tag{A1}$$

We assume that $\psi_{n,0}$ can be written

$$\psi_{n,0} = \mathfrak{N} \exp[E(\tau)q_n^2 + B(\tau)q_n + C(\tau)] . \tag{A2}$$

For convenience, we choose $B(\tau)=0$ or alternatively choose a new q_n with no linear term in the exponent of Eq. (A2). The Schrödinger equation (37) restricts $E(\tau)$ and $C(\tau)$ to satisfy the equations

$$ip_\lambda \frac{dE}{d\tau} = -2E^2 + \tfrac{1}{2} n^2 e^{4\tau} \tag{A3a}$$

and

$$ip_\lambda \frac{dC}{d\tau} = -E . \tag{A3b}$$

Equations (A3) are easily solved[21] to yield

$$\psi_{n,0} = \mathfrak{N}\left[Z_0\left(\frac{|n|}{2p_\lambda} e^{2\tau}\right)\right]^{-1/2} \times \exp\left[-\tfrac{1}{2}|n| e^{2\tau} Z_1\left(\frac{|n|}{2p_\lambda} e^{2\tau}\right) q_n^2 \Big/ Z_0\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\right], \tag{A4}$$

where $Z_\nu(x)$ is any solution of the νth-order Bessel's equation.

Let us assume that ψ is a solution of Eq. (37). We wish to construct an operator

$$A_n \equiv \gamma(\tau)\hat{q}_n + \beta(\tau)\partial/\partial q_n \tag{A5}$$

such that $A_n\psi$ is also a solution of Eq. (37). For $A_n\psi$ to be a solution, we require

$$ip_\lambda \frac{d\gamma}{d\tau} = -n^2 e^{4\tau}\beta \tag{A6a}$$

and

$$ip_\lambda \frac{d\beta}{d\tau} = -\gamma . \tag{A6b}$$

Equations (A6) are easily solved to yield

$$A_n = i|n| e^{2\tau} Z_1\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\hat{q}_n - Z_0\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\frac{\partial}{\partial q_n}. \tag{A7}$$

We also see that the Hermitian conjugate of Eq. (A7)

$$A_n^\dagger = -i|n| e^{2\tau} Z_1^*\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\hat{q}_n - Z_0^*\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\frac{\partial}{\partial q_n} \tag{A8}$$

yields $A_n^\dagger\psi$ as a solution of Eq. (37) for ψ a solution. [$Z_\nu^*(x)$ is the complex conjugate of $Z_\nu(x)$ when x is real.]

Operating with A_n on $\psi_{n,0}$ yields

$$A_n\psi_{n,0} \equiv 0 \tag{A9}$$

for any choice of Bessel function Z, confirming our calling $\psi_{n,0}$ the ground state and allowing us to call A_n an annihilation operator. We shall call $A_n^\dagger$ a creation operator and define in analogy with the usual harmonic oscillator

$$\psi_{n,N} \equiv (N!)^{-1/2}(A_n^\dagger)^N \psi_{n,0} . \tag{A10}$$

The boundary condition (38) requires that we choose

$$Z_\nu = H_\nu^{(1)} \tag{A11}$$

as the appropriate solution of Bessel's equation, where $H_\nu^{(a)}$ is the νth-order Hankel function of the ath kind. (Note: $Z_\nu^* = H_\nu^{(2)}$.)

APPENDIX B: THE N-QUANTUM WAVE FUNCTIONS

From Eq. (A10) we obtain the analogous recurrence relations

$$\psi_{n,N+1} = (N+1)^{-1/2} A_n^\dagger \psi_{n,N} \tag{B1a}$$

and

$$\psi_{n,N-1} = (N)^{-1/2} A_n \psi_{n,N} . \tag{B1b}$$

Equations (B1) will be used to find the explicit form for $\psi_{n,N}$.

The form [Eq. (45)] of the operators A_n and $A_n^\dagger$ forces $\psi_{n,N}$ to be written as

$$\psi_{n,N}(q_n,\tau) = \mathfrak{N}_N(\tau) P_N(q_n,\tau) \exp\left\{ -\tfrac{1}{2} H_1^{(1)}\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\left[H_0^{(1)}\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\right]^{-1} |n| e^{2\tau} q_n^2 \right\}, \tag{B2}$$

where $\mathfrak{N}_N$ is some function of time and P_N is a polynomial of order N in q_n with time-dependent coefficients.

The recurrence relations (B1) yield

$$P_{N+1} = (N+1)^{-1/2}(\mathfrak{N}_{N+1})^{-1}\mathfrak{N}_N i \left\{ \left(\frac{8p_\lambda}{\pi}\right)^{1/2}\left[H_0^{(1)}\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\right]^{-1} P_N q_n - \left(\frac{\pi}{8p_\lambda}\right)^{1/2} H_0^{(2)}\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\frac{\partial P_N}{\partial q_n}\right\} e^{-i\pi/4} \tag{B3a}$$

and

$$P_{N-1}=(N)^{-1/2}(\mathfrak{N}_{N-1})^{-1}\mathfrak{N}_N(-i)\left(\frac{\pi}{8p_\lambda}\right)^{1/2}e^{+i\pi/4}H_0^{(1)}\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\frac{\partial P_N}{\partial q_n}, \tag{B3b}$$

which combine to yield the second-order equation

$$2NP_N-2q_n\frac{\partial P_N}{\partial q_n}+\frac{\pi}{4p_\lambda}\left|H_0^{(1)}\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\right|^2\frac{\partial^2 P_N}{\partial q_n^2}=0. \tag{B4}$$

The solution of Eq. (B4) which reduces to the correct limit as $\tau\to\infty$ is

$$P_N(q_n,\tau)=\mathcal{H}_N\left(2\left(\frac{p_\lambda}{\pi}\right)^{1/2}q_n\Big/\left|H_0^{(1)}\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\right|\right), \tag{B5}$$

where $\mathcal{H}_N$ is the Hermite polynomial of order N.

The ratio $\mathfrak{N}_N/\mathfrak{N}_{N-1}$ is determined by requiring Eqs. (B3) to be the usual recurrence relations for Hermite polynomials. We find that

$$\mathfrak{N}_{N+1}(\tau)=2^{-1/2}(N+1)^{-1/2}(+i)\left[H_0^{(2)}\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\right]^{1/2}e^{-i\pi/4}\left[H_0^{(1)}\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\right]^{-1/2}\mathfrak{N}_N(\tau). \tag{B6}$$

By comparing Eq. (B2) for $N=0$ with the known result [Eq. (43)] for $\psi_{n,0}$ we see that

$$\mathfrak{N}_0=\left(\frac{2}{\pi}\right)^{1/2}p_\lambda^{1/4}e^{-i\pi/8}\left[H_0^{(1)}\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\right]^{-1/2}. \tag{B7}$$

Thus

$$\mathfrak{N}_N(\tau)=\left(\frac{2}{\pi}\right)^{1/2}e^{-iN\pi/4}e^{-i\pi/8}2^{-N/2}(+i)^N(N!)^{-1/2}p_\lambda^{1/4}\left[H_0^{(2)}\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\right]^{N/2}\left[H_0^{(1)}\left(\frac{|n|}{2p_\lambda}e^{2\tau}\right)\right]^{-(N+1)/2}. \tag{B8}$$

Substituting Eqs. (B8) and (B5) in Eq. (B2) yields Eqs. (40) as the N-quantum wave function.

APPENDIX C: THE OVERCOMPLETE FAMILY OF STATES (OFS)

In his work on continuous-representation theory[12] Klauder has defined an overcomplete family of states as normalized vectors in a Hilbert space which lie arbitrarily close to each other (nonorthogonal) and span the Hilbert space. As an example of an OFS he gives[12] one-dimensional phase space characterized by

$$[\hat{q},\hat{p}]=i, \tag{C1}$$

where the caret denotes operator. The elements of this OFS are the vectors in the Hilbert space of solutions of the free-particle Schrödinger equation

$$i\partial\psi/\partial t=-\tfrac{1}{2}\partial^2\psi/\partial q^2. \tag{C2}$$

We choose a fiducial normalized state Φ_0 which is a solution of Eq. (C2) and construct a unitary operator $U(p_0,q_0)$ such that

$$\Phi(p_0,q_0)=U(p_0,q_0)\Phi_0 \tag{C3}$$

is also a solution of Eq. (C2). For

$$U(p_0,q_0)\equiv e^{i\alpha}e^{-iq_0\hat{p}}e^{ip_0\hat{q}} \tag{C4}$$

the $\Phi(p_0,q_0)$ constitute an OFS for any choice of real α.[12]

For the Gowdy T^3 universe, we wish to show that the $|p_n^0q_n^0\rangle$ defined by Eq. (56) constitute an OFS (for each n,p_λ). For the choice of fiducial vector

$$\Phi_0=|p_n^0=0q_n^0=0\rangle\equiv\lim_{\tau\to-\infty}\psi_{n,0}, \tag{C5}$$

we can show that

$$|p_n^0q_n^0\rangle=U(p_n^0,q_n^0)\Phi_0 \tag{C6}$$

for

$$U=\exp[-\tfrac{1}{2}(p_n^0)^2\bar{\tau}/p_\lambda-i(q_n^0+p_n^0\bar{\tau}/p_\lambda)\hat{p}_n+ip_n^0\hat{q}_n] \tag{C7}$$

at each value $\bar{\tau}$. Comparing Eqs. (C7) and (C4) shows that there is a 1 to 1 correspondence between the solution states for the Gowdy T^3 universe near $\tau=-\infty$—i.e., between solutions of Eq. (53)—and phase space—Eqs. (C1)-(C4). Thus the $|p_n^0q_n^0\rangle$ constitute an OFS. This is not surprising since Eq. (C2) is the same as Eq. (53) for $t=\bar{\tau}/p_\lambda$.

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[1]R. H. Gowdy, Phys. Rev. Lett. 27, 826 (1971); Ann. Phys. (N. Y.) 83, 203 (1974).

[2]B. K. Berger, Ph.D. thesis, Univ. of Maryland Center for Theoretical Physics Report No. 73-024, (1972) (unpublished).

[3]B. K. Berger, Ann. Phys. (N. Y.) 83, 458 (1974).

[4]C. W. Misner, Phys. Rev. D 8, 3271 (1973). The notation of this paper will be used in most of the following.

[5]C. W. Misner, in *Confrontation of Cosmological Theories with Observational Data*, I. A. U. Symposium No. 63, edited by M. S. Longair (Reidel, Dordrecht, 1974).

[6]L. Parker, Phys. Rev. 183, 1057 (1969); Phys. Rev. Lett. 28, 705 (1972), and the references therein.

[7]Ya. B. Zel'dovich and A. A. Starobinsky, Zh. Eksp. Teor. Fiz. 61, 2161 (1971) [Sov. Phys.—JETP 34, 1159 (1972)]. See also B.-L. Hu, Phys. Rev. D 9, 3263 (1974).

[8]The problem of the effect of created particles on the metric has been studied by L. Parker and S. A. Fulling, Phys. Rev. D 7, 2357 (1973), and V. N. Lukash and A. A. Starobinsky, Order of Lenin Institute of Applied Mathematics, Academy of Sciences, U. S. S. R., Report No. 23, 1974 (unpublished).

[9]Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. Pis'ma Red. 12, 443 (1970) [JETP Lett. 12, 307 (1971)].

[10]D. R. Brill and R. H. Gowdy, Rep. Prog. Phys. 33, 413 (1970).

[11]E. Salusti and F. Zirilli, Nuovo Cimento Lett. 4, 999 (1970).

[12]J. R. Klauder, J. Math. Phys. 4, 1055 (1963); 4, 1058 (1963).

[13]The results of Sec. II follow those of Misner in Ref. 4.

[14]See Refs. 2, 3, and 4 for a discussion of the interpretation of the variables p_λ and $|n|e^{2\tau}$.

[15]See, for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1955), 2nd ed.

[16]See, for example, E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1961).

[17]See, for example, C. W. Misner, Phys. Rev. 186, 1319 (1969), and Refs. 2 and 3.

[18]B. K. Berger, Phys. Rev. D (to be published).

[19]See, for example, J. A. Wheeler, in *Batelle Rencontres*, edited by C. M. DeWitt and J. A. Wheeler (Benjamin, New York, 1968).

[20]Such mode interactions have been discussed with reference to particle creation in B.-L. Hu, S. A. Fulling, and L. Parker, Phys. Rev. D 8, 2377 (1973).

[21]Since Eq. (A3a) is a Ricatti equation, Eq. (A4) is not the general solution. The general solution for arbitrary ω_n is given in Ref. 10. It can be shown that the boundary condition (38) restricts the solution such that the remaining terms do not contribute.