Magnetic-monopole solution of non-Abelian gauge theory in curved spacetime

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A magnetic-monopole solution of a non-Abelian gauge theory as proposed by 't Hooft is studied in curved spacetime. Einstein's equations are solved for the case of a magnetic point charge yielding a metric identical to the Reissner-Nordström metric, except that a nonvanishing cosmological constant is invoked.

I. INTRODUCTION

Recently there has been considerable interest in classical solutions of non-Abelian gauge theories. One of the main triggers was the observation by 't Hooft and others[1] that particlelike solutions, having the properties of magnetic monopoles, might exist in those theories with spontaneous symmetry breaking. One reason for studying these monopoles is their possible relevance for the theory of strong interactions.[2] In this paper we investigate 't Hooft-type solutions in the arena of curved spacetime and obtain a solution of Einstein's equations for the magnetic monopole. First we give the covariant Lagrangian; then we derive the equations of motion and show that 't Hooft-type solutions still exist in curved spacetime. After that we solve for the stress-energy of the system. We find a solution for Einstein's equations which is identical to the Reissner-Nordström metric (the spherically symmetric static solution for a radial electric field), in agreement with an extension of Birkhoff's theorem.[3] In addition, our solution invokes a nonvanishing cosmological constant. Finally we consider implications of these results to other related problems.

II. THE EQUATIONS OF MOTION

Following 't Hooft, we consider an SO(3) gauge theory with a triplet of Yang-Mills fields

W_μ^a ($\mu = 0, \ldots, 3$ spacetime indices; $a = 1, 2, 3$ isospace indices)

and a Higgs triplet

φ^a $(a = 1, 2, 3)$.

The covariant Lagrangian is given by

$$L = \int \frac{\sqrt{-g}}{8\pi}\left[-\tfrac{1}{2}g^{\alpha\beta}g^{\sigma\delta}\mathcal{G}^a_{\alpha\sigma}\mathcal{G}^a_{\beta\delta} - g^{\alpha\beta}(D_\alpha\varphi^a)(D_\beta\varphi^a) - \mu^2\varphi^2 - \frac{\lambda}{4}\varphi^4\right]d^4x\,, \qquad (2.1)$$

with

$$\mathcal{G}^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + e\epsilon_{abc}W^b_\mu W^c_\nu\,,$$
$$D_\mu\varphi^a = \partial_\mu\varphi^a + e\epsilon_{abc}W^b_\mu\varphi^c\,, \qquad (2.2)$$

where ϵ_{abc} is the totally antisymmetric tensor. Note that the (geometrical) covariant derivatives have reduced to partial derivatives; this implies that the general covariant Lagrangian still exhibits gauge invariance. Choosing μ^2 negative corresponds to a broken symmetry of the vacuum where the Higgs field has a nonzero vacuum expectation value:

$$|\langle\varphi\rangle| = F\,, \qquad (2.3)$$

where

$$F^2 = -2\mu^2/\lambda\,. \qquad (2.4)$$

The mass of the Higgs particle is given by

$$M_H = \sqrt{\lambda}\,F\,. \qquad (2.5)$$

In the gauge where the Higgs field points along the positive z axis of isospace, two components of the vector triplet acquire a mass

$$M_{W_{1,2}} = eF\,, \qquad (2.6)$$

whereas the third component corresponds to the usual photon field.

Varying the Lagrangian with respect to the fields, we obtain the following equations of motion:

$$\frac{1}{\sqrt{-g}}\partial^\nu(\mathcal{G}^a_{\mu\nu}\sqrt{-g}) - \epsilon_{abc}g^{\alpha\beta}\mathcal{G}^b_{\mu\alpha}W^c_\beta - e\epsilon_{abc}(\partial_\mu\varphi^b)\varphi^c + e^2\varphi^2W^a_\mu - e^2W^b_\mu\varphi^b\varphi^a = 0\,,$$

$$\frac{1}{\sqrt{-g}}\partial^\mu[(\partial_\mu\varphi^a + e\epsilon_{abc}W^b_\mu\varphi^c)\sqrt{-g}] - e^2\delta^{da}_{bc}g^{\alpha\beta}W^b_\alpha W^d_\beta\varphi^c + e\epsilon_{abc}g^{\alpha\beta}(\partial_\alpha\varphi^c)W^b_\beta - \mu^2\varphi^a - \frac{\lambda}{2}\varphi^2\varphi^a = 0\,. \qquad (2.7)$$

We look for solutions of the type[4]

$$W_\mu^a = \epsilon_{\nu\mu ab}\eta_\nu r^b W(r)\,, \qquad \varphi^a = r^a\varphi(r)\,, \tag{2.8}$$

where $\eta_\nu = (1, 0, 0, 0)$ is the timelike unit vector. Furthermore, we assume a spherically symmetric time-independent metric:

$$g_{\mu\nu} = \begin{bmatrix} -e^{2\phi} & 0 & 0 & 0 \\ 0 & e^{2\Lambda} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}, \tag{2.9}$$

with ϕ and Λ functions of r only.

The functions $W(r)$ and $\varphi(r)$ can then be determined by solving the resulting coupled radial equations:

$$rW'' + 4W' - 3erW^2 - e^2r^3W^3 - er\varphi^2 - e^2r^3W\varphi^2 + (rW' + 2W)(\phi' + \Lambda') = 0\,,$$
$$r\varphi'' + 4\varphi' + \frac{\lambda}{2} rF^2\varphi - \frac{\lambda}{2} r^3\varphi^3 - 2e^2r^3W^2\varphi - 4er\varphi W + (r\varphi' + \varphi)(\phi' + \Lambda') = 0\,. \tag{2.10}$$

An exact, but singular, solution of these equations is given by

$$\varphi(r) = \frac{F}{r}, \quad \varphi^a = \frac{r^a}{r} F\,, \tag{2.11}$$

and

$$W(r) = -\frac{1}{er^2}\,, \quad W_\mu^a = -\epsilon_{\nu\mu ab}\eta_\nu \frac{r^b}{r}\frac{1}{er}\,. \tag{2.12}$$

That this solution corresponds to a magnetic monopole can be seen if one inserts it into 't Hooft's gauge-invariant generalization of the electromagnetic field tensor:

$$F_{\mu\nu} = \frac{\varphi_a}{|\varphi|}\mathcal{G}_{\mu\nu}^a - \frac{1}{e}\epsilon_{abc}\frac{\varphi^a}{|\varphi|}(D_\mu\varphi^b)(D_\nu\varphi^c)\,, \tag{2.13}$$

which becomes

$$F_{\mu\nu} = -\epsilon_{\mu\nu a}\frac{r^a}{r^3}\,. \tag{2.14}$$

(Obviously $F_{\mu\nu}$ satisfies the general covariant Maxwell equations, except at the origin.)

This corresponds to a radial magnetic field

$$\vec{B} = g\frac{\vec{r}}{r^3} \tag{2.15}$$

of a magnetic point charge with

$$g = 1/e\,. \tag{2.16}$$

III. STRESS-ENERGY TENSOR

Varying the Lagrangian with respect to the metric yields the stress-energy tensor

$$T_{\alpha\beta} = -2\frac{\delta\mathcal{L}}{\delta g^{\alpha\beta}} + g_{\alpha\beta}\mathcal{L}\,, \tag{3.1}$$

which in our case is

$$T_{\alpha\beta} = (1/4\pi)\{g^{\mu\nu}\mathcal{G}_{\alpha\mu}^a\mathcal{G}_{\beta\nu}^a - \tfrac{1}{4}g_{\alpha\beta}g^{\mu\nu}g^{\sigma\tau}\mathcal{G}_{\mu\sigma}^a\mathcal{G}_{\nu\tau}^a + (D_\alpha\varphi^a)(D_\beta\varphi^a) - \tfrac{1}{2}g_{\alpha\beta}g^{\mu\nu}(D_\mu\varphi^a)(D_\nu\varphi^a) - g_{\alpha\beta}[\tfrac{1}{2}\mu^2\varphi^2 + (\lambda/8)\varphi^4]\}\,.$$

Inserting the solutions (2.11) of the field equations, we obtain the following expression for the gauge field tensor:

$$\mathcal{G}_{\mu\nu}^a = \frac{g}{r^4}(-2\epsilon_{\mu\nu a}r^2 - 2\epsilon_{\mu ab}r_b r_\nu + 2\epsilon_{\nu ab}r_b r_\mu + \epsilon_{\mu\nu b}r_a r_b)\,.$$

To evaluate the stress-energy tensor, we must transform $\mathcal{G}_{\mu\nu}^a$ into spherical coordinates. Since

$$\mathcal{G}_{0\nu}^a = \mathcal{G}_{\nu 0}^a = \mathcal{G}_{mm}^a = 0\,,$$

we get, for example,

$$\mathcal{G}_{\theta\varphi}^a = \mathcal{G}_{xy}^a\left(\frac{dx}{d\theta}\frac{dy}{d\varphi} - \frac{dy}{d\theta}\frac{dx}{d\varphi}\right) + \mathcal{G}_{xz}^a\left(\frac{dx}{d\theta}\frac{dz}{d\varphi} - \frac{dz}{d\theta}\frac{dx}{d\varphi}\right) + \mathcal{G}_{yz}\left(\frac{dy}{d\theta}\frac{dz}{d\varphi} - \frac{dz}{d\theta}\frac{dy}{d\varphi}\right)\,.$$

The transformed tensors become

$$\mathcal{G}_{\theta\varphi}^x = -g\sin^2\theta\cos\varphi\,, \qquad \mathcal{G}_{\theta\varphi}^y = -g\sin^2\theta\sin\varphi\,, \qquad \mathcal{G}_{\theta\varphi}^z = -g\sin\theta\cos\theta\,. \tag{3.2}$$

All the others vanish. Putting these into our expression for the stress-energy tensor, we find the following result:

$$T_{tt} = \frac{1}{8\pi}e^{2\phi}\left(\frac{g^2}{r^4} - \beta\right), \quad T_{rr} = -\frac{1}{8\pi}e^2\left(\frac{g^2}{r^4} - \beta\right), \quad T_{\theta\theta} = \frac{1}{8\pi}\left(\frac{g^2}{r^2} + \beta r^2\right), \quad T_{\varphi\varphi} = T_{\theta\theta}\sin^2\theta\,, \tag{3.3}$$

where we have set

$$\beta \equiv \mu^4/\lambda = \tfrac{1}{4} g^2 M_H{}^2 M_W{}^2 \ . \qquad (3.4)$$

Note that, apart from the contribution of the scalar field due to symmetry breaking, the stress-energy of the system comes out the same as in Abelian electrodynamics. The reason is that, for our solution, $D_\mu \varphi^a$ vanishes everywhere (even though $\partial_\mu \varphi$ does not) and therefore does not contribute to the stress-energy.

IV. SOLUTION OF EINSTEIN'S EQUATIONS

We now solve Einstein's equations

$$G_{\mu\nu} + \gamma g_{\mu\nu} = 8\pi k T_{\mu\nu} \ , \qquad (4.1)$$

where $G_{\mu\nu}$ is the Einstein tensor, k is the gravitational constant, and where we have explicitly included the cosmological constant γ for the following reason. Let us write $T_{\mu\nu}$ as

$$T_{\mu\nu} = T^*_{\mu\nu} + T^{(\text{vac})}_{\mu\nu} \ , \qquad (4.2)$$

where

$$T^{(\text{vac})}_{\mu\nu} = \frac{\beta}{8\pi} g_{\mu\nu} + T^0_{\mu\nu} \ . \qquad (4.3)$$

We interpret $T^{(\text{vac})}_{\mu\nu}$ as the total stress-energy of the vacuum, with $(\beta/8\pi) g_{\mu\nu}$ as the contribution induced by spontaneous symmetry breaking. In field theory, the energy of the vacuum is an arbitrary constant since it does not appear in any observable quantity, and one conventionally chooses the stress-energy of the symmetric vacuum $T^0_{\mu\nu}$ to be zero. This leads in our case[5] to the relation

$$\gamma/k = \beta \ , \qquad (4.4)$$

which shows that the stress-energy of the vacuum with broken symmetry is directly related to the observable cosmological constant.[6] From observations[7] one obtains an upper bound of 10^{-29} g/cm^3 for γ/k, corresponding in our units ($\hbar = c = 1$) to $\sim 10^{-47}$ GeV^4. The quantity β contains the masses of the scalar and vector fields which are as yet unknown. Though the mass of the vector boson can be estimated from theory, it is very model dependent. We chose for simplicity the SO(3) model corresponding to the Georgi-Glashow theory of electromagnetic and weak interactions. This model, however, has turned out to be inconsistent with observations because it does not include neutral currents.[8] On the other hand, in the favorite Weinberg SU(2)×U(1) model, the magnetic monopole solution does not exist because the gauge group is not compact.[1] This theory would have to be extended to a bigger compact group—for example, Georgi and Glashow have more recently suggested SU(5) for a combined theory of strong, electromagnetic, and weak interactions.[9] An implication of this particular model, however, is that the mass of the heaviest vector boson would become at least 10^{10} GeV, instead of ~20 GeV in the above theories.

Now one can use relation (4.4) to find an upper limit for the mass of the Higgs particle[10] (we take $M_W \simeq 20$ GeV): $M_H \lesssim 10^{-19}$ GeV. Being practically massless, this Higgs particle would give rise to a long-range scalar coupling many orders of magnitude greater than the gravitational tensor coupling, and clearly inconsistent with experiment.

Another possibility is to abandon the assumption that $T^0_{\mu\nu}$ is zero, and thus relax our restriction on the mass of the Higgs particle. This alternative implies a large negative curvature for the early universe, before a possible transition to the present state with broken symmetry.[11]

Returning to Einstein's equations, we must now solve

$$G_{\mu\nu} = 8\pi k T^*_{\mu\nu} \ . \qquad (4.5)$$

Because of the spherical symmetry of $T^*_{\mu\nu}$, Birkhoff's theorem[3] predicts that the solution will be a piece of the Reissner-Nordström spacetime (4.10). In spherical coordinates, the equations (4.5) take the following simple form:

$$\begin{aligned} &e^{2(\phi-\Lambda)}\left(\frac{2\Lambda'}{r} - \frac{1}{r^2}\right) + \frac{e^{2\phi}}{r^2} = \frac{g^2 k}{r^4} e^{2\phi} \ , \\ &\frac{2\phi'}{r} + \frac{1}{r^2} - \frac{e^{2\Lambda}}{r^2} = -\frac{g^2 k}{r^4} e^{2\Lambda} \ , \\ &r^2 e^{-2\Lambda}\left[\phi'^2 - \phi'\Lambda' + \phi'' + \frac{1}{r}(\phi' - \Lambda')\right] = \frac{g^2 k}{r^2} \ . \end{aligned} \qquad (4.6)$$

The first equation can be solved for $e^{-2\Lambda}$:

$$\begin{aligned} &\frac{d}{dr}(re^{-2\Lambda}) = 1 - \frac{g^2 k}{r^2} \ , \\ &e^{-2\Lambda} = 1 - \frac{Ak}{r} + \frac{g^2 k}{r^2} \ . \end{aligned} \qquad (4.7)$$

Clearly, the second and third equations are satisfied by

$$\phi = -\Lambda \ . \qquad (4.8)$$

By looking at orbits of a neutral test particle, one can determine the constant A as

$$A = -2M_{\text{monopole}} \ . \qquad (4.9)$$

Hence, the metric for the magnetic monopole in curved spacetime is

$$ds^2 = -\left(1 - \frac{2M_m k}{r} + \frac{g^2 k}{r^2}\right) dt^2 + \left(1 - \frac{2M_m k}{r} + \frac{g^2 k}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \ . \qquad (4.10)$$

Since

$$\frac{g^2}{kM_m^{\,2}} \sim 10^9 \gg 1 \ ,$$

the solution has no horizons.

V. CONCLUSION

We have found that an explicit solution of Einstein's equations for a non-Abelian gauge theory yields the same result as one would expect for a point charge in the Abelian theory. There are two features of the system we studied which caused this simplicity. First of all, the Lagrangian of the system only contains a scalar field and antisymmetric combinations of the momenta of the vector fields, so that the affine connections vanish as we noted following Eq. (2.2). Secondly, since the singular solution of the non-Abelian field equations represents a purely electromagnetic solution and hence the stress-energy possess spherical symmetry and time independence, it follows that

$$\phi = -\Lambda \ . \tag{5.1}$$

This means that the Einstein equations (4.6) decouple from the field equations (2.10); the latter then become identical to the flat-space equations. (Note that in our particular solution, this decoupling takes place independently of the relation between ϕ and Λ.)

Another monopole solution was proposed by 't Hooft which behaves in flat space for large r like the solution discussed here, but which is nonsingular at the origin. This solution is more physical since it has a finite self-energy. It too will exist in curved spacetime, but it will be essentially different to construct. Since this solution is not purely electromagnetic, one expects that Einstein's equations will not decouple from the other field equations as was the case in our calculation.

Finally, for the dyon[12] (a pole with both electric and magnetic charge) with spherically symmetric charge distributions about a common origin, solutions of Einstein's equations outside the distributions will take the form (4.10) with

$$g^2 \to g^2 + e^2 \ .$$

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[4] When we write $\epsilon_{\mu\nu ab}$ it is understood that the Greek indices are raised and lowered with the spacetime metric and the Latin indices with the Euclidean metric.

[5] If one chooses $\gamma g_{\mu\nu} = 8\pi k T^0_{\mu\nu}$ the metric (4.7) acquires an extra term:

$$e^{2\phi} = 1 - \frac{2mk}{r} + \frac{g^2 k}{r^2} + \frac{\beta^3 k r^2}{3} \ .$$

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