Faraday rotation near charged black holes and other electrovacuum geometries

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In space permeated by a steady background electromagnetic field a gravitational wave and an electromagnetic wave not only undergo beat frequency oscillations, but the linear polarizations of these waves undergo Faraday rotations as well. The beating and the Faraday rotations are inextricably related. The classification of these phenomena requires three parameters, the three Euler parameters of SU(2). They specify in a more general sense the "polarization" of an electrograviton mode. The evolution of the beat frequency oscillations and the Faraday rotations along a propagating wave front is described as a moving point in SU(2). Consequently, a charged black hole serves not only as a catalyst for converting suitably directed electromagnetic radiation into gravitational radiation, but also as an agent that randomized the linear polarizations of radiation emerging from it. An assessment of these phenomena in relation to the origin of Weber's signals is given.

I. INTRODUCTION AND SUMMARY

Discussions about the propagation of gravitational or electromagnetic waves in a gravitational field are mostly presented within the context where any effects due to a steady background electromagnetic field are ignorable. Although for most earthbound or astrophysical wave phenomena such a selective focus of attention is probably often quite adequate, the recent discovery of black holes makes it difficult to ignore steady background electromagnetic fields associated with these objects. Under suitable conditions rotating black holes[1] (and neutron stars[2]) are capable of acquiring a net charge and thus can be endowed with an ambient steady electromagnetic field.

That electromagnetic waves propagating in such a background change the electromagnetic stress energy and thereby are (according to the Einstein field equations) a cause of gravitational waves with identical propagation properties probably comes with little surprise.[3] That gravitational waves propagating in such a background act back on the electromagnetic stress energy and thereby cause perturbations in it—perturbations which according to Maxwell's field equations are electromagnetic waves with identical propagation properties—is true and very consequential.

The interaction between the two types of waves with identical propagation properties—an interaction evidently caused by the steady background electromagnetic stress energy—gives rise to the "resonance phenomenon" between coupled systems. The "resonance phenomenon" between simple systems of a single degree of freedom is well known.[4] Here, however, the two systems are waves—not merely scalar waves, but even waves with polarization properties. All evolutionary properties of interacting waves can be found once the normal modes are determined. For high frequencies, however, the physically most interesting features are discussed not directly in terms of normal modes but rather in terms of the individual waves "resonating" (i.e., exchanging energy) among each other.

The purpose of this article is to determine the normal modes of the interacting gravitational and the electromagnetic waves and, hence, to exhibit the associated resonance phenomenon. It manifests itself, as we shall see, either as (1) beat frequency oscillations between the two waves,[5] or as (2) Faraday rotations of their polarization, or as (3) a very classifiable mixture of the two phenomena.

The Faraday rotations seem to be a new effect.[6]

The beating phenomenon is characterized by (i) a periodic total interconversion of gravitons and photons and (ii) the parallel transport of the respective polarizations of the two waves, a transport which conserves the total number of electrogravitons consisting of linearly polarized photons and gravitons of one type and also the total number of electrogravitons consisting of linearly polarized photons and gravitons of another (orthogonal to the first) type.

The Faraday rotations are characterized by (i) a periodic total interconversion of vertically and horizontally polarized photons (and also of up-down and diagonally polarized gravitons) and (ii) a transport which conserves the total photon number and also the total graviton number.

The mixture of these two phenomena is characterized by (i) a periodic partial interconversion of gravitons and photons combined with (ii) Faraday rotations of linear polarizations of the waves traveling through the electromagnetic background field.

The rates of (1) the interconversion, (2) the Faraday rotation of the linear polarizations of each wave, and (3) the mixed processes are all identical and are strictly locally (eventwise) determined. The periodic evolution of these phenomena along a null ray depends on the local background electromagnetic field and the propagation direction along the ray history of a wave; it is not defined in terms of an asymptotically Minkowskian frame.

Regardless of which mode one is considering, beat frequency, Faraday rotations, or a mixture of the two, the coupling between the linearized Einstein equations and the perturbed Maxwell equations is such that there is only a pairwise energy exchange among suitable linear combinations of the following four waves: (1) horizontally (linearly) polarized electromagnetic waves, (2) vertically (linearly) polarized electromagnetic waves, (3) up-down (linearly) polarized gravitational waves, and (4) diagonally (linearly) polarized gravitational waves.

This article can be summarized briefly by saying that we solve the coupled Einstein-Maxwell wave equations in the high-frequency (WKB) approximation and observe that their solutions, which govern all the above-mentioned phenomena, are most efficiently exhibited and classified as members of the group SU(2).

In Sec. II the linearized coupled Einstein-Maxwell equations are exhibited in vector (and tensor) component form.

In Sec. III scalar wave functions and the corresponding linear polarizations are introduced for the vector (electromagnetic) and tensor (gravitational) waves. This introduction generalizes an analogous approach by Isaacson[7] and Misner[7] to the uncoupled Maxwell field equations and linearized Einstein field equations. The coupled equations are then rewritten in terms of two complex scalar wave equations.

In Sec. IV the two coupled complex Einstein-Maxwell equations first are written more compactly in terms of a complex spinor wave equation, then are decoupled [in the short-wavelength (geometrical optics, WKB) approximation] by introducing their normal modes, and thereby are rewritten in terms of (1) the standard wave equation [Eq. (12)] which governs the propagation properties of the waves and (2) the evolution equation [Eq. (13)] for the interaction transformation, which relates the normal modes to the complex Einstein-Maxwell wave function. Solutions to this evolution equation are discussed in terms of the group SU(2) and a typical solution is exhibited [Eq. (15)].

In Sec. V the normal modes and hence the solutions to the coupled Einstein-Maxwell equations are written down. The solutions are actually exhibited for the class of special cases mentioned in Sec. IV.

In Sec. VI the normal modes are classified in terms of curve segments in SU(2). The classification (Fig. 1 and Fig. 2) embodies the three general features of electromagnetic and gravitational waves propagating through a steady electromagnetic and gravitational background: (1) beat frequency oscillations, (2) Faraday rotations, and (3) a mixture of these two phenomena.

In Sec. VII Weber's signals are assumed to have their origin in our galactic center; in light of this assumption some of those chief difficulties which are removed and also those which are created by attributing the signals as coming from a charged black hole are delineated and discussed.

II. PERTURBED EINSTEIN-MAXWELL FIELD EQUATIONS

Focus attention on the linearized coupled Einstein-Maxwell equations in the WKB approximation[5,7,8]:

$$\nabla^m\nabla_m\psi^a = -F_{pm}\nabla^m\psi^{pa} \ ,$$

$$\nabla^m\nabla_m\psi_{ab} = -\frac{16\pi G}{c^4}\,\frac{-1}{4\pi}\,(F_{am}\nabla^m\psi_b + F_{bm}\nabla^m\psi_a + \nabla_a\psi^m F_{mb} + \nabla_b\psi^m F_{ma} + g_{ab}F_{mn}\nabla^m\psi^n) \ .$$

These equations govern the propagation of disturbances of a steady background vacuum geometry g_{ab} permeated by a steady background electromagnetic field F_{am}. The disturbances are related to the above gravitational potentials ψ_{ab} and the electromagnetic potentials ψ_a by

$$\delta F_{ab} = \nabla_a\psi_b - \nabla_b\psi_a \ ,$$

$$\delta g_{ab} = \psi_{ab} - \tfrac{1}{2}\, g_{ab}\psi_m{}^m \ .$$

The equations are most easily dealt with in terms of dimensionless potentials

$$\bar\psi_a = \left(\frac{8G}{c^4}\right)^{1/2}\psi_a \ ,$$

$$\bar\psi_{ab} = \psi_{ab} \ ,$$

and the electromagnetic background in geometrical units (cm^{-1}),

$$\bar F_{ab} = \left(\frac{8G}{c^4}\right)^{1/2} F_{ab} \ .$$

Introducing these variables and dropping the bars, the equations reduce to

$$\nabla^m\nabla_m\psi^a = -F_{pm}\nabla^m\psi^{pa} \qquad (1a)$$

for the perturbed Maxwell field equations and

$$\nabla^m \nabla_m \psi_{ab} = \tfrac{1}{2}(F_{am}\nabla^m \psi_b + F_{bm}\nabla^m \psi_a) + \tfrac{1}{2}(F_{ma}\nabla_b \psi^m + F_{mb}\nabla_a{}^m) - \tfrac{1}{2} g_{ab} F_{mn}\nabla^n \psi^m \tag{1b}$$

for the perturbed Einstein field equations. It is evident that these equations preserve, as they must, both the traceless nature of ψ_{ab}, i.e., $\psi^b_b = 0$, and the Lorentz conditions $\nabla^a \psi_{ab} = 0$ and $\nabla^a \psi_a = 0$.

III. WKB FORMULATION OF THE COUPLED EINSTEIN-MAXWELL WAVE EQUATIONS

Generalizing Isaacson's and Misner's WKB treatment[7] of the perturbed Einstein field equation and the Maxwell field equations, focus first on a Maxwell wave whose propagation vector is k_a. That wave is described completely by specifying (a) its propagation vector field and (b) its two scalar amplitudes for its two linear polarizations along two spacelike unit vectors parallel propagated along k_a; in other words, a Maxwell wave and an Einstein wave are given, respectively, by

$$\psi^a = \psi^1_E e^a_1 + \psi^2_E e^b_2 \tag{2a}$$

and

$$\psi^{ab} = \psi^1_G e^{ab}_1 + \psi^2_G e^{ab}_2 \ . \tag{2b}$$

Here the basis vectors and tensors are orthonormal (thus, $e^{ab}_1 e_{1ab} = 1$, $e^a_1 e_{1a} = 1$, etc.) and are parallel propagated along the mutual wave propagation vector k_a of ψ^a and ψ^{ab}. They are related to one another by

$$\begin{aligned} e^{ab}_1 &= \frac{1}{\sqrt{2}}(e^a_1 e^b_1 - e^a_2 e^b_2) \ , \\ e^{ab}_2 &= \frac{1}{\sqrt{2}}(e^a_1 e^b_2 + e^b_1 e^a_2) \ , \\ e_{1a} e^{ab}_1 &= e_{2a} e^{ab}_2 = \frac{1}{\sqrt{2}} e^b_1 \ , \\ e_{1a} e^{ab}_2 &= - e_{2a} e^{ab}_1 = \frac{1}{\sqrt{2}} e^b_2 \ . \end{aligned} \tag{3}$$

The linear polarization amplitudes in Eqs. (2a) and (2b) are, of course, given by transvection against the respective basis vectors and tensors. Thus, in the absence of a background electromagnetic field, the unit polarization vectors and tensors are parallel transported along the direction of the propagating wave,[7] i.e., the polarizations coincide with parallel-transported basis vectors, and the ratio of the two polarization amplitudes is a constant along the null ray. If on the other hand a background electromagnetic field is present, then Eqs. (1a) and (1b) examined by the usual WKB approach[7] reveal that in general neither the electromagnetic nor the gravitational polarization is parallel-transported along the history of a wave crest. It follows that the ratio of the polarization amplitudes in each of Eqs. (2a) and (2b) is not a constant. To determine how they vary introduce these equations into the coupled Einstein-Maxwell wave equations (1a) and (1b), use Eqs. (3), and obtain the four coupled Einstein-Maxwell wave equations[9,10] for the scalars ψ^1_E, ψ^2_E (electromagnetic) and ψ^1_G, ψ^2_G (gravitational):

$$\nabla^m \nabla_m \psi^1_E = -\frac{i}{\sqrt{2}}(\psi^1_G e_1 \cdot f + \psi^2_G e_2 \cdot f) \ , \tag{4a}$$

$$\nabla^m \nabla_m \psi^2_E = -\frac{i}{\sqrt{2}}(-\psi^1_G e_2 \cdot f + \psi^1_G e_1 \cdot f) \ , \tag{4b}$$

$$\nabla^m \nabla_m \psi^1_G = +\frac{i}{\sqrt{2}}(\psi^1_E e_1 \cdot f - \psi^2_E e_2 \cdot f) \ , \tag{5a}$$

$$\nabla^m \nabla_m \psi^2_G = +\frac{i}{\sqrt{2}}(\psi^1_E e_1 \cdot f + \psi^2_E e_2 \cdot f) \ . \tag{5b}$$

Here the projections of the vector $F_{am}k^m$ onto the two basis vectors e^a_1 and e^a_2 are given by the inner products

$$e_1 \cdot f = e^a_1 F_{am} k^m \ ,$$
$$e_2 \cdot f = e^a_2 F_{am} k^m \ .$$

The wave propagation vector k^m arises from the WKB approximation for the gradient of the four scalar wave functions ψ^1_E, ψ^2_E, ψ^1_G, ψ^2_G, as for example in

$$\nabla_m \psi^1_E = i k_m \psi^1_E, \text{ etc.}$$

To solve the Einstein-Maxwell scalar equations (4) and (5) introduce the complex-wave-function electromagnetic and gravitational wave functions

$$\Phi_E = \psi^1_E + i\psi^2_E \ , \tag{6a}$$

$$\Phi_G = \psi^1_G + i\psi^2_G \ . \tag{6b}$$

In terms of these the Einstein-Maxwell wave equations (4) and (5) become

$$\nabla^m \nabla_m \Phi_E = -i\frac{|f|}{\sqrt{2}} e^{-i\alpha}\Phi_G \ , \tag{7a}$$

$$\nabla^m \nabla_m \Phi_G = +i\frac{|f|}{\sqrt{2}} e^{i\alpha}\Phi_E \ . \tag{7b}$$

Here we introduce the angle α between the vector $f_a = F_{am}k^m$ and the parallel transported basis vector e^a_1 and also the magnitude $|f|$ of that vector[11]:

$$\begin{aligned} e_1 \cdot f &= e^a_1 F_{am} k^m = |f| \cos\alpha \ , \\ e_2 \cdot f &= e^a_2 F_{am} k^m = |f| \sin\alpha \ . \end{aligned} \tag{8}$$

IV. EQUATIONS DECOUPLED BY NORMAL MODES

The solutions to the coupled complex Einstein-Maxwell wave equations (7a) and (7b),

$$\nabla^m\nabla_m \begin{pmatrix} \Phi_E \\ \Phi_G \end{pmatrix} = \frac{|f|}{\sqrt{2}} \begin{pmatrix} 0 & -ie^{-i\alpha} \\ ie^{i\alpha} & 0 \end{pmatrix} \begin{pmatrix} \Phi_E \\ \Phi_G \end{pmatrix}, \tag{9}$$

are found in terms of the normal modes Φ_1 and Φ_2,

$$\begin{pmatrix} \Phi_E \\ \Phi_G \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \equiv T \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \tag{10}$$

In the WKB approximation the transformation matrix T has only "slowly" varying entries. Only the wave functions contain the rapidly varying dynamical phase factor $\exp(iS)$. In terms of the normal modes the coupled Einstein-Maxwell system, Eq. (9), is

$$T\nabla^m\nabla_m \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = -2ik_m \nabla^m T \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} + \frac{|f|}{\sqrt{2}} \begin{pmatrix} 0 & -ie^{-i\alpha} \\ ie^{i\alpha} & 0 \end{pmatrix} T \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}. \tag{11}$$

The functions Φ_1 and Φ_2 constitute a normal mode if and only if T is that nonsingular transformation which makes the right-hand side a multiple of the right-hand side of Eq. (10). But one can, and we shall, require without loss of generality that this multiple (the "eigenvalue") be equal to zero. Consequently, the coupled equations (11) become

$$\nabla^m\nabla_m \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = 0, \tag{12}$$

provided the transformation T, which relates any of the normal modes to the complex Einstein-Maxwell modes, satisfies the differential equation

$$\frac{dT}{df} + \frac{i}{2}(\vec{\sigma}\cdot\vec{n})\,T = 0. \tag{13}$$

Here the derivative with respect to the parameter f,

$$\frac{dT}{df} = \frac{dx^m}{df}\nabla_m T = \frac{k^m}{(\frac{1}{2}F^a_{\ m}F_{an}k^m k^n)^{1/2}}\nabla_m T,$$

has an invariant significance and evidently is independent of the scale of the components k^m. As we shall see, it measures the evolution of (i) the beat frequency oscillations, or (ii) the Faraday rotation, or (iii) a mixture of these two phenomena along a null ray determined by a complex normal mode satisfying Eq. (12). The matrix $\vec{\sigma}\cdot\vec{n}$ is the Hermitian matrix exhibited in Eq. (11). In terms of the Pauli spin matrices it is

$$\vec{\sigma}\cdot\vec{n} = -\sin\alpha\,\sigma_x + \cos\alpha\,\sigma_y. \tag{14}$$

Consequently, the solution to Eq. (13) constitutes a one-parameter family of unitary 2×2 matrices, i.e., a curve in SU(2) whose tangents[12] are given by Eq. (14). Observe, for example, that if the spatial vector field $F_{am}k^m$ is parallel along a given null ray, i.e., if the angle α in Eq. (8) does not change along the null ray,[13] then $\vec{\sigma}\cdot\vec{n}$ is constant and the solutions to Eq. (13) constitute simply segments of great circles[13] of SU(2),

$$T(f) = e^{-i\vec{\sigma}\cdot\vec{n}f/2}. \tag{15}$$

The starting point of this segment is the identity matrix. The end point of the segment is determined by the limiting value of the parameter that measures the progress of the beating and Faraday rotation along the null ray,

$$f_{\text{final}} = \int_{\text{null ray}} \left(\tfrac{1}{2}F^a_{\ m}F_{an}\frac{dx^m}{d\lambda}\frac{dx^n}{d\lambda}\right)^{1/2} d\lambda. \tag{16}$$

It goes without saying that for nonconstant angles α in Eq. (14) the solutions to Eq. (13) are no longer closed great circles in SU(2) but rather curves that presumably fill the SU(2) manifold in some ergodic fashion, provided of course that the curve parameter[14] in Eq. (16) is long enough.

V. SOLUTIONS TO THE COUPLED EINSTEIN-MAXWELL WAVE EQUATIONS

Having solved Eq. (13) for the unitary transformation matrix, we can exhibit the complex Einstein-Maxwell field, Eq. (10), in terms of the normal modes. The general normal mode, a solution to Eq. (12), is the product of a constant complex spinor with a real-valued wave function, Φ, which of course also satisfies the wave equation $\nabla_m\nabla^m\Phi = 0$:

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \begin{pmatrix} a+ib \\ c+id \end{pmatrix}\Phi. \tag{17}$$

Here the complex spinor with constant entries is normalized to unity,

$$a^2+b^2+c^2+d^2=1.$$

This normalization is preserved by the transformation matrices $T(f)$. Such a complex spinor can be effectively parametrized by three Euler parameters s, t, and u:

$$\begin{pmatrix} a+ib \\ c+id \end{pmatrix} = e^{i\sigma_y s/2}e^{i\sigma_z t/2}e^{i\sigma_y u/2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{18}$$

Consequently, the general solution to the coupled Einstein-Maxwell wave equation is

$$\begin{pmatrix}\Phi_E\\ \Phi_G\end{pmatrix} = T(f)\, e^{i\sigma_y s/2}\, e^{i\sigma_z t/2}\, e^{i\sigma_y u/2}\begin{pmatrix}\Phi\\ 0\end{pmatrix} \equiv T(f)\, T_0 \begin{pmatrix}\Phi\\ 0\end{pmatrix}. \quad (19)$$

To illustrate what physical phenomena this solution embodies, consider the special case where the evolution matrix $T(f)$ is given by Eq. (15). In that case the vector $F_{am}k^m$ is parallel all along the null ray. Without loss of generality one may set the constant angle $\alpha=0$. According to Eqs. (14), (15), and (19), the complex waves are therefore

$$\Phi_E = [\cos\tfrac{1}{2}t\cos\tfrac{1}{2}(s-f+u) + i\sin\tfrac{1}{2}t\cos\tfrac{1}{2}(s-f-u)]\Phi \equiv [\cos\theta\, e^{i\delta_E}]\Phi\,, \quad (20a)$$

$$\Phi_G = [-\cos\tfrac{1}{2}t\sin\tfrac{1}{2}(s-f+u) - i\sin\tfrac{1}{2}t\sin\tfrac{1}{2}(s-f-u)]\Phi \equiv [\sin\theta\, e^{i\delta_G}]\Phi\,. \quad (20b)$$

By comparing this expression with Eqs. (6a) and (6b) and equating real and imaginary parts, we obtain the scalar wave functions for the four respective linear polarizations of a Maxwell-Einstein wave,

$$\psi_E^1 = \cos\theta\cos\delta_E\Phi\,,$$
$$\psi_E^2 = \cos\theta\sin\delta_E\Phi\,,$$
$$\psi_G^1 = \sin\theta\cos\delta_G\Phi\,,$$
$$\psi_G^2 = \sin\theta\cos\delta_G\Phi\,.$$

Here we have introduced not only the angles δ_E and δ_G for the linear polarizations of the electromagnetic and the gravitational waves, respectively, but also the beating phase angle θ, which determines the ratio of the electromagnetic to the gravitational amplitude. Reference to Eqs. (20) now makes evident the fact that the evolution, the beat phenomenon, and the Faraday rotation of the polarization can be most generally characterized as the motion of a point on the three-sphere, the group SU(2). As the wave crest of an electrograviton mode traces out its history along a null ray, Eq. (19) asserts that there is a corresponding point, $T(f)T_0$, in SU(2) which moves along a curve determined by Eq. (13). For the illustrative case under consideration, Eqs. (20a) and (20b), the point moves along a great circle; how far is determined by the value of the final curve parameter given by Eq. (16).

VI. CLASSIFICATION OF SOLUTIONS

Given a specific null ray, how do we classify the various beat frequency and Faraday rotation (normal) modes of the waves whose propagation vector k^m lies along the null ray? A glance at Eqs. (20a) and (20b) reveals that the two appropriate parameters that distinguish one normal mode from another are the Euler parameters u and t. Their values distinguish one great circle from another. The third Euler parameter, $s-f$, measures the position of the moving point along a given circle. How does one best concretize these circles in terms of the evolution of the amplitude and the linear polarization of the electromagnetic and gravitational waves? Consider in SU(2) any two-dimensional surface spanned by the two Euler parameters u and t, a torus. Every possible normal mode is characterized by a curve that pierces this torus. It follows that each and every propagating mode under consideration is represented by a point on the torus. A chart of this surface together with the essential physical attributes that each point refers to is given in Fig. 1 and Fig. 2. These figures and the above discussions can be summarized by the following remarks:

(1) In the absence of a background electromagnetic field electromagnetic waves and gravitational waves propagate independently of each other. The linear polarizations of these waves are parallel propagated with a wave crest (along a null ray). Consequently, their polarizations can be depicted on two circles as static points characterized by two angles.

(2) In the presence of a steady background electromagnetic field the electromagnetic and gravitational waves beat against each other. Their relative amplitude changes periodically. Furthermore, the two linear polarizations rotate slowly as they are carried by a wave crest along its null ray. This rotation of the polarizations is a cumulative effect just as Faraday rotation in a plasma permeated by a steady magnetic field is cumulative. Evidently the two phenomena, Faraday rotation of gravitational as well as electromagnetic polarized radiation and the beating phenomenon, are inextricably related. Their classification requires three parameters, the Euler parameters. They specify in a more general sense the "polarization" of an electrograviton mode. That "polarization" is initially represented by a complex spinor, Eq. (18), of unit magnitude. Its change along the null ray is depicted in Eq. (19) as a point moving in SU(2).

VII. DISCUSSION

Should Weber's signals actually be due to bursts of gravitational radiation coming from our galactic center[15] then it follows that this radiation must be due to synchrotron modes of excitation.[16] The most likely candidate for such an excitation mechanism examined so far is that of a particle spiraling[17–20] into a rotating massive uncharged black hole. The results are that (a) the radiation must be highly polarized and (b) the radiated power for such (tensor) radiation is distressingly small. These two consequences of the synchrotron mechanism are in direct conflict with experiment, which requires the gravitational radiation to have (a) a polarization not in excess of 40% (see Ref. 21) and (b) a power much larger than that attainable from any of the uncharged rotating black holes examined so far.[17]

The results of this paper indicate that at least the first and possibly both of the two aforementioned difficulties do not arise if one considers charged black holes instead. In other words, (a) the polarizations of suitably excited gravitational and electromagnetic waves undergo Faraday rotations in the vicinity of the charged black hole and (b) accompanying the Faraday rotations, the two waves undergo an already-reported mutual interconversion process[5] (beat frequency oscillations) which keeps interchanging the energies in the two waves as long as they are traveling in space permeated by a steady electromagnetic field. The Faraday rotations randomize the radiation emerging from the vicinity of the charged black hole.

However, the removal of these difficulties is obtained at the seemingly very heavy price of understanding how a large charged black hole can be formed or how, once formed, it acquired additional charge. Assume the price has been paid. Even then, understanding the radiation mechanics presents a very nontrivial problem. In order to reconcile the strength of Weber's signals with ob-

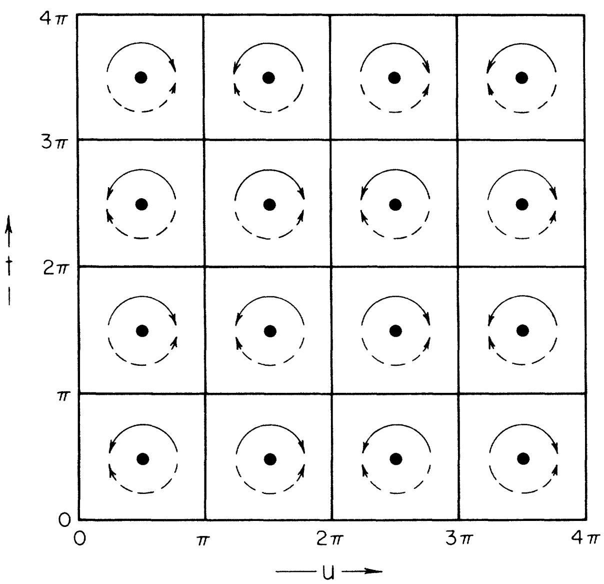

FIG. 1. Space of solutions to the coupled Einstein-Maxwell wave equations. The u and the t coordinates are the Euler parameters that subtend a two-dimensional submanifold, a torus, of SU(2). Each point on this torus is the initial point of a curve that describes the evolution of the beating and the Faraday rotations along a wave's history. The solid lines constitute the starting points for the 100% beat frequency modes (no Faraday rotation) already found in Ref. 9 where the linear polarization angles δ_E and δ_G are constants for the electromagnetic and the gravitational wave along the whole given null ray. The heavy dots, on the other hand, are the starting points for the 100% Faraday rotation modes which undergo no beating. The arrows around these dots refer to the changing of the linear polarizations as a wave crest travels with the speed of light along the null ray. The solid arrow refers to the changing angle δ_E of the electromagnetic polarization; the dashed arrow refers to the changing δ_G of the gravitational polarization. For details of the classification see Fig. 2.

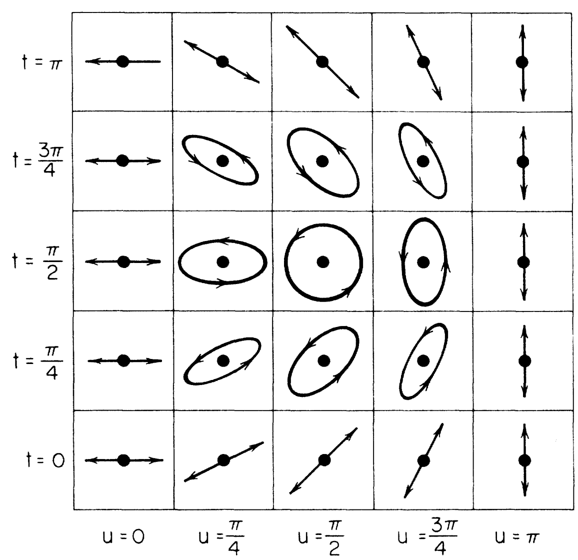

FIG. 2. A collection of 25 tracings of the tip of the electromagnetic linear polarization. The to-be-imagined x and y axes in each picture are proportional to the real scalar amplitudes ψ_E^1 and ψ_E^2 of the two linear polarizations. Each picture portrays one of the solutions characterized by the Euler parameters u and t, a typical ellipse referring to the electromagnetic part of an electrograviton mode. The linear polarization not only rotates slowly (Faraday rotation) along the history (null ray) of a wave crest, but the whole amplitude of the electromagnetic wave varies in proportion to the distance from the central dot of the picture to the elliptical trace of the tip of the slowly rotating linear polarization. If the vector $F_{am}k^m$ in Eq. (8) does not stay parallel along the null ray, then the coefficient matrix of the differential Eq. (13) is not constant. Consequently, the curves in SU(2) are no longer great circles and the ellipses depicted here become Lissajous figures.

servational limits on the mass loss from the galactic center, it is necessary that the radiation be beamed in the galactic plane. Gravitational synchrotron radiation has precisely this property. However, such radiation requires that the moving source be highly relativistic (at least near an uncharged rotating black hole) in a well-defined sense,[19] and no one, it seems, has yet been able to identify an astrophysically reasonable process which furnishes such relativistic sources. Whether this identification is made easier by virtue of the fact that one is focusing on charged matter moving in the geometry of a charged or even a rotating charged black hole is not obvious.

In any case, should the net charge of a black hole play an important role in furnishing the (presumably necessary) ultrarelativistic sources of synchrotron-type radiation, then one is immediately confronted with very rapid charge neutralization of the black hole. This would definitely be the case for nonrotation black holes. For a rotating charged black hole, however, there is the possibility that rotation may prevent charge neutralization or may even be responsible for the acquisition of a net charge by means of an as-yet-unspecified process.[2,22]

If such a process exists, it could very well solve both problems, i.e., it could be responsible (a) for the creating and maintenance of a net charge of a rotating black hole and (b) for the furnishing of the ultrarelativistic sources. The attractiveness of such a process has its roots in the huge amount of extractable energy stored in a rotating black hole,[23] if Weber's experiments in fact do indicate a black hole at the center of our galaxy. The essence of that to-be-discovered process is that it is expected to be a catalytic mechanism which trades the rotational energy of the black hole for electrical energy,[24] i.e., angular momentum for charge. It would presumably be the net charge of the black hole that is responsible for putting particles into ultrarelativistic orbits, which would radiate electromagnetic and/or gravitational synchrotron radiation in the galactic plane.

How much charge must a black hole have in order that the Faraday rotations and the interconversion process be appreciable? Consider a charged Kerr black hole with mass M^*, charge Q^*, and rotation parameter a^* (geometrical units) situated in the center of our galaxy. A straightforward analysis based on Eq. (16), or on Eq. (26) in Ref. 5, reveals that the total amount of beating phase and/or Faraday rotation angle that a null ray undergoes as it emerges from its radius of closest approach r near the black hole to a distant observer is

$$\Delta\theta,\ \Delta\delta_E,\ \Delta\delta_G \geq \frac{Q^*}{r^2} M^* .$$

If the signal emerges from the ergosphere, $r \sim M^*$, then it follows that a substantial beating and Faraday rotation requires that the net charge, Q^*/M^*, be substantial (unless, of course, most of the null rays start out by spiraling out from their unstable quasibound orbits). Parenthetically, one may notice that a Kerr black hole endowed with a substantial amount of charge, say $Q^*/M^* \geq 0.5$, must have a mass of at least $M^* \geq 3.36\times10^6 M_\odot^*$ (solar masses) if the vacuum near the event horizon is not to break down because of spontaneous electron-positron formation.

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[1] R. M. Wald, Phys. Rev. D 10, 1680 (1974).

[2] For ways of overcoming charge neutralization of Newtonian neutron stars see P. Goldreich and W. H. Julian, Astrophys. J. 157, 869 (1968); R. Ruffini and A. Treves, Astrophys. Lett. 13, 109 (1973).

[3] M. E. Gertzenshtein (Zh. Eksp. Teor. Fiz. 41, 113 (1961) [Sov. Phys.—JETP 14, 84 (1962)]) seems to be the first one to recognize the importance of resonance in converting electromagnetic into gravitational radiation.

[4] For a discussion of the notion "resonance" between different degrees of freedom see, for example, L. Pauling and L. B. Wilson, *Introduction to Quantum Mechanics* (McGraw-Hill, New York, 1935).

[5] U. H. Gerlach, Phys. Rev. Lett. 32, 1023 (1974).

[6] The author has been kindly informed that N. R. Sibgatullin (Zh. Eksp. Teor. Fiz. 66, 1187 (1974) [Sov. Phys.—JETP (to be published)]) has also considered the "interaction between short gravitational and electromagnetic waves in arbitrary external electromagnetic fields." The primary differences between his work and ours lie in the facts that (1) the basic equations that govern beat frequency oscillations and Faraday rotations are not arrived at by starting with a generalization of Isaacson's and Misner's WKB approach, but rather by starting with the Newman-Penrose tetrad formalism in the WKB approximation; and (2) we give an SU(2) classification of the solutions to the coupled Einstein-Maxwell system.

[7] R. A. Isaacson, Phys. Rev. 166, 1263 (1969).

[8] In these perturbed equations we have already assumed that the potentials on the left-hand side satisfy the Lorentz gauge and that the gravitational potential is traceless. In this paper Latin subscripts range over 0, 1, 2, 3.

[9] Equations (4) and (5) together with Eqs. (2) (all in the present paper) generalize an analogous set of equations in Ref. 5 [Eqs. (14) and (6), respectively].

[10] There is a numerical error in an analogous set of equations given in Ref. 5. Equations (14a) and (14b)

should not have the factor of 2 given there. The result of that change is that the beating phase should be only one half that given by Eq. (22) in that reference. Furthermore, in Eqs. (11a) and (11b), contrary to remarks preceding those equations, the resonance propagation vector α_m is constructed out of real (i.e., linear) polarizations only.

[11]The squared magnitude $f_a f^a$ of f_a reduces to $f_a(e_1^a e_1^b + e_2^a e_2^b) f_b$, which is the squared magnitude of the projection of f_a onto the plane spanned by e_1^a and e_2^a.

[12]For a discussion of the structure and geometry of SO(3), which is locally isomorphic to SU(2), see, for example, Problems 10.17 and 11.12 in C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[13]In this case we align, for convenience, the parallel transported vector e_1^a in Eq. (2a) and in Eq. (8) with f_a so that $\alpha = 0$.

[14]For the case of a pure beat frequency (photon-graviton interconversion) mode this parameter is the *beating* phase introduced in Ref. 5 [Eq. (22)]. There, however, the more general case (where the vector $f_a = F_{am} k^m$ does not keep its direction constant in the parallel-transported e_1^a–e_2^a plane) is considered.

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