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Neutrino opacity of neutron-star matter *

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Scattering of neutrinos from neutron-star matter is calculated for the case of very high neutron-star temperatures, under the assumption of a neutral-current interaction of the form given by the Weinberg-Salam theory. The resulting neutrino opacity depends in an interesting way upon the equation of state of the matter. The maximum possible opacity of a thin layer of matter is estimated for the case of an arbitrary equation of state.

I. INTRODUCTION

The neutrino opacity of matter under extreme conditions of pressure and temperature plays an important role in supernova theory[1] and in the theory of the development of a newly formed neutron star, where the damping mechanism for the radial oscillations is thought to be neutrino emission.

The present note will be concerned with that neutrino opacity which originates in the neutral-current coupling of neutrinos to nucleons, in the case in which the neutrino wavelength is larger than the internucleon spacing, in the dense interiors of neutron stars, for example, for neutrinos of energy up to 20 MeV.

Under conditions of very high temperature which may exist in the interiors of neutron stars at the time of formation, $(10^{10})^\circ < T < (10^{11})^\circ$, we shall find that neutrino opacity depends in an interesting way on the equation of state of the matter in the star. For some possible equations of state the opacity will be significantly larger than the opacity that one would calculate for the case of a free Fermi gas of neutrons.

This can be seen qualitatively for the case of a pure neutron gas and a vector weak, neutral, neutron current. In this case, at high temperatures, we can describe the scattering of neutrinos as being from the thermal density fluctuations of the nuclear medium. In a local region in the star in which there is a softening of the equation of state (that is, a large compressibility of the matter) these fluctuations will become quite large, compared to those of a free Fermi gas at the same density and temperature.

Our aim, in the present work, is to estimate the neutrino opacities which are due to the coherent scattering from these fluctuations. For this purpose we shall neglect the Gamow-Teller part (if any) of the weak neutral baryon current. For the case of scattering off of a free neutron gas the Gamow-Teller terms could contribute as much as the Fermi terms, but they will not be enhanced by the softening of the equation of state, as will the Fermi terms.

We shall present opacity profiles for a model of neutron-star matter, based on the Fermi part of the neutral-current interaction of neutrons with neutrinos. We shall discuss the possibility of a fairly thin shell of neutron matter within the star developing a very large opacity through a local softening of the equation of state. A limit will be set on the maximum opacity of such a shell, this limit depending only on the radius of the shell and on the amount of matter contained in the sphere of which the shell forms the surface.

II. SCATTERING OF NEUTRINOS FROM A NEUTRON GAS

For the Fermi part of the neutral-current coupling of neutrons to neutrinos we choose

$$H_w = \frac{G_w \lambda}{2\sqrt{2}} \int [\bar{\nu}\gamma_\alpha(1-\gamma_5)\nu][\bar{n}\gamma_\alpha n]d^3x\ , \tag{1}$$

where G_w is the weak coupling constant of ordinary beta decay. In the Salam-Weinberg theory[2] the parameter λ is given by unity. We shall retain λ as an undetermined parameter in displaying our results, since at the present time there are no direct data on the reaction $\nu + n \to \nu + n$ (or $\bar{\nu} + n \to \bar{\nu} + n$). However, there is evidence for the existence of $\Delta S = 0$ neutral-current couplings of about the strength predicted by the Weinberg theory.[3]

In the nonrelativistic limit for the neutrons we can replace the form $\bar{n}\gamma_\alpha n$ by the neutron density operator

$$\bar{n}\gamma_\alpha n = \delta_{\alpha 0}\rho_n(x)\ . \tag{2}$$

If we regard $\rho_n(x)$ as a classical density of neutrons we obtain, for the differential cross section for scattering through an angle, θ, with momentum transfer $\vec{q}$, the result

$$\frac{d\sigma}{d\cos\theta}=\frac{E^2}{8\pi}G_w{}^2\lambda^2(1+\cos\theta)|\rho(\vec{q})|^2\ , \tag{3}$$

where E is the energy of the neutrino and

$$\rho(\vec{q})=\int d^3x\,\rho(\vec{x})\exp(i\vec{q}\cdot\vec{x})\ . \tag{4}$$

Taking $G_w=1.01\times10^{-5}(M_{\text{proton}})^{-2}$ and expressing E in MeV, we obtain

$$\frac{d\sigma}{d\cos\theta}=E^2\lambda^2(1+\cos\theta)|\rho(\vec{q})|^2\times0.19\times10^{-44}\ \text{cm}^2\ . \tag{5}$$

It is worth noting that Eq. (3) gives the usual incoherent sum of cross sections when $\rho(x)$ is given by a sum of incoherent δ functions, and that it gives Freedman's expression[4] for coherent scattering from nuclei when $\rho(x)$ is taken as the density distribution within a nucleus (with the modifications necessary to take into account the presence of protons in nuclei and the isospin dependence of the weak neutral-current interactions).

For the uniform neutron liquid we shall take $|\rho(\vec{q})|^2$ to be given by the mean-square density fluctuation which can be calculated classically from the equation of state and the temperature. The conditions under which this calculation is correct are discussed in Appendix A. These conditions depend on the values of neutron density, neutrino wavelength, scattering angle, and temperature. In most of the domain of the other parameters relevant to early neutron-star physics the condition is roughly $k_BT>1$ MeV.

The classical result for the mean-square density fluctuation, for the case of small $|\vec{q}|$ and in a piece of matter of volume V, is[5]

$$|\delta\rho(\vec{q})|^2=(k_BT)\rho^2V(K_T)^{-1}\ , \tag{6}$$

where K_T is the isothermal bulk modulus, $K_T=-VdP/dV$.

We shall be concerned with a temperature region ($k_BT<20$ MeV) sufficiently low compared to the Fermi energy such that we can use the equation of state at zero temperature to calculate K_T and sufficiently high ($k_BT>1$ MeV) for the scattering to be dominated by the classical fluctuations.

From (5) and (6) we obtain a mean free path, a, for the neutrino,

$$a^{-1}=E^2\lambda^2(k_BT)\rho^2(K_T)^{-1}\times0.38\times10^{-5}\ \text{cm}^{-1}\ , \tag{7}$$

where we have now taken E and k_BT in units of MeV, K_T in MeV/F^3, and ρ in units of F^{-3}.

The mean free path given by Eq. (7) depends on the equation of state of the matter through the bulk modulus, K_T. If K_T becomes small as a result of nuclear interactions, then the neutrino cross sections will become enhanced. For very small K_T, Eq. (7) must be modified in a way which we shall discuss in Sec. IV.

III. OPACITY OF NEUTRON STARS

As an example we have computed the bulk modulus K_T, as a function of density, using the equation of state of Baym, Bethe, and Pethick[6] in the lower-density region and Pandharipande's model C of neutron-star matter[7] in the higher-density region. We obtain the results shown in Table I for the factor $\rho^2K_T{}^{-1}$ which appears in expression (7) for a^{-1}. We see that in the lower-density regime, and for $\lambda=1$, the mean free path can become less than one km only in the upper regions of temperature and neutrino energy under consideration, E^2 (MeV^2)$\times k_BT$ (MeV)>500 (MeV^3). At the higher densities which are encountered in the core of the neutron star we see that opacities decrease markedly, the increasing density being more than

TABLE I. The factor $\rho^2K_T{}^{-1}$ which appears in Eq. (7), as computed from Pandharipande's equation of state "C" for densities greater than 0.3 F^{-3} and from the equation of state of Baym, Bethe, and Pethick for lower densities.

ρ (F^{-3})	$\rho^2K_T{}^{-1}=\rho(dP/d\rho)^{-1}$ (10^{-3} MeV/F^3)
0.05	5.9
0.07	6.6
0.09	7.2
0.11	7.1
0.13	6.0
0.15	5.8
0.17	5.7
0.19	5.6
0.21	5.4
0.23	5.3
0.25	4.4
0.28	4.0
0.32	4.2
0.36	5.2
0.40	6.3
0.45	6.2
0.50	5.3
0.55	4.6
0.60	4.7
0.70	3.1
0.80	2.7
0.90	2.4
1.0	2.2
1.1	2.1
1.25	1.9
1.4	1.8
1.55	1.6
1.7	1.4
1.9	1.3
2.1	1.2
2.3	1.1
2.6	1.0
2.9	0.9

compensated by the increase in K_T coming from the increasing importance of the short-range repulsive forces.

In other models of neutron-star matter, K_T can be significantly smaller. For example, in a model of pion-condensed neutron-star matter considered by Hartle, Sawyer, and Scalapino[8] there is great softening of the equation of state at densities slightly above nuclear density. In this model K_T not only becomes small just above nuclear densities, but goes negative at higher densities, corresponding to a pressure dip in the plot of P against ρ, then rising at the highest densities much as in the model shown in Table I. When a star is constructed using this equation of state, the Maxwell equal-area construction dictates a density discontinuity such that a broad density region, extending beyond the region of pressure dip, is excluded from the star. In the model of Hartle, Scalapino, and Sawyer this has the effect of excluding the entire domain in which the value of K_T has become significantly smaller than the values shown in Table I, and the opacity profile for the star as a whole is similar to the one calculated from Table I.

However, it would be easy to construct models of neutron matter in which there is a thin shell of matter with relatively small K_T and correspondingly large opacity.[9] In the absence of a definitive equation of state it is interesting to ask how opaque a thin layer can become, under the most extreme conditions.

The maximum opacity of a layer is limited by two considerations:

(a) When K_T is very small, K_T^{-1} in Eq. (7) should be replaced by $K_T + q^2 b$, where the parameter b measures the resistance to formation of very small scale density fluctuations (b is the coefficient of a $\vec{\nabla}\rho \cdot \vec{\nabla}\rho/\rho^2$ term in the effective energy density).

(b) Under the gravitational loading present in a star a layer of very small K_T must also be very thin. We can see this in the following way: Suppose that the (nucleon) density ρ changes by $\delta\rho$ in a layer of thickness δz. The pressure change across this layer is

$$\delta P = \frac{K_T \delta\rho}{\rho} . \tag{8}$$

Hydrostatic equilibrium dictates that this pressure change be equal to the gravitational force per unit area on the shell,

$$\delta P = \delta z M(R) G \rho m_N R^{-2} , \tag{9}$$

where R is the radius of the shell and $M(R)$ is the mass of that portion of the star inside the inner surface of the shell. G is the gravitational constant. For the purpose of our rough estimate we have taken the matter energy density as given by ρm_N where m_N is the nucleon mass (thus neglecting interaction energy in the calculation of the gravitation). Equating the two expressions for δP gives

$$K_T = M(R) G \rho^2 m_N (\delta z) R^{-2} (\delta\rho)^{-1} . \tag{10}$$

Putting (10) into (7) we obtain an expression for the number of mean free paths, N, within the layer of thickness δz,

$$N = a^{-1}\delta z = 0.38 \times 10^{-44} E^2 \lambda^2 R^2 (\delta\rho)(k_B T) M(R)^{-1} G^{-1} m_N^{-1} . \tag{11}$$

The constant in (11) has been chosen such that all quantities except E are measured in units of some power of cm in a system in which $\hbar = c = 1$. In this system we have $G = 2.6 \times 10^{-66}$ cm^2, $m_N = 4.8 \times 10^{13}$ cm^{-1}, $M_\odot = 5.7 \times 10^{70}$ cm^{-1}, $k_B = 4.4$ cm^{-1} (°K)$^{-1}$. E is measured in MeV.

The parameter $\delta\rho$ in (11) should be taken as the width of the region in density in which $K_T(\rho)$ is anomalously small. The density change over this soft layer is at most of the order of ρ_I, the density on the inner surface of the layer. Thus we arrive at an upper limit for N in terms of ρ_I,

$$N \leq 2.3 \times 10^{-24} E^2 \lambda^2 R^2 \rho_I T M(R)^{-1} , \tag{12}$$

where we have now introduced units such that R is to be measured in cm, ρ_I in F^{-3}, T in °K, and M in solar masses.

As an example of possible interest let us consider a star of mass $\frac{1}{2}M_\odot$ and of radius 10 km and suppose that a layer near the surface has a neutron density of 0.5 F^{-3}. At $T = (10^{11})°$, in this case, a thin layer could be at most 20 mean free paths thick to a neutrino of 10 MeV.

Next we must ask whether the value of K_T implicitly assumed in the last estimate is so small that the bq^2 term in the denominator, described above in (a), must be taken into account. This depends on how thick the layer is taken to be. To obtain the same opacity in a thinner layer requires reducing the value of K_T. Thus the size of the b coefficient controls how thin the layer can be while maintaining the maximum opacity allowed by (12).

We make an *a priori* estimate of the parameter, b, as being at most $b \approx (1\ \mathrm{F})^{-2}$, since the fermi is the scale of every dimensional quantity in neutron-star matter theory. For this value of b, the bq^2 term is sufficiently small to allow the upper limit to be achieved in (12) as long as the layer is thicker than about 100 m.

To summarize the example which we worked out:

It is possible, through a local softening of the equation of state, to produce a thin layer, near the surface of a very hot neutron star, which is quite opaque to neutrinos of some energy. The neutrino scattering is elastic and predominantly at large angles.

IV. ISOSPIN DENSITY FLUCTUATIONS

The theory presented in Sec. II was simplified by the assumption that the matter is composed of neutrons only. This is nearly enough true, in conventional theories of nuclear matter, up to nucleon densities $\rho \approx 1\ F^{-3}$, although an alternative theory has large numbers of protons and pions.[10] However, for higher densities, which occur in the interiors of large neutron stars, there are expected to be significant numbers of other baryons, Λ, Σ^-, p. This complicates the picture enormously, since the neutrino current will probably couple differently to each baryon. Only if the baryon current were purely isoscalar and universal among baryons could we use our formula, which depends on the equation of state only.

We cannot, therefore, make reliable predictions in the domain of high densities. Indeed, the Pandharipande equation of state used in calculating the numbers in Table I already involves large numbers of Λ, Σ^-, and p at the high end of the density scale. Therefore, our results for the opacity in this region are correct only for the case of universal neutrino coupling.

In the Weinberg-Salam model, however, the neutral vector baryon current is predominantly the isospin current. Thus if our model contained, say, protons, neutrons, and Σ^-'s as constituents, the isospin density fluctuations, rather than the baryon density fluctuations, would enter the expression for the opacity.

This would have the effect of increasing the opacities in the higher range of density, as can be understood in the following way: In the lower range of densities, where we have almost pure neutron matter, the baryon density fluctuation is exactly twice the isospin density fluctuation, in magnitude, and we can use the results of Sec. II for any isotopic-spin structure of the neutral-current interaction, provided the constant G stands for the coupling strength to neutrons. Above densities of about $0.7\ F^{-3}$ (in the Pandharipande model) the opacity predicted in Sec. II falls fairly sharply with increasing density. This is the result of the decrease in the compressibility due to the rapid increase of potential energy associated with the short-range repulsion.

At the same time, the other baryons are becoming significant in this region. Since the short-range repulsions are thought to be universal, an isospin fluctuation at constant baryon density costs little, if any, potential energy, although it will cost some kinetic energy. Thus we anticipate that when the compressibility becomes dominated by the short-range repulsions, the isospin density fluctuations will be much larger than the baryon density fluctuations and the neutrino opacities will no longer decrease with increasing density as predicted by Table I.

In Appendix B an idealized, but quantitative, model is worked out which involves n, p, and Σ^- particles and which illustrates the way in which isospin density fluctuations can be expected to dominate baryon density fluctuations at very high densities.

V. DISCUSSION

It should be emphasized that the considerations of the present note apply only in very special circumstances, namely, at a very high temperature and in very dense matter. There will probably be a few moments after the creation of a neutron star in which these conditions are realized. The neutrino opacity in these moments may be of great interest in determining the later history of the star, as well as determining what kind of neutrino pulses, if any, should be observed from the earth. It would seem particularly interesting to consider the consequences of a possible thin shell of greatly enhanced opacity.

An important result of our work is the demonstration that the effects of nuclear forces cannot be neglected, even in a rough estimate of the opacity. Estimates based on a free Fermi gas can be wrong by orders of magnitude. In limiting the conditions to those under which the opacity is dominated by neutral-current effects, and in which the scattering is correctly described by the scattering off of the thermal density fluctuations, we have chosen that case in which the effects of the nuclear forces can be calculated in terms of the equation of state of the matter.

Many of the things which we have not calculated, e.g., the neutrino opacity due to the ordinary charged-current coupling, or that due to the Gamow-Teller part of the matrix element, or the opacity at lower temperature, also should be very dependent on the nuclear forces, or wave functions. For these cases, however, it will probably prove necessary to work out the complete theory of the neutrino scattering within the interacting nuclear medium. Our estimates are that the other sources of neutrino opacity are small compared to the neutral-current source, under the conditions of temperature and neutrino energy assumed in this paper.

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APPENDIX A: THE DOMAIN OF VALIDITY OF EQUATION (7)

As a rough guide in determining the domain of parameters in which the classical thermal density fluctuations of Eq. (6) will determine the neutrino scattering, we consider the free Fermi gas model. Neutrino differential cross sections are obtained by replacing the square of the Fourier transform of the density, $|\rho(q)|^2$, in (3) by the appropriate density-density correlation function in the Fermi gas,

$$|\rho(\vec{q})|^2 \to \langle \rho(\vec{q})\rho(-\vec{q})\rangle = (2\pi)^{-3} V \int n(\vec{p})[1-n(\vec{p}+\vec{q})]\,d^3p\ , \quad \text{(A1)}$$

where $n(p)$ is the Fermi distribution function.

In the limit of small q, keeping temperature fixed, but with $k_B T \ll \epsilon_F$ (the Fermi energy) we obtain

$$\langle \rho(\vec{q})\rho(-\vec{q})\rangle \to (k_B T) V \left(\frac{\partial^2 \epsilon}{\partial \rho^2}\right)^{-1}\ , \quad \text{(A2)}$$

where $\epsilon(\rho)$ is the energy per unit volume of the Fermi gas of density ρ, evaluated at zero temperature. This is exactly the classical result (3), with $K_T = \rho^2(\partial^2\epsilon/\partial\rho^2)$.

On the other hand, if $\vec{q}$ is kept fixed while $k_B T$ approaches zero, we obtain (when $q \ll p_F$)

$$\langle \rho(\vec{q})\rho(-\vec{q})\rangle = \tfrac{3}{4}\, q p_F^{-1} \rho V\ . \quad \text{(A3)}$$

Comparison of (A2) with (A3) reveals that the criterion for the classical term to dominate is

$$q p_F^{-1} < 2(k_B T)(\epsilon_F)^{-1}\ . \quad \text{(A4)}$$

We have chosen the domain of values in the present note that this condition is satisfied. In the case of greatest interest, in which the interactions reduce the value of K_T to an amount smaller than the free Fermi value, the dominance of the classical terms can be expected to be more complete.

APPENDIX B: A THREE-SPECIES CALCULATION OF ISOSPIN DENSITY FLUCTUATIONS

We consider a mixture of neutrons, protons, and Σ particles, with a common mass, for simplicity, and an energy per unit volume given by

$$E(\rho_n,\rho_p,\rho_\Sigma) = K(\rho_n)+K(\rho_p)+K(\rho_\Sigma) + V(\rho_n+\rho_p+\rho_\Sigma)\ , \quad \text{(B1)}$$

where $K(\rho)$ is the kinetic energy density of a particular species, which is a function of the density of that species only. We have assumed that the potential-energy density depends only on the total baryon density, as it would in the case of a universal force between baryons if exchange energies were neglected. At equilibrium, at a given total baryon density, there will be equal numbers of each species

$$\rho_n = \rho_p = \rho_\Sigma = \rho/3\ .$$

For the neutrino scattering we choose

$$H_w = G\lambda 2^{-3/2} \int d^3x [\bar{\nu}\gamma_\alpha(1+\gamma_5)\nu] \times (\bar{n}\gamma_\alpha n - \bar{p}\gamma_\alpha p + 2\bar{\Sigma}_-\gamma_\alpha\Sigma_-)\ , \quad \text{(B2)}$$

which couples the neutrino current to the vector isospin current of the baryons. Thus we wish to calculate the mean value of

$$|2\delta\rho_I(\vec{q})|^2 = |\delta\rho_n(\vec{q}) - \delta\rho_p(\vec{q}) + 2\delta\rho_\Sigma(\vec{q})|^2\ ,$$

which is to replace $|\rho(\vec{q})|^2$ in Eq. (3).

This mean value can be calculated directly from the probability distribution for fluctuations $\delta\rho_p(\vec{q})$, $\delta\rho_n(\vec{q})$, $\delta\rho_\Sigma(\vec{q})$,

$$P(\delta\rho) = \mathfrak{N} \exp\left[-(2k_B T V)^{-1} \sum_{i,j=p,n,\Sigma} \frac{\partial^2 E}{\partial\rho_i \partial\rho_j} \sum_{\vec{q}} \delta\rho_i(\vec{q})\delta\rho_j(-\vec{q})\right]\ , \quad \text{(B3)}$$

where V is the volume. In calculating expectation values we must take into account the reality of the density fluctuation $\delta\rho_i(\vec{x})$: $\delta\rho_i(\vec{q}) = \delta\rho_i^*(-\vec{q})$.

We obtain

$$2\langle|\delta\rho_I(\vec{q})|^2\rangle = k_B T V D^{-1}\left(\frac{\partial}{\partial a_{pp}} + \frac{\partial}{\partial a_{nn}} + 4\frac{\partial}{\partial a_{\Sigma\Sigma}} - 2\frac{\partial}{\partial a_{np}} + 4\frac{\partial}{\partial a_{n\Sigma}} - 4\frac{\partial}{\partial a_{p\Sigma}}\right) D\ , \quad \text{(B4)}$$

where we have defined

$$a_{ij} = \frac{\partial^2 E}{\partial\rho_i\partial\rho_j}$$

and

$$D = \mathrm{Det}\, a_{ij}\ . \quad \text{(B5)}$$

From (B1) we then obtain

$$2\langle|\delta\rho_I(\vec{q})|^2\rangle = k_B T V (6K'' + 14\Lambda'')(K'')^{-1}(K''+3\Lambda'')^{-1}\ , \quad \text{(B6)}$$

where K'' is the second derivative of the kinetic-energy function $K(\rho_i)$ introduced in (B1), evaluated at $\rho_i = \rho/3$, and Λ'' is the second derivative of the potential-energy function.

As we move into the density region in which the short-range potential dominates we find that $\Lambda'' \gg K''$, and the isospin density fluctuations become

$$2\langle |\delta\rho_I(\vec{q})|^2\rangle \to \tfrac{14}{3} k_B T V(K'')^{-1}\,, \tag{B7}$$

as contrasted to the baryon density fluctuations

$$\langle |\delta\rho_B(\vec{q})|^2\rangle = k_B T V(\tfrac{1}{3}K'' + \Lambda'')^{-1} \to k_B T V(\Lambda'')^{-1}\,. \tag{B8}$$

Thus, in this model, in the high-density limit, when $\Lambda'' \gg K''$ the isospin density fluctuations are much bigger than the baryon density fluctuations.

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