Affine-projective field laws

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The general topic of geometric unified field theories is discussed in the first section. Some reasons are given for pursuing such theories, and some criticisms are considered. The second section develops the fundamental equations of a purely affine theory which is invariant under projective transformations of the affine connection. This theory is a generalization of that of Schrödinger. Possible identifications for the space-time metric are considered in Sec. III. Sections IV and V deal with the limits of pure gravitation and electrodynamics. In the symmetric limit, Einstein's vacuum equations with cosmological term are recovered. The theory also contains a generalized electrodynamic set of equations which is very similar to the Born-Infeld set. In the weak-field approximation, a finite mass must be attributed to the photon. The problem of motion for charges is discussed here, and it is argued that criticisms of unified field theories because of a supposed inability to produce the Lorentz force law are probably not justified. Three more speculative sections deal with possible explanations of nuclear forces, the spin-torsion relation, and particle structure.

I. PROS AND CONS OF UNIFIED FIELD THEORIES

Einstein's general theory of relativity has been important for at least two reasons. It is still the most satisfactory theory of graviation and, perhaps more importantly, the success of dynamic non-Euclidean geometry has destroyed the idea that Euclidean geometry is a necessary truth. But Einstein's theory is not a complete geometrization of physics, for it describes only gravitation in purely geometric terms. Other interactions, and the structure of matter itself, require the introduction of nongeometric variables and laws.

From the mathematical point of view, there is no more reason for Riemannian geometry to be regarded as a necessary truth than there was for Euclidean geometry. Why should we assume from the outset that lengths are integrable, or that torsion vanishes? Perhaps we will find such restrictions to obtain in nature, but to assume them at the beginning is to make the object of our study what we would like the world to be, rather than what it is.

These considerations suggest that we should consider more general geometries, with the hope that the new geometric entities which arise will be appropriate for the description of nongravitational phenomena. This does not mean to add assumptions to general relativity, but to eliminate the assumptions which restrict the geometry to that of Riemann.

Many attempts of this sort have been made. Weyl[1] suggested that the length of a vector might change on parallel transport, and identified the field causing such changes with the electromagnetic field. This idea was generalized by Einstein and Eddington in the 1920's,[2] and the electrodynamics to which such attempts give rise was studied by Born and Infeld.[3] In the 1940's, Einstein[4] and Schrödinger[5] took the further step of allowing a nonsymmetric affinity. These theories have incurred a number of specific criticisms, which I will deal with when encountered in the development of the present theory. There are, however, some general criticisms which should be discussed first.

The work of Rainich and of Misner and Wheeler[6] (RMW) has produced the claim that Einstein-Maxwell theory is an "already unified" theory, and that no further unification is needed. Similar work has been done for other phenomena besides electrodynamics.[7] But the conditions which have to be imposed on the curvature to obtain the correct equations of motion, while geometric, do not follow from any basic principle. They are designed only to give previously known results. For example, no one would dream of imposing something as cumbersome as the RMW differential condition on Riemannian geometry if he did not know exactly what he wanted to get out of the theory beforehand. Thus these formulations cannot really be regarded as natural geometric theories.

Most modern physicists are suspicious of unified field theories because of the belief that they stand in opposition to quantum theory. Working on them is regarded as roughly equivalent to working on phlogiston. This belief is, of course, associated with Einstein's well-known dislike of quantum theory. But there is no necessary incompatibility. Perhaps orthodox quantization techniques will have to be applied to even a good classical theory, though it may be hoped that a geometrical formulation of quantum dynamics can be found. Of course one must have an open mind. Dogmatism about the Copenhagen orthodoxy, or any other interpretation of quantum theory, is not consistent with a serious attempt to analyze the foundations of physics.

We cannot afford to be dogmatic about the geometrization of physics, or insist that a particular geometric formulation must be final. Even a theory which is a considerable improvement on general relativity will probably be incomplete. There are also nongeometric unified field theories, such as the *Urmaterie* theory of Heisenberg.[8] Such a theory involving a basic spinor field might be provisionally combined with a geometric theory. In fact, I will point out later just where this connection could be made.

The notation in this paper will be fairly standard. Greek indices range over 0, 1, 2, 3. Gothic letters indicate tensor densities *and* determinants, so that, e.g., $\mathfrak{g}=\det(g_{\mu\nu})$. Parentheses about two indices indicate symmetrization, and square brackets indicate antisymmetrization.

II. GEOMETRY AND DYNAMICS

In this section I present the basic formulas of the present theory, which is a modified version of Schrödinger's. To avoid undue prolixity, I refer the reader to previous work for much of the mathematical background and for details of proofs and calculations.[9]

An affinity $\Gamma^{\alpha}_{\beta\gamma}$ gives a mapping of the tangent space at P onto that at $P+dP$ via the fundamental formula for parallel transport of a vector A^{σ} through a displacement dx^{τ},

$$\delta A^{\mu}=-\Gamma^{\mu}_{\sigma\tau}A^{\sigma}dx^{\tau}\,. \qquad (1)$$

(This is not the most general structure providing such a mapping and the possibility for further generalizations, to Finsler space, e.g., should be noted.) The lower indices of $\Gamma^{\alpha}_{\beta\gamma}$ play different roles in (1), and there is no obvious reason to assume the affinity symmetric. The skew part, $\Gamma^{\alpha}_{[\beta\gamma]}\equiv S^{\alpha}_{\beta\gamma}$, is the torsion, whose presence means that a displacement of dx^{μ} through δx^{ν} will not give the same result as displacement of δx^{ν} through dx^{μ}. (I will use only coordinate bases in this paper.)

From an affinity the curvature tensor

$$R^{\sigma}{}_{\kappa\lambda\mu}=-\Gamma^{\sigma}_{\kappa\lambda,\mu}+\Gamma^{\sigma}_{\kappa\mu,\lambda}+\Gamma^{\sigma}_{\alpha\lambda}\Gamma^{\alpha}_{\kappa\mu}-\Gamma^{\sigma}_{\alpha\mu}\Gamma^{\alpha}_{\kappa\lambda} \qquad (2)$$

can be constructed. It has two independent contractions,

$$P_{\kappa\lambda}\equiv R^{\beta}{}_{\kappa\lambda\beta}=-\Gamma^{\beta}_{\kappa\lambda,\beta}+\Gamma^{\beta}_{\kappa\beta,\lambda}+\Gamma^{\beta}_{\alpha\lambda}\Gamma^{\alpha}_{\kappa\beta}-\Gamma^{\beta}_{\alpha\beta}\Gamma^{\alpha}_{\kappa\lambda},$$
$$Q_{\kappa\lambda}\equiv R^{\beta}{}_{\beta\kappa\lambda}=-\Gamma^{\beta}_{\beta\kappa,\lambda}+\Gamma^{\beta}_{\beta\lambda,\kappa}\,. \qquad (3)$$

Metric has not yet been defined, so we cannot say that geodesics are curves which extremize the intervals between points, but parallelism of vectors with respect to a curve can be defined by considering the transport of a vector parallel to itself along the curve. A necessary and sufficient condition for the vectors $A^{\mu}(t)$ to be parallel with respect to the curve $x^{\mu}=x^{\mu}(t)$ is

$$A^{\mu}(dA^{\nu}/dt+\Gamma^{\nu}_{\sigma\tau}A^{\sigma}dx^{\tau}/dt)=A^{\nu}(dA^{\mu}/dt+\Gamma^{\mu}_{\sigma\tau}A^{\sigma}dx^{\tau}/dt)\,. \qquad (4)$$

In particular, a *path* is a curve whose tangents are parallel with respect to itself. If $A^{\mu}=dx^{\mu}/dt$, (4) requires

$$d^2x^{\mu}/dt^2+\Gamma^{\mu}_{\sigma\nu}(dx^{\sigma}/dt)(dx^{\nu}/dt)=\phi(t)dx^{\mu}/dt\,, \qquad (5)$$

with $\phi(t)$ an arbitrary function of the parameter t. An appropriate change of parameter to $s=s(t)$ will make this

$$d^2x^{\mu}/ds^2+\Gamma^{\mu}_{\sigma\nu}(dx^{\sigma}/ds)(dx^{\nu}/ds)=0\,. \qquad (6)$$

Clearly s is determined only up to a linear transformation with constant coefficients.

It is natural to ask if there are transformations of the affinity, besides those induced by coordinate transformations, which do not alter the property of parallelism with respect to an arbitrary curve. Can (4) be satisfied for both of the connections $\Gamma^{\alpha}_{\beta\gamma}$ and $\Gamma'^{\alpha}_{\beta\gamma}=\Gamma^{\alpha}_{\beta\gamma}+\Theta^{\alpha}_{\beta\gamma}$ and for arbitrary A^{μ} and dx^{μ}/dt? The most general such transformation is given by

$$\Gamma'^{\alpha}_{\beta\gamma}=\Gamma^{\alpha}_{\beta\gamma}+2\delta^{\alpha}_{\beta}\psi_{\gamma}\,, \qquad (7)$$

with ψ_{γ} an arbitrary vector field. (The factor of 2 is simply for convenience.) Such transformations are called projective because, as a special case, they map paths onto paths. However, they are not the most general transformations which preserve paths, for the torsion drops out of (4) when $A^{\mu}=dx^{\mu}/dt$, and any change in the skew part of the affinity will therefore not affect the paths.

Projective transformations do alter the "preferred" parameter s for which the path equations take the form (6), at least along most paths. I will discuss this in Sec. III.

The transformation (7) produces in the curvature tensor the change

$$R'^{\mu}{}_{\nu\sigma\tau}=R^{\mu}{}_{\nu\sigma\tau}+2\delta^{\mu}_{\nu}(\psi_{\tau,\sigma}-\psi_{\sigma,\tau})\,, \qquad (8)$$

and its contractions become

$$P'_{\nu\sigma}=P_{\nu\sigma}+2(\psi_{\nu,\sigma}-\psi_{\sigma,\nu})\,,$$
$$Q'_{\nu\sigma}=Q_{\nu\sigma}+8(\psi_{\nu,\sigma}-\psi_{\sigma,\nu})\,. \qquad (9)$$

In order to use Hamilton's principle to derive field equations, we must have a Lagrangian for the variational principle. The simplest scalar density at this stage, since we have no way of raising and lowering indices, is the square root of the determinant of a covariant second-rank tensor, and a linear combination of $P_{\mu\nu}$ and $Q_{\mu\nu}$ comes to mind. Of course such a choice is far from unique, and Pauli[10] regarded this fact as a defect of affine theories, but it is not. A Lagrangian is never unique, and some criterion of simplicity is always used. So we put

$$\Lambda g_{\mu\nu} \equiv P_{\mu\nu} + \Upsilon Q_{\mu\nu} \ , \tag{10}$$

with Λ a constant having the units of an inverse length squared, and Υ a pure number, and consider the Lagrangian density

$$\begin{aligned} \mathfrak{L} &= 2\Lambda(-\mathfrak{g})^{1/2} \\ &= 2\Lambda(-\det g_{\mu\nu})^{1/2} \\ &= (2/\Lambda)[-\det(P_{\mu\nu} + \Upsilon Q_{\mu\nu})]^{1/2}. \end{aligned} \tag{11}$$

The action is to be varied with respect to $\Gamma^{\alpha}_{\beta\gamma}$. The field equations thus obtained are

$$\begin{aligned} \mathfrak{g}^{\lambda\tau}{}_{,\kappa} - \delta^{\tau}_{\kappa}\mathfrak{g}^{\lambda\sigma}{}_{,\sigma} + \mathfrak{g}^{\lambda\alpha}\Gamma^{\tau}_{\kappa\alpha} + \mathfrak{g}^{\alpha\tau}\Gamma^{\lambda}_{\alpha\kappa} \\ -\mathfrak{g}^{\lambda\tau}\Gamma^{\alpha}_{\kappa\alpha} - \mathfrak{g}^{\alpha\beta}\Gamma^{\lambda}_{\alpha\beta}\delta^{\tau}_{\kappa} + 2\Upsilon\delta^{\lambda}_{\kappa}\mathfrak{g}^{[\tau\alpha]}{}_{,\alpha} = 0 \ . \end{aligned} \tag{12}$$

Here $g^{\alpha\beta}$ is the matrix inverse of $g_{\alpha\beta}$ and $\mathfrak{g}^{\alpha\beta} = (-\mathfrak{g})^{1/2}g^{\alpha\beta}$. The last term in (12) does not appear if we choose $\Upsilon = 0$, as Schrödinger did.

The set (12) can be manipulated to yield $(1-4\Upsilon)\mathfrak{g}^{[\alpha\beta]}{}_{,\beta} = 0$. If $\Upsilon \neq \frac{1}{4}$, then we get the identity $\mathfrak{g}^{[\alpha\beta]}{}_{,\beta} = 0$, but, if $\Upsilon = \frac{1}{4}$, we obtain only $0 = 0$. And this latter case is exactly the one for which $\mathfrak{L}$ is projectively invariant, as is seen from equations (9). In this case the vector density $\mathfrak{g}^{[\alpha\beta]}{}_{,\beta}$ is undetermined by the variational principle.

Even if $\Upsilon \neq \frac{1}{4}$, $\mathfrak{L}$ is invariant under the subgroup of (7) for which ψ_ν is a gradient, for such transformations do not even affect the full curvature tensor. (These are Einstein's "λ-transformations."[11]) It is important to realize that the requirement of projective invariance, with whatever restrictions on ψ_ν (except $\psi_\nu \equiv 0$), makes the use of a torsion-free affinity impossible. It has been argued that the use of "reducible" quantities like an affinity possessing torsion presents a strong objection against nonsymmetric field theories.[12] It is true that, under coordinate transformations, the symmetric and skew parts of an affinity transform in different ways, the former as an affinity and the latter as a third-rank tensor. But with projective invariance the affinity is not reducible with respect to the full symmetry group of the theory.

It may be worthwhile to note a similarity between the present theory and that of a complex vector field in this regard.[13] Vector mesons can be described by means of the Lagrangian $\mathfrak{L}' = -\frac{1}{2}f^{*}_{\mu\nu}f^{\mu\nu} - m^2\phi^{*}_{\mu}\phi^{\mu}$, where $f_{\mu\nu} = \phi_{\mu,\nu} - \phi_{\nu,\mu}$ and m is the meson mass. From this results $f^{\mu\nu}{}_{,\nu} = -m^2\phi^{\mu}$ and, since $f_{\mu\nu}$ is skew-symmetric, $m^2\phi^{\mu}{}_{,\mu} = 0$. If $m \neq 0$, $\phi^{\mu}{}_{,\mu} = 0$. On the other hand, we can put $m = 0$ and gain an additional symmetry, the invariance of the Lagrangian under the gauge group $\phi_\nu \to \phi_\nu + \chi_{,\nu}$. But then the scalar $\phi^{\mu}{}_{,\mu}$ is not determined.

In the following work I will put $\Upsilon = \frac{1}{4}$, so that the theory is projectively invariant, but then some decision has to be made about the vector density $\mathfrak{J}^{\mu} \equiv \mathfrak{g}^{[\mu\nu]}{}_{,\nu}$. $\mathfrak{J}^{\mu}$ could be constrained to be anything, and it might seem natural to make it vanish, as it would if we started with a general value of Υ and then took the limit $\Upsilon \to \frac{1}{4}$. Again, the treatment of photons as vector mesons with $m \to 0$ comes to mind.

But we should bear in mind the fact that the present theory is probably incomplete, and that a later development may determine $\mathfrak{J}^{\mu}$. In particular, it is not easy to see how spinor fields will come into the theory in an unforced way and, as a temproary expedient, we might let $\mathfrak{J}^{\mu}$ be the current density of a fundamental spinor field like that of Heisenberg. It seems best to leave $\mathfrak{J}^{\mu}$ free, at least where it does not unduly complicate matters.

I should point out that the idea of projective invariance has been treated in the context of Einstein's mixed affine-metric theory by Chau.[14]

The system (12) can be simplified by a projective transformation to Schrödinger's "star affinity,"

$$^{*}\Gamma^{\mu}_{\nu\sigma} = \Gamma^{\mu}_{\nu\sigma} + \tfrac{2}{3}\delta^{\mu}_{\nu}\Gamma^{\rho}_{[\sigma\rho]} \ , \tag{13}$$

with the important property that $^{*}\Gamma^{\sigma}_{[\sigma\rho]} = 0$. (This is what leads to the simplification.) The field equations become

$$\begin{aligned} \mathfrak{g}^{\kappa\lambda}{}_{,\alpha} + \mathfrak{g}^{\sigma\lambda}\,{}^{*}\Gamma^{\kappa}_{\sigma\alpha} + \mathfrak{g}^{\kappa\sigma}\,{}^{*}\Gamma^{\lambda}_{\alpha\sigma} - \mathfrak{g}^{\kappa\lambda}\,{}^{*}\Gamma^{\sigma}_{\sigma\alpha} \\ = \tfrac{1}{6}(\delta^{\lambda}_{\alpha}\mathfrak{J}^{\kappa} - 3\delta^{\kappa}_{\alpha}\mathfrak{J}^{\lambda}) \ , \end{aligned} \tag{14}$$

and can be converted to

$$\begin{aligned} g_{\rho\beta,\alpha} - g_{\kappa\beta}\,{}^{*}\Gamma^{\kappa}_{\rho\alpha} - g_{\rho\kappa}\,{}^{*}\Gamma^{\kappa}_{\alpha\beta} \equiv g_{\rho(+)\beta(-);\alpha} \\ = L_{\rho\beta\alpha\mu}I^{\mu} \ , \end{aligned} \tag{15}$$

where

$$\begin{aligned} L_{\rho\beta\alpha\mu} = \tfrac{1}{12}[-2g_{\mu\beta}g_{\rho\alpha} + 6g_{\alpha\beta}g_{\rho\mu} \\ + g_{\rho\beta}(g_{\mu\alpha} - 3g_{\alpha\mu})] \ , \end{aligned} \tag{16}$$

and $I^{\mu} = \mathfrak{J}^{\mu}/(-\mathfrak{g})^{1/2}$. The semicolon denotes the covariant derivative with respect to the star affinity, but the + and − signs in (15) indicate the change in index position in the different terms.

The system (15) has been studied extensively for the case $I^{\mu} = 0$. Here it will suffice to note the results of Hlavatý. We use the notation $g_{(\mu\nu)} \equiv \boldsymbol{h}_{\mu\nu}$, $g_{[\mu\nu]} \equiv k_{\mu\nu}$ and $K_{\kappa\mu\lambda} \equiv k_{\lambda\mu|\kappa} + k_{\kappa\lambda|\mu} + k_{\kappa\mu|\lambda}$, the vertical stroke denoting the covariant derivative with respect to the Christoffel affinity,

$$\{{}_\beta{}^\alpha{}_\gamma\} \equiv \tfrac{1}{2} h^{\alpha\tau} (h_{\beta\tau,\gamma} + h_{\gamma\tau,\beta} - h_{\beta\gamma,\tau}) .$$

The affinity has the form

$$*\Gamma^{\sigma}_{\iota\kappa} = \{{}_\iota{}^\sigma{}_\kappa\} + S^{\sigma}{}_{\iota\kappa} + 2h^{\sigma\lambda} S^{\beta}{}_{\lambda(\iota} k_{\kappa)\beta} , \quad (17)$$

with $S^\alpha{}_{\beta\gamma}$ the torsion. The complete solution of (15) with $I^\mu = 0$ is then

$$*\Gamma^{\nu}_{\lambda\mu} = \{{}_\lambda{}^\nu{}_\mu\} + \tfrac{1}{2} K_{\lambda\mu}{}^{\nu} + k^{\alpha}{}_{[\mu} K_{\lambda]\alpha}{}^{\beta} k^{\nu}{}_{\beta} + h^{\sigma\nu} k_{\delta(\beta} K_{\alpha)\gamma}{}^{\delta} [\delta^{\gamma\alpha\beta}_{\sigma(\lambda\mu)} - 2\delta^{\alpha}_{\sigma} k^{\gamma}{}_{(\lambda} k_{\mu)}{}^{\beta} - 2\delta^{\beta}_{(\lambda} k_{\mu)}{}^{\alpha} k_{\sigma}{}^{\gamma}] . \quad (18)$$

This gives the affinity in terms of the sixteen components of $g_{\mu\nu}$ and their derivatives. But it must not be concluded from this that the content of the theory will be exhausted by gravitation and electrodynamics, which in Einstein-Maxwell theory require just sixteen components. Or perhaps I should say that gravitation and electromagnetism may play some quite unexpected roles in a truly unified theory.

Energy-momentum complexes can be found either from the canonical formalism or by appealing to the invariance of the action. The latter procedure was used by Schrödinger to calculate the conserved quantities in his theory, and it is easy to modify his procedure to include the effect of the tensor $Q_{\mu\nu}$ in the present theory.[15] However, I will not go into the details here.

III. THE SPACE-TIME METRIC

The present approach can be regarded as a physical version of Klein's *Erlangen Programm*.[16] He considered the various levels of geometry to be characterized by properties which were unchanged by different transformation groups. For example, the concepts of lines and points as their intersections are unchanged by projective transformations, and are therefore meaningful quantities in *projective* geometry. A structure of parallelism can be added to yield *affine* geometry, and *metric* properties of more or less specialization can be introduced. At each step we restrict further the group which the geometry allows.

I began the previous section with the concept of affine connection, but then noted that it is possible to find a function of the affinity, the tensor $g_{\mu\nu}$, which is unchanged by the projective transformations which preserve paths, (though not the most general ones with arbitrary change of torsion). But in order to tie this theory in with other physical theories, a metric structure must be defined. This is not hard to do. In fact, there is a superfluity of metrics.

Any affinity gives space-time a path structure, and a length can be defined, up to a linear transformation, independently on each path by the parameter s of Eq. (6). A projective transformation will change this measure of length along each path. Under (7), the path equation becomes

$$d^2x^\mu/ds^2 + \Gamma'^{\mu}_{\nu\sigma}(dx^\nu/ds)(dx^\sigma/ds) = 2\psi_\sigma (dx^\sigma/ds)(dx^\mu/ds).$$

A transformation $s' = s'(s)$ with

$$ds'/ds = A \exp\left[2 \int \psi_\sigma dx^\sigma\right], \quad (19)$$

A being a constant, will bring the path equation back to the form (6) with s replaced by s'.

There will be no surprise in this modification of length if you think of elementary examples of projective transformations. But it is possible in projective geometry to define an invariant distance function in terms of cross ratio.[17] Here we can define a new path parameter p which satisfies

$$dp/ds = B \exp\left[-\tfrac{1}{4} \int \Gamma^{\mu}_{\mu\lambda} dx^\lambda\right], \quad (20)$$

with B a constant. It is easy to show that p is not changed by a projective transformation. However, it is not invariant under coordinate transformations because of the nontensorial transformation law for the affinity, while ds is a coordinate scalar.

Any path parameter, including s and p, gives a metric only along one path, and the parameters along different paths are independent. We can assign an interval measure to *any* displacement by using a symmetric second-rank tensor as a metric tensor, and the symmetric part of $g_{\mu\nu}$ defined by (10) will serve. It is projectively invariant for any value of Υ because projective transformations affect only $Q_{\mu\nu}$ and $P_{[\mu\nu]}$. Thus we can define

$$d\tau^2 = g_{(\mu\nu)} dx^\mu dx^\nu = (1/\Lambda) P_{(\mu\nu)} dx^\mu dx^\nu . \quad (21)$$

Naturally we would like to have some relation between $d\tau^2$ and ds^2 or dp^2. Schrödinger discussed the general problem of "compatibility" between the s metric and the τ metric, and gives the condition for it as[18]

$$\Gamma^{\mu}_{\nu\sigma} = \{{}_\nu{}^\mu{}_\sigma\} + h^{\mu\rho} T_{\rho\nu\sigma}, \quad (22)$$

with $T_{\rho[\nu\sigma]} = 0$ and $T_{\rho\nu\sigma} + T_{\nu\sigma\rho} + T_{\sigma\rho\nu} = 0$. It is easy to show that these conditions obtain for affinities satisfying (17).

Thus the metric defined by (21) is always compatible with the s metric defined by the star affinity when $I^\mu = 0$. The path equation

$$d^2x^\mu/ds^2 + \{{}_\nu{}^\mu{}_\sigma\}(dx^\nu/ds)(dx^\sigma/ds) + 2h^{\mu\lambda} k_{\beta\sigma} S_{\nu\lambda}{}^{\beta} (dx^\nu/ds)(dx^\sigma/ds) = 0,$$

on inner multiplication by $h_{\mu\alpha} dx^\alpha/ds$, gives

$$d[h_{\mu\nu}(dx^{\mu}/ds)(dx^{\nu}/ds)]/ds=0,$$

$$d\tau/ds=\text{constant}.$$

We do not, in general, have compatibility for affinities obtained from the star affinity by projective transformations, nor when $I^{\mu}\neq 0$. But then the τ metric should be regarded as fundamental, since it is unaltered. So far, no useful physical interpretation of the p metric has occurred to me.

IV. GRAVITATION AND COSMOLOGY

I now consider various limits of the general theory, with the hope that the physical interpretation of the various geometric entities will become clearer. Such identifications will have only approximate validity. For example, the fact that a certain skew tensor can be identified with the electromagnetic field in a certain limiting case does not mean that it will always have that, and only that, meaning.

The most obvious limiting case is that in which the torsion vanishes and $I^{\mu}=0$. Then (17) shows that the affinity is given by the Christoffel brackets, and (10) then requires that $k_{\mu\nu}=0$, so that

$$B_{\mu\nu}\equiv P_{\mu\nu}(\{_{\beta}{}^{\alpha}{}_{\gamma}\})=\Lambda h_{\mu\nu}. \tag{23}$$

These are Einstein's vacuum equations with a cosmological term. Furthermore, any space-time for which the torsion has the form $S^{\kappa}{}_{\lambda\sigma}=\delta^{\kappa}_{[\lambda}F_{\sigma]}$ can be reduced to this case by a projective transformation.

In this limit, the physical interpretation of the geometric entities used in general relativity can be transferred to the present theory. The derivation of equations of motion by treating particles as singularities can be used if we are willing to admit singular solutions, and there seems to be no reason not to in this approximation. "Singular" regions are simply those in which $S^{\alpha}{}_{\beta\gamma}$ or I^{μ} do not vanish. (The general case is different, as I will argue later.) In particular, test particles will follow the paths (geodesics, in a Riemannian space) of the external field.[19]

The constant Λ cannot vanish in this theory, and the necessity for a cosmological term was considered sufficient cause for rejecting affine theories by Pauli.[20] The generally negative attitude toward the cosmological term requires some comment.

The arguments which can be mustered against the cosmological term in orthodox relativistic cosmology are, I think, successfully disposed of by McVittie and Rindler,[21] and I will make only a couple of brief points. First, the current severe disagreement between the mass density in the universe and the deacceleration parameter makes it quite impossible to rule out a reasonable value of Λ on observational grounds.[22] And secondly, the fact that opponents of the cosmological term are often reduced to appealing to the authority of Einstein shows the weakness of their case.[23] (Quantum mechanics can be refuted in the say way.)

Most arguments against the Λ term are philosophical ones: It is said that this term does not do what Einstein introduced it to do, and that it complicates the field equations.[24] Now for whatever force such arguments may have in ordinary general relativity, they are quite irrelevant in a new theory. Why should affine theories be ruled out for all time simply because Einstein, perhaps unwisely, put an extra term in his equations a long time ago to obtain a static universe? If he had never done this, someone would have eventually "discovered" the cosmological constant in an attempt to construct a unified field theory.

The simplest solution of the set (23) is not Minkowski space, but the de Sitter space-time whose metric can be written in the form

$$h_{00}=+1,\quad h_{ij}=-\delta_{ij}\exp[(4\Lambda/3)^{1/2}ct]. \tag{24}$$

(With this signature, $\Lambda>0$ gives an expanding universe.) This and other forms of the line element are treated very nicely by Schrödinger.[25] The de Sitter group, rather than the Lorentz group, should be considered the basic space-time symmetry group for elementary particles.[26]

V. ELECTRODYNAMICS

We now reinstitute torsion and attempt to find electromagnetic equations. Consider the case in which $I^{\mu}=0$, and write $\Phi_{\mu\nu}\equiv P_{[\mu\nu]}+\frac{1}{4}Q_{\mu\nu}$. Two sets of our equations are then

$$\mathfrak{g}^{[\mu\nu]}{}_{,\nu}=0 \tag{25a}$$

and

$$\Phi_{\nu\rho,\mu}+\Phi_{\mu\nu,\rho}+\Phi_{\rho\mu,\nu}=2\Lambda({}^*S_{\mu\rho\nu}+{}^*S_{\rho\nu\mu}+{}^*S_{\nu\mu\rho}), \tag{25b}$$

where ${}^*S_{\alpha\beta\gamma}=h_{\kappa\gamma}{}^*\Gamma^{\kappa}_{[\alpha\beta]}$. Equations (25) can be interpreted as a Maxwellian set in two different ways. I shall follow what has become the conventional interpretation in a rough way by identifying $\Phi_{\sigma\tau}$ with the dual of the displacement $(\vec{D},\vec{H})$ tensor density and $\mathfrak{g}^{\sigma\tau}$ with the dual of the field $(\vec{E},\vec{B})$ tensor.[27] This means that the third-rank tensor on the right-hand side of (25b) is the dual of an electric current density. The vector I^{μ} would represent a magnetic current, but would also introduce extra terms into (25b). The opposite identification has been considered recently[28] but, without wishing to rule this out permanently, I will adopt the usual scheme here.

It must be remembered that the association of a tensor and a tensor density is not as obvious in nonsymmetric theories as it is with Riemannian geometry. This is because the square root of the determinant of any of the tensors $g_{\mu\nu}$, $h_{\mu\nu}$, or $k_{\mu\nu}$ are scalar densities, and a number of different relative tensors of weight ±1 can be formed from a given tensor by multiplication or division with them.[29] The relation between these determinants can be written[30]

$$\mathfrak{g}/\mathfrak{h} \equiv \Delta = 1 + \tfrac{1}{2} k_{\alpha\beta} k^{\alpha\beta} + \mathfrak{k}/\mathfrak{h} \ , \tag{26}$$

indices being raised and lowered with $h_{\mu\nu}$.

We shall also need the totally skew relative pseudotensors of weights ±1, $\mathfrak{E}^{\alpha\beta\gamma\delta}$, and $\mathfrak{e}^{\alpha\beta\gamma\delta}$. Furthermore,

$$g^{[\lambda\nu]} = (1/\mathfrak{g})[\mathfrak{h} k^{\lambda\nu} + (\kappa/2)(\mathfrak{k})^{1/2} \mathfrak{E}^{\alpha\mu\lambda\nu} k_{\alpha\mu}] \ , \tag{27a}$$

and

$$g^{(\lambda\nu)} = (1/\Delta)[h^{\lambda\nu}(1 + k_{\alpha\beta}k^{\alpha\beta}/2) + k^{\lambda\alpha} k_{\alpha}{}^{\nu}] \ , \tag{27b}$$

with κ the sign of $\mathfrak{E}^{\alpha\beta\gamma\delta} k_{\alpha\beta} k_{\gamma\delta}$. We then define

$$F_{\sigma\tau} = (b/2)\, \mathfrak{e}_{\sigma\tau\mu\nu}\, \mathfrak{g}^{[\mu\nu]} \ ,$$
$$\mathfrak{H}^{\alpha\beta} = -(b/2\Lambda) \mathfrak{E}^{\alpha\beta\gamma\delta} \Phi_{\gamma\delta} \ , \tag{28}$$

b being a constant with units of an electromagnetic field strength to convert from geometric units to conventional ones. Then

$$F_{\mu\nu,\sigma} + F_{\sigma\mu,\nu} + F_{\nu\sigma,\mu} = 0 \ , \tag{29a}$$

and

$$\mathfrak{H}^{\kappa\lambda}{}_{,\lambda} = \mathfrak{B}^{\kappa} \ , \tag{29b}$$

with

$$\mathfrak{B}^{\sigma} = -(b/3) \mathfrak{E}^{\sigma\mu\nu\rho} ({}^*S_{\mu\rho\nu} + {}^*S_{\rho\nu\mu} + {}^*S_{\nu\mu\rho}) \ . \tag{30}$$

The "constitutive relations" can be written as

$$(-\mathfrak{h})^{1/2} F_{\alpha\beta} = \frac{\mathfrak{H}_{\alpha\beta} + (\kappa/2b^2)(\mathfrak{H})^{1/2} \mathfrak{e}_{\alpha\beta\sigma\tau} \mathfrak{H}^{\sigma\tau}}{[1 - (1/2b^2) H_{\sigma\tau} H^{\sigma\tau} + \mathfrak{H}/\mathfrak{h} b^4\ ]^{1/2}} \ , \tag{31}$$

where $\mathfrak{H} = \det H_{\mu\nu}$, $H_{\mu\nu} = \mathfrak{H}_{\mu\nu}/(-\mathfrak{h})^{1/2}$.

Equations (29) and (31), with $\mathfrak{B}^{\sigma} = 0$, are those of the Born-Infeld electrodynamics, if care is taken with the different sign conventions.[31] Thus the Born-Infeld theory should provide quidance for us in some situations. But note the following qualifications.

(i) The set (25) is not a complete statement of the field equations of the projective-affine theory. The other equations will place additional constraints on their solutions.

(ii) $h_{\mu\nu}$ ultimately depends on "electromagnetic" variables, and the current density $\mathfrak{B}^{\mu}$, which does not occur in the Born-Infeld theory, depends on the torsion, which also enters into the fields $F_{\alpha\beta}$ and $\mathfrak{H}^{\sigma\tau}$.

I will defer to a later section discussion of the agreement of the Born-Infeld theory and the results of this section to the known facts concerning the electromagnetic structure of "elementary" particles.

Note that I have not said that $g_{[\mu\nu]}$ is the dual of the displacement field, but that $\Phi_{\mu\nu}$ is. In view of (10), the distinction might seem pointless, but it is not, and an investigation of this will provide additional insight into the physical meaning of our equations. A slight digression is needed to explain this.

Let us begin by asking what quantity in conventional Maxwell electrodynamics is the analog of the metric tensor in general relativity. We often consider the metric $h_{\sigma\tau}$ and the vector potential A_{ρ} as analogous "potentials," but the analogy is weak when we consider physical significance. In electrodynamics, the physically meaningful quantity is the Maxwell field, $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$, which gives the force on a charged particle. For gravitation, the meaningful quantity is the curvature tensor which, according to the equation of geodesic deviation, gives tidal accelerations between pairs of test particles. It is made up essentially of *second* derivatives of $h_{\mu\nu}$, for the terms containing first derivatives can be transformed away, while $F_{\mu\nu}$ is made up of *first* derivatives of A_{ρ}.

In electrodynamics one can define covariant *Hertz potentials*[32] with a skew tensor $Z_{\mu\nu}$, from which the vector potential is calculated according to $A_{\mu} = Z_{\mu\nu}{}^{,\nu}$. (The formulas here are for flat space for simplicity.) $F_{\mu\nu}$ will be made up of second derivatives of $Z_{\mu\nu}$ so, as far as order of differentiation is concerned, $h_{\mu\nu}$ and $Z_{\mu\nu}$ correspond.

There is a disruption of this correspondence with the usual formulation of the field equations: $F^{\mu\nu}{}_{,\nu} = 0$ is of third order in $Z_{\mu\nu}$, while $B_{\mu\nu} = 0$ is of second order in $h_{\mu\nu}$. But (23) is a more general formulation of the gravitational equations. We can exploit the $h_{\mu\nu} \leftrightarrow Z_{\mu\nu}$ analogy by replacing the Maxwellian $F^{\mu\nu}{}_{,\nu} = 0$ with

$$F_{\mu\nu} = \alpha\Lambda Z_{\mu\nu} \ , \tag{32}$$

where α is a pure number. Then we find

$$\Box A_{\mu} + \alpha\Lambda A_{\mu} = 0 \ . \tag{33}$$

Equation (32) thus results in a nonzero mass for the photon. We shall find shortly that α is of orde. unity, so that (33) will be approximated by Maxwell's equations for distances much smaller than $\Lambda^{-1/2}$. Present observational limits on the photon rest mass are many orders of magnitude greater than required by this theory.[33] Equation (33) is also a covariant statement of London's empirical supercurrent formula.[34] Equations of the type (current) = (constant) × (vector potential) occurred a long time ago in affine theories.[35]

Now consider (10) and (25) in the light of this analogy. We have already identified $\Phi_{\mu\nu}$ with the dual of the displacement tensor density, which is the same as the field tensor for weak fields. It then seems that we should identify $k_{\mu\nu}$ with the dual of the Hertz potential.

In order to see how this should be done, let us look at the weak-field approximation. If (18) is used for the affinity, and only linear terms in $k_{\mu\nu}$ are retained, (10) yields (23) and

$$(\Box + 2\Lambda)k_{\kappa\lambda} = (k^{\alpha}{}_{\kappa|\lambda} - k^{\alpha}{}_{\lambda|\kappa})|_{\alpha} . \qquad (34)$$

(The equation $\mathfrak{J}^{\lambda} = 0 \simeq [(-\mathfrak{h})^{1/2}k^{\lambda\nu}]_{,\nu} = (-\mathfrak{h})^{1/2}k^{\lambda\nu}|_{\nu}$ is used to get this.)

Now we introduce the tensor $Z_{\mu\nu}$ by means of

$$k_{\mu\nu} = \alpha e_{\mu\nu\sigma\tau} Z^{\sigma\tau}/2 , \qquad (35)$$

$e_{\alpha\beta\gamma\delta}$ and $E^{\alpha\beta\gamma\delta}$ being pseudotensors rather than densities. Inner multiplication of (34) by $E^{\kappa\lambda\mu\nu}$ produces

$$(\Box + \Lambda)Z_{\mu\nu} = \tfrac{1}{2}(A_{\mu,\nu} - A_{\nu,\mu}) , \qquad (36)$$

A_{ρ} being defined by $A_{\rho} \equiv Z_{\rho\sigma}{}^{,\sigma}$. On the other hand, the usual Maxwell set

$$F^{\mu\sigma}|_{\sigma} \equiv J^{\mu} , \quad F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}$$

in general relativity yields[36]

$$\Box F_{\mu\nu} = J_{\nu,\mu} - J_{\mu,\nu} - B_{\mu}{}^{\epsilon}F_{\epsilon\nu} + B_{\nu}{}^{\epsilon}F_{\epsilon\mu} + 2B_{\mu\nu\alpha\epsilon}F^{\epsilon\alpha} . \qquad (37)$$

Here $B^{\mu}{}_{\nu\sigma\tau}$ is the curvature tensor formed from $h_{\alpha\beta}$. Equations (23) and (37) together give

$$(\Box + 2\Lambda)F_{\mu\nu} = J_{\nu,\mu} - J_{\mu,\nu} + 2B_{\mu\nu\alpha\epsilon}F^{\epsilon\alpha} . \qquad (38)$$

Put $F_{\mu\nu} = \alpha\Lambda Z_{\mu\nu}$ and $J_{\mu} = \alpha\Lambda A_{\mu}$ in (36) and subtract this equation from (38). The result is

$$F_{\mu\nu} = \tfrac{3}{2}(J_{\nu,\mu} - J_{\mu,\nu}) + 2B_{\mu\nu\alpha\epsilon}F^{\epsilon\alpha} . \qquad (39)$$

This looks rather strange at first, and it is instructive to work it out for the de Sitter metric (24). We have $B_{\mu\nu\alpha\epsilon} = \Lambda(h_{\mu\nu}h_{\alpha\epsilon} - h_{\mu\alpha}h_{\nu\epsilon})/3$, and obtain

$$A_{\mu,\nu} - A_{\nu,\mu} = F_{\nu\mu} = (9/2\Lambda)(J_{\mu,\nu} - J_{\nu,\mu}) , \qquad (40)$$

so that $\alpha = \frac{2}{9}$. The situation is more complicated if the curvature is not uniform, but then the considerations of the first part of this section will be of more interest than the present analogy.

A general comment on such weak-field approximations should be made here. Hlavatý[37] points out that perturbation methods will generally distort the algebraic classification of the tensor $k_{\alpha\beta}$. In the first place, it would have more meaning to work with an algebraic classification of the full curvature tensor, as one does with the Petrov-Pirani classification in general relativity. Secondly, such a classification may not always be of most importance. For example, the Weizsäcker-Williams method is useful for the description of atomic scattering problems, even though the field of a moving charge does not belong to the same algebraic class as does that of a plane wave.

I have not yet discussed electromagnetic fields as sources of the gravitational field. Linear terms in $k_{\mu\nu}$ do not appear in the symmetric part of (10) but, in higher approximation, there are quadratic terms. These do not seem to form any kind of recognizable energy-momentum tensor for the electromagnetic field.

There are no data on the *active* gravitational mass of electromagnetic fields. But the correct energy-momentum tensor is needed if the usual approximation methods of the relativistic theory of motion are to yield the Lorentz force law.[38] This is a serious problem. Indeed, the inability of such methods to derive the correct equations of motions for charges has furnished a powerful argument against unified field theories.[39]

But it is doubtful whether or not approximation methods which make use of singularities to describe particles are compatible with the type of theory which we are trying to develop here. In general relativity, such singular regions are really idealizations of regions in which the energy-momentum tensor does not vanish, but one of the purposes of a unified theory is to represent matter geometrically, and to eliminate the need to introduce *ad hoc* energy-momentum tensors. Thus singular solutions are not, in general, justified. As Wheeler has pointed out, to allow singular solutions in a field theory is really to allow anything at all.[40] In the present version of the projective-affine theory, singularities representing magnetic monopoles are permissible. This is because the field equations do not determine the current I^{μ}.

It is also important to note that C. R. Johnson has, in a recent series of papers,[41] given an approximation method by which the equations of motion for charges can be derived from Einstein's mixed affine-metric theory. That theory is closely related to the present one. The calculations are very lengthy, and I will not attempt to summarize them here. However, I think that this work is quite important, in that it disposes of what has come to be used as a standard "disproof" of nonsymmetric theories.

In connection with this problem of electromagnetic interactions, the exchange between C. P. Johnson and Einstein should be mentioned.[42] Johnson argued that the homogeneity of Einstein's theory allowed scale transformations of the form $\bar{x}^{\mu} = \beta x^{\mu}$ (with β a constant), and displayed an inconsistency between Newton's law of gravitational motion and Coulomb's law because of the different ways in which mass and charge would have to transform. Einstein's reply showed the tentative nature of such arguments, but was not entirely convincing. In affine theory, no such argument can be brought, because the field equations with a cosmological term do not allow such scale transformations. I feel that this is an argument in favor of the purely affine approach.

VI. STRONG INTERACTION

I have already cautioned against the idea that the content of this theory is exhausted by gravitation and electrodynamics. The previous section indicates that the theory will be even more complex than a theory of interacting Einstein and Born-Infeld fields. At the same time, it must be confessed that these next considerations will be more speculative than the preceding ones. They are only of the nature of suggestions on places where new phenomena may be found.

We know that the equations of motion of a test particle in a gravitational field are contained in Einstein's equations, but originally Einstein had to postulate that a test particle followed the geodesics of an external field. At the present stage of development it seems reasonable to examine the paths of our non-Riemannian space-time to see if they might serve as the world lines of uncharged test particles. The path equation does not involve the torsion directly, but it does involve the non-Christoffel part of $\Gamma^{\sigma}_{(\mu\nu)}$.

The solution (18) for the affinity yields a path metric which is compatible with that defined by $g_{\mu\nu}$, but this solution does not hold if $I^{\mu} \neq 0$, or if a projective transformation is made from the star affinity. It is therefore useful to consider a case in which these metrics are not compatible.

$C^{\sigma}_{\ \tau\rho} \equiv \Gamma^{\sigma}_{(\tau\rho)} - \{{}_{\tau}{}^{\sigma}{}_{\rho}\}$ can be written in such a way as to yield a vector force which is proportional to the inertial mass of a test particle, but independent of any other property of the particle, and which cannot be removed by a coordinate transformation. If

$$C^{\sigma}_{\ \tau\rho} = \phi^{\sigma} h_{\tau\rho} - \delta^{\sigma}_{(\rho}\phi_{\tau)}\,, \tag{41}$$

with ϕ^{μ} a vector field, we obtain an acceleration

$$\begin{aligned} a^{\mu} &= -C^{\mu}_{\ \sigma\tau} U^{\sigma} U^{\tau} \\ &= -\phi^{\kappa}(\delta^{\mu}_{\kappa} - U_{\kappa} U^{\mu}), \end{aligned} \tag{42}$$

where U^{ν} is the four-vector velocity of the particle. This has the necessary property that $U_{\mu} a^{\mu} \equiv 0$.

This force might represent a very simple-minded classical version of the nuclear interaction, as Schrödinger suggested.[43] If ϕ^{μ} were a gradient, we would obtain a scalar or pseudoscalar mesonic interaction. For this approach to be of any value, it would have to be shown that the scalar field obeyed an equation with some resemblance to the Klein-Gordon equation on the basis of the projective-affine field equations.

VII. SPIN AND TORSION

A number of writers have considered the possibility that the torsion tensor is related to the spin of material systems.[44] The work of Sciama, in which "vierbeins" are used, makes clear the physical and mathematical ideas involved in such an approach. But it is a little difficult to compare this work, which uses a mixed affine-metric formalism, with the present purely affine one. (Although Sciama speaks of "the Einstein-Schrödinger theory," the approaches are not at all the same.) I have not been able to cast the Lagrangian (11) into such a mixed form without destroying its most attractive features. Therefore, rather than present a mutilated theory, I will only sketch some ideas here.

Sciama introduces "vierbeins" and a *vierbein* connection to define their parallel transport, and relates these elements to the space-time affinity. The Einstein-Palatini action[45] is written in terms of "vierbeins" and their connection, and is varied with respect to both. Variation with respect to the connection yields relations leading to the Christoffel affinities, while variation of the *vierbein* fields leads to the vacuum Einstein equations.

If a matter Lagrangian, involving matter variables and the "vierbeins," but not their connection, is added, the usual energy-momentum tensor appears in the Einstein equations, while the affinity is unchanged. Such a matter Lagrangian will not be invariant under local (nonrigid) rotations of the "vierbeins." It is possible to remedy this by intoducing into the matter Lagrangian terms which involve the *vierbein* connection and whose changes under local Lorentz rotations will compensate for those produced by the derivatives of the *vierbein* fields. But the Christoffel relations will then no longer hold, and the torsion will not vanish. Sciama showed how this torsion could be related to the material spin.

The purely affine theory already possesses torsion, and it should be possible to match up with such a spin-torsion theory in some appropriate limit. The torsion would then enter into both the spin and the electric current given by (30). The suggestion of a relation between spin and isospin, and thus charge, has been made by Corben.[46]

In order for this idea to become more than wishful thinking, the projective-affine Lagrangian will have to be dealt with in terms of the *vierbein* connection, without destroying its purely affine character. In any case, it seems to me that the present approach is superior to one in which torsion is simply tacked on to the Christoffel affinity.

VIII. PARTICLE STRUCTURE

Can a nonquantum geometric theory of this type make any contribution to our knowledge of the ultimate structure of matter? Certainly a final solution cannot be reached without some form of quantum dynamics, but an improvement of our knowledge of the geometry of space-time may be essential for progress.

The Born-Infeld electrodynamics has many attractive features which are lacking in the linear Maxwell theory. In particular, a static, spherically symmetric solution with finite total energy exists.[47] The mass for this particlelike solution depends on the critical field strength b [cf. Eq. (31)]. To within factors of order unity, $mc^2 \simeq e^2/r_0$ and $e/r_0{}^2 \simeq b$, so $b \simeq m^2c^4/e^3$, with r_0 a length which specifies the extent of the effective ("bound") charge distribution.

Born and Infeld identified these solutions with electrons, so that r_0 is on the order of the classical electron radius, but this seems to disagree with experiment. The departure from a pure Coulomb interaction, equivalent to an extra vacuum polarization, should contribute to the Lamb shift. The effect would be of the same order of magnitude as that due to a finite nuclear size, and first-order perturbation theory gives an estimate of

$$\Delta\nu \simeq (|E_2|/h)(r_0/a)^2 \simeq 1 \text{ MHz},$$

for the additional $2s_{1/2} - 2p_{1/2}$ frequency shift, a being the Bohr radius. Such a value is inconsistent with present measurements of the Lamb shift in hydrogen.[48]

But the identification of these solutions with the electron is open to question on other grounds. These solutions have no spin while, of course, intrinsic angular momentum is a fundamental feature of real electrons. And the limit to the localizability of the electron is its Compton wavelength, 137 times its classical radius. Any attempt to measure the electron's position more precisely than this will produce real pairs.

Perhaps the particlelike solutions should be identified with classical (and so, *inter alia,* spinless) nucleons. This would mean that r_0 should be on the order of 10^{-16} cm. New phenomena would then be expected at such distances.

Then what about electrons? Perhaps they have to be represented by some sort of spinor *Urmaterie,* as in Heisenberg's theory. The current density of this field could be identified with $\mathfrak{J}^\mu$. Real nucleons then might be a combination of the spinless Born-Infeld solution and the spinor electron solution. Both theories are highly nonlinear, so such a "combination" would be very complicated. And it must be remembered that the Born-Infeld theory is only an approximation to the present one.

IX. CONCLUSION

In the previous sections I have outlined the simplest affine-projective theory, and have indicated some of the possible physical consequences. There are many ways in which the theory could be altered. For example, the skew part of $P_{\mu\nu} + \frac{1}{4}Q_{\mu\nu}$ could be given an arbitrary dimensionless coefficient in the Lagrangian. In order to get the correct coupling between gravitation and electromagnetism, I suspect that this will be necessary. However, only a more detailed treatment of electromagnetic interactions can decide this. Quantization might take care of this automatically.

I hope to deal with exact solutions, the theory of motion, conservation laws and the spin-torsion relation in more detail later.

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