Influence of gravitation on the propagation of electromagnetic radiation*

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The existence of a general helicity-rotation coupling is demonstrated for electromagnetic waves propagating in the field of a slowly rotating body and in the Gödel universe. This coupling leads to differential focusing of circularly polarized radiation by a gravitational field which is detectable for a rapidly rotating collapsed body. The electromagnetic perturbations and their frequency spectrum are given for the Gödel universe. The spectrum of frequencies is bounded from below by the characteristic rotation frequency of the Gödel universe. If the universe were rotating, the differential focusing effect would be extremely small due to the present upper limit on the anisotropy of the microwave background radiation.

I. INTRODUCTION

Some of the well-known tests of Einstein's theory of gravitation (bending of light by the sun, gravitational redshift, and radar time delay) deal with electromagnetic theory in curved spacetime in the approximation of geometrical optics. Harwit *et al.*[1] have recently initiated some interesting experiments which could in principle serve as a test of Einstein's theory beyond the geometrical-optics limit.[2] It is the purpose of the present paper to discuss further the physical-optics effects in a gravitational field. Such effects are of fundamental interest since they involve the wave nature of light and thus go beyond the principle of equivalence.

Electromagnetic waves in a gravitational field can be thought of as propagating in flat spacetime but in the presence of a "medium" whose properties are determined by conformally invariant quantities constructed from the metric tensor of the Riemannian spacetime in Cartesian coordinates. In a Cartesian frame the electromagnetic field tensor is decomposed,[3] $F_{\mu\nu} \to (\vec{E}, \vec{B})$, and $(-g)^{1/2}F^{\mu\nu} \to (-\vec{D}, \vec{H})$ to yield the usual Maxwell's equations in a medium together with the constitutive relations[4]

$$D_i = \epsilon_{ij} E_j - (\vec{G}\times\vec{H})_i \,, \tag{1}$$

$$B_i = \mu_{ij} H_j + (\vec{G}\times\vec{E})_i \,, \tag{2}$$

where $\epsilon_{ij} = \mu_{ij} = -(-g)^{1/2} g^{ij}/g_{00}$ and $G_i = -g_{0i}/g_{00}$. It is now possible to write the electromagnetic equations in a form which is particularly suitable for the discussion of wave phenomena in a gravitational field. If the vectors $\vec{F}^{\pm} = \vec{E} \pm i\vec{H}$ and $\vec{S}^{\pm} = \vec{D} \pm i\vec{B}$ are introduced, then

$$S_i^{\pm} = \epsilon_{ij} F_j^{\pm} \pm i(\vec{G}\times\vec{F}^{\pm})_i \tag{3}$$

and the equations[5,6]

$$\frac{1}{i}\,\vec{\nabla}\times\vec{F}^{\pm} = \pm\frac{\partial}{\partial t}\,\vec{S}^{\pm}, \tag{4}$$

$$\vec{\nabla}\cdot\vec{S}^{\pm} = 0 \tag{5}$$

determine the properties of waves in a gravitational field.

In a gravitational field there is an infinity of possible Cartesian systems that can be used for the decomposition of the field tensors discussed above. Of special interest are the systems that are related to each other by $t' = t + \phi(\vec{x})$ and $\vec{x}' = \vec{x}$, which form a subgroup of the group of transformations between the different Cartesian frames. It can be shown that such a transformation induces in turn the gauge transformation $G_i' = G_i + \partial\phi/\partial x^i$ and $\epsilon_{ij}' = \epsilon_{ij}$ under which $\vec{F}^{\pm}$ remains invariant, $\vec{F}'^{\pm}(t', \vec{x}) = \vec{F}^{\pm}(t, \vec{x})$. Thus $\vec{G}$ and $U_{ij} = \delta_{ij} - \epsilon_{ij}$ have the interpretation of vector and tensor potentials, respectively.

A classical massless spinning test particle follows a null geodesic with its spin either parallel or antiparallel to the direction of motion. Moreover, the helicity is conserved in an orientable spacetime.[7] This classical picture for the photon agrees with the geometrical-optics limit where a ray follows a null geodesic regardless of its polarization state and the polarization vector is parallel transported along the ray. It can be easily shown that the law of conservation of helicity holds in general for the photon. In the propagation of polarized electromagnetic radiation in a gravitational field (asymptotically flat spacetime with no other electromagnetic fields present) the right-circularly-polarized (RCP) radiation ($\vec{F}^- = 0$, polarization properties determined at infinity) decouples from the left-circularly-polarized (LCP) radiation ($\vec{F}^+ = 0$) according to (3)–(5); hence the helicity of the photon is conserved. Moreover, if the gravitational field is matter-free and spherically symmetric the scattering amplitude is independent of the photon helicity and the polarization state of the incident wave is unaltered by the gravitational field. In general, however, the scattering amplitude is helicity dependent. This is investigated in this paper for wave propagation in the field of a slowly rotating body and in the rotating cosmological model due to Gödel. The existence of a helicity-rotation coupling is explicitly demonstrated. Though this effect has been shown only for these particular cases, it is conjectured to be of general validity. It is further shown (Appendix A) that a very similar effect occurs in the propagation of high-frequency electromagnetic waves in a magnetoactive plasma. The astrophysical implications of the results are discussed and it is pointed out that the gravitational and plasma effects can be experimentally distinguished since they vary differently with the wavelength of the radiation.

II. DIFFERENTIAL GRAVITATIONAL DEFLECTION OF POLARIZED RADIATION

The position of the image of a point source whose waves pass through the gravitational field of a rotating body in general depends on the circular polarization state of the radiation. An estimate of the angular separation of the two images will be given in this section for initially unpolarized radiation. The propagation equation for waves of frequency ω in the exterior field of a slowly rotating body of mass m and angular momentum $\vec{J}=J\hat{x}_3$ can be written as[6]

$$\left(\frac{1}{i}\vec{\nabla}-\omega\vec{G}\right)\times\vec{f}^{\pm}=\mp i\omega N\vec{f}^{\pm}, \qquad (6)$$

where $\vec{F}^{\pm}=\vec{f}^{\pm}(\vec{x})\exp(-i\omega t)$, $\vec{G}=-2\vec{J}\times\vec{x}/|\vec{x}|^3$, and $N\simeq 1+2m/|\vec{x}|$ for $|\vec{x}|\gg m$. The differential focusing effect for circularly polarized radiation is due to the presence of $\vec{G}$ since the index of refraction N causes only a general deflection for any wave packet.[8] Thus consider the solution of (6) with $N=1$ and for a wave packet of frequency ω, $\omega m\gg 1$, propagating nearly in the x_2-x_3 plane. The influence of the rotating body on the packet is considered at a distance D ($D\gg m$) along the x_3 axis, where $\vec{G}$ can be written as $G_1\simeq\eta x_2$, $G_2\simeq-\eta x_1$, and $G_3=0$, with $\eta=2J/D^3$. The dimensions of the packet are assumed to be much smaller than D. Now introduce the coordinate transformation $t'=t-\eta x_1x_2$ which induces the gauge transformation $\vec{\Psi}^{\pm}=\exp(-i\omega\eta x_1x_2)\vec{f}^{\pm}$ under which (6) can be written as

$$\left(\frac{1}{i}\vec{\nabla}+\omega\vec{A}\right)\times\vec{\Psi}^{\pm}=\mp i\omega\vec{\Psi}^{\pm}, \qquad (7)$$

with $A_1=A_3=0$ and $A_2=2\eta x_1$. This wave equation can be easily solved with

$$\vec{\Psi}^{\pm}=\vec{\psi}^{\pm}(x_1)\exp(iK_2^{\pm}x_2+iK_3^{\pm}x_3). \qquad (8)$$

If a new variable ξ is introduced,

$$\xi^{\pm}=(2\eta\omega)^{-1/2}(2\eta\omega x_1+K_2^{\pm}), \qquad (9)$$

then $\psi_3^{\pm}(\xi^{\pm})$ satisfies a harmonic-oscillator wave equation, which, assuming that $\psi_3^{\pm}$ is finite everywhere, implies that

$$\omega^2=(K_3^{\pm})^2\mp 2\eta K_3^{\pm}+2\eta\omega(2n^{\pm}+1) \qquad (10)$$

for $n^{\pm}=0,1,2,\ldots,$ and that

$$\psi_3^{\pm}(n^{\pm};\xi^{\pm})=C^{\pm}(n^{\pm})\exp[-\tfrac{1}{2}(\xi^{\pm})^2]H_{n^{\pm}}(\xi^{\pm}), \qquad (11)$$

where $H_n(\xi)$ is a Hermite polynomial and

$$C^{\pm}(n)=[2(n+1)]^{1/2}C^{\pm}(n+1). \qquad (12)$$

It can also be shown that

$$\psi_1^{\pm}(n^{\pm};\xi^{\pm})=iA^{\pm}\psi_3^{\pm}(n^{\pm}+1;\xi^{\pm})+iB^{\pm}\psi_3^{\pm}(n^{\pm}-1;\xi^{\pm}), \qquad (13)$$

$$\psi_2^{\pm}(n^{\pm};\xi^{\pm})=A^{\pm}\psi_3^{\pm}(n^{\pm}+1;\xi^{\pm})-B^{\pm}\psi_3^{\pm}(n^{\pm}-1;\xi^{\pm}), \qquad (14)$$

where $A^{\pm}$ and $B^{\pm}$ are given by

$$(\omega\pm K_3^{\pm})A^{\pm}=\pm[\eta\omega(n^{\pm}+1)]^{1/2}, \qquad (15)$$

$$(\omega\mp K_3^{\pm})B^{\pm}=\pm[\eta\omega n^{\pm}]^{1/2}. \qquad (16)$$

The presence of a helicity–angular-momentum coupling is evident in Eq. (10). It is clear that when the direction of propagation is orthogonal to the rotation vector there is no differential focusing effect to first order in the angular momentum of the body. It follows from (10) that for waves of the same frequency following approximately the same path

$$K_3^{+}-K_3^{-}=\gamma\eta, \qquad (17)$$

where γ is a function of the frequency of the wave and its propagation vector.[9] Equation (17) implies that for an unpolarized wave packet propagating in the field of a rotating body the different helicity states generally separate within the wave packet, which amounts to a differential focusing angle of

$$\delta\simeq|\gamma(\omega,\theta)|\,\omega^{-1}\eta\sin\theta \qquad (18)$$

where θ is the angle between the direction of propagation and the rotation vector and $K_2^{+}=K_2^{-}$. An estimate of δ can be obtained from $\delta_0=\lambda J/D_0^{\,3}$, where λ is the wavelength of the radiation and D_0 is the distance of closest approach of the beam with respect to the body. For a maximally rotating black hole of $m=6M_{\odot}$, $\nu=8$ GHz, and $D_0=10m$ one obtains $\delta_0=8.7\times10^{-4}$ arc sec, which is well within the observational possibilities.[10] Observation of binary systems with a collapsed component is therefore promising for the detection of the new effect. There can be difficulties, however, with the presence of magnetic fields (plasma birefringence). These problems can be surmounted by performing the experiments at different wavelengths. Appendix A develops a close analogy between the gravitational effects discussed here and the influence of a collisionless magnetoactive plasma on the propagation of electromagnetic waves. It is shown there that (in the quasilongitudinal approximation) the plasma double refraction depends on the wavelength of the radiation as λ^3 whereas the differential focusing effect varies linearly with λ. This may then be used to distinguish the two effects.

It should be emphasized that our discussion so far concerns the circular polarization states of the radiation. An initially linearly polarized beam becomes, in general, elliptically polarized because of scattering by a gravitational field, and the differential cross section for either state of initially orthogonal linear polarization is equal to that of initially unpolarized radiation. These general results can be simply obtained by using the fact that the scattering matrix is diagonal in the circular polarization basis. Thus the differential focusing phenomenon is absent for linearly polarized states and a null test of the theory is possible for this case.[11]

It is of interest to consider the propagation of electromagnetic waves along the rotation axis of the body ($K_2^{\pm}=0,\ K_3^{\pm}>0$). If we assume that $2\eta n^{\pm} \ll \omega$, then we can write Eq. (10) as

$$\omega = K_3^{\pm} + \eta(2n^{\pm}+1) \mp \eta \tag{19}$$

to first order in η. It follows from (15) and (16) that $A^+ \ll B^+$ and $B^- \ll A^-$, so

$$\psi_1^+(n^+;\xi) \simeq -i\psi_2^+(n^+;\xi) \simeq iB^+\psi_3^+(n^+-1;\xi), \tag{20}$$

$$\psi_1^-(n^-;\xi) \simeq i\psi_2^-(n^-;\xi) \simeq iA^-\psi_3^-(n^-+1;\xi). \tag{21}$$

Thus if a linearly polarized wave starts at $x_3 \simeq D$ and propagates out to infinity along the x_3 axis we must have $n^+ - 1 = n^- + 1$, so (19) implies

$$K_3^- - K_3^+ = 2\eta, \tag{22}$$

which amounts to a rotation of the plane of linear polarization in the same direction as the body given by

$$\tfrac{1}{2}\int_D^{\infty} (K_3^- - K_3^+)\,dx_3 = J/D^2. \tag{23}$$

This is a geometrical-optics result analogous to Faraday rotation and can also be obtained from the fact that the polarization vector is parallel propagated along a null geodesic.[12] Moreover, it is independent of the wavelength λ and hence can be distinguished from the Faraday effect, which varies as λ^2. A more thorough discussion of the rotation of the plane of linear polarization is given in Appendix B, where some conflicting results of different authors are discussed.

III. ELECTROMAGNETIC WAVES IN A ROTATING UNIVERSE

The helicity-angular-momentum coupling effect described in the previous section is expected to be present in a similar way in the propagation of electromagnetic waves in a rotating cosmological model. This is demonstrated here for the model discovered by Gödel.[13] It turns out that electromagnetic perturbation equations can be solved exactly in the Gödel universe, and the dispersion relation for the continuous spectrum of the modes shows the desired effect.

The metric of the stationary and spatially homogeneous solution found by Gödel can be written in a Cartesian frame as

$$\begin{aligned}-ds^2 = -dt^2 + 2\sqrt{2}\,[1-\exp(\sqrt{2}\,\Omega x_1)]\,dt\,dx_2 \\ + dx_1^2 + [4\exp(\sqrt{2}\,\Omega x_1) - \exp(2\sqrt{2}\,\Omega x_1) - 2]\,dx_2^2 \\ + dx_3^2,\end{aligned} \tag{24}$$

where $\Omega \geq 0$ is a constant parameter related to the vorticity of matter and (24) reduces to the flat-spacetime metric for $\Omega \to 0$. Now consider the coordinate transformation $\vec{x}\,' = \vec{x}$, and

$$t' = t - \sqrt{2}\,x_2, \tag{25}$$

which leaves the Cartesian nature of the system invariant as discussed previously. Then the metric takes the form

$$\begin{aligned}-ds^2 = -dt'^2 - 2\sqrt{2}\exp(\sqrt{2}\,\Omega x_1)\,dt'\,dx_2 \\ + dx_1^2 - \exp(2\sqrt{2}\,\Omega x_1)\,dx_2^2 + dx_3^2,\end{aligned} \tag{26}$$

which is closely related to the one originally given by Gödel. The Einstein tensor for (26) is

$$G_{\mu\nu} = \Omega^2(2u_\mu u_\nu + g_{\mu\nu}), \tag{27}$$

where $u^\mu = \delta_0^\mu$ is the timelike Killing vector. Thus, with u_μ as the velocity vector of matter, (26) can be interpreted as the metric of a universe with cosmological constant $\Lambda = -\Omega^2$ and pressure-free matter of constant density.[14] The vorticity vector of matter is given by twice the angular velocity vector

$$\Omega^\mu = \tfrac{1}{2}(-g)^{-1/2}\epsilon^{\mu\nu\rho\sigma}u_\nu u_{\rho;\sigma}, \tag{28}$$

which turns out to be $\Omega^\mu = \Omega\delta_3^\mu$; Ω is related to the density of matter ρ by $\Omega^2 = 4\pi\rho$. The Gödel universe has some remarkable properties: An absolute time coordinate cannot be defined over all of the spacetime, and there exist closed timelike lines. These have interesting consequences for the propagation of waves in this model.

The asymptotically flat nature of a spacetime is crucial in the interpretation of (4) and (5) as helicity wave equations. For the Gödel universe this interpretation can still be retained in the sense that (4) and (5) reduce to the corresponding helicity wave functions in the limit $\Omega \to 0$.[15] The vector and tensor potentials for the metric (26) are given by $G_1 = G_3 = 0$, $G_2 = -\sqrt{2}\, u^{-1}$, $\epsilon_{11} = \epsilon_{33} = u^{-1}$, and $\epsilon_{22} = u$, where the function $u = \exp(-\sqrt{2}\,\Omega x_1)$, $0 \leq u < \infty$, has been introduced. The electromagnetic wave equations can be solved by assuming a solution of the form

$$\vec{F}^{\pm} = \vec{\phi}^{\pm}(u) \exp(iK_2^{\pm} x_2 + iK_3^{\pm} x_3 - i\omega t'), \tag{29}$$

where u is treated as a "radial coordinate." It is then simple to show that

$$\phi_1^{\pm} + iu\phi_2^{\pm} = a^{\pm}\phi_3^{\pm} + b^{\pm} u \frac{d}{du}\phi_3^{\pm}, \tag{30}$$

$$\phi_1^{\pm} - iu\phi_2^{\pm} = \tilde{a}^{\pm}\phi_3^{\pm} - \tilde{b}^{\pm} u \frac{d}{du}\phi_3^{\pm}, \tag{31}$$

where $a^{\pm}$ and $b^{\pm}$ are given by

$$(\omega \pm K_3^{\pm}) a^{\pm} = \pm i(\sqrt{2}\,\omega + uK_2^{\pm}), \tag{32}$$

$$(\omega \pm K_3^{\pm}) b^{\pm} = \pm\sqrt{2}\, i\Omega \tag{33}$$

and $\tilde{a}^{\pm}$ and $\tilde{b}^{\pm}$ can be obtained from $a^{\pm}$ and $b^{\pm}$ by simply replacing $K_3^{\pm}$ by $-K_3^{\pm}$ in (32) and (33), respectively. The wave equation for $\phi_3^{\pm} = \chi$ is of the form

$$\left[\frac{d^2}{du^2} - \alpha^2 - \frac{\beta}{u} - \frac{\zeta(\zeta+1)}{u^2}\right]\chi = 0, \tag{34}$$

where $\alpha = K_2^{\pm}/\sqrt{2}\,\Omega$, $\beta = \sqrt{2}\,\omega K_2^{\pm}/\Omega^2$, and

$$\zeta = -\frac{1}{2} + \left[\frac{\omega^2 + (K_3^{\pm} \mp \Omega)^2}{2\Omega^2} - \frac{1}{4}\right]^{1/2}. \tag{35}$$

The perturbations have to be finite everywhere (and for all time); therefore ω, $K_2^{\pm}$, and $K_3^{\pm}$ are real and only those solutions of (34) for which $\vec{\phi}^{\pm}$ is bounded are acceptable. It is clear from (34) that no bounded solution exists for radiation propagating in the x_3 direction ($K_2^{\pm} = 0$). Consider next electromagnetic waves with $K_2^{\pm} < 0$. A detailed examination of (30), (31), and (34) shows that the only possible solution is

$$\chi = c e^{-v/2} v^{1+\zeta} F(-n, 2+2\zeta; v), \tag{36}$$

where $v = -2\alpha u$, $n = 0, 1, 2, \ldots$, $\zeta \geq 0$, F is a confluent hypergeometric function, and c is a constant. The frequency ω is given by

$$\omega = (1 + n + \zeta)\Omega, \tag{37}$$

so $\omega \geq \Omega$. It is interesting to note that no electromagnetic waves with frequency smaller than Ω can propagate in the Gödel universe. The relation (37) can also be written in the form

$$\omega^2 = (K_3^{\pm})^2 \mp 2\Omega K_3^{\pm} + 2\omega\Omega(2n^{\pm}+1) - 2\Omega^2 n^{\pm}(n^{\pm}+1), \tag{38}$$

with $n^{\pm} = 0, 1, 2, \ldots$. It is evident from (38) that, as expected, there is a helicity–angular-velocity coupling in the propagation of photons in this rotating-universe model. A similar analysis shows that for $K_2^{\pm} > 0$ the only possible solution is the same as (36) except with $v = 2\alpha u$. Thus in this case the spectrum is given by

$$\omega = -(1 + n + \zeta)\Omega, \tag{39}$$

so $\omega \leq -\Omega$ and the dispersion relation is the same as (38) except for $\omega \to -\omega$. Electromagnetic waves traveling forward in time can therefore propagate only backward along the x_2 direction. This circumstance is apparently a manifestation of the curious nature of time in the Gödel solution. The solutions with negative frequency can be interpreted as positive-frequency waves traveling backward in time. In this way one can recover the waves propagating forward along the x_2 direction with the same dispersion relation as (38). This interpretation is in accord with the existence of closed null curves and the fact that there is no absolute time coordinate in the Gödel universe.[16]

In the propagation of an unpolarized wave packet of frequency ω in the Gödel universe the different helicity states generally separate, so that

$$K_3^{+} - K_3^{-} = \Gamma\Omega, \tag{40}$$

where Γ is a coefficient which depends on ω, Ω, and $K_3^{\pm}$. The general nature of the coupling between helicity and angular velocity suggests that in any rotating universe a relation of the form (40) might hold. The isotropy of the microwave background radiation places a stringent upper limit on the rotation of the universe.[17] If the universe were rotating, the image of a distant point source viewed in one circularly polarized radiation would be displaced when the orthogonal circular polarization was used for observation. However, the angular separation of the images $\delta \simeq |\Gamma|\omega^{-1}\Omega \sin\theta$, where θ is the angle between the rays and the axis of rotation, would be exceedingly small, if it existed at all.[18]

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APPENDIX A: ANALOGY WITH PROPAGATION OF ELECTROMAGNETIC WAVES IN A MAGNETOACTIVE PLASMA

There exists a certain analogy between electromagnetic wave propagation in the field of a rotating body and in a magnetoactive plasma. This is due to the presence of a coupling between the helicity of the photon and the magnetic field which can be easily demonstrated in the propagation of electromagnetic waves in a collisionless uniform plasma in the presence of a constant external magnetic field. It turns out[19] that the waves have a dispersion relation of the form

$$\omega^2 = K^2 + \frac{\omega_p^2}{1-\mu}\,, \tag{A1}$$

where ω_p is the plasma frequency and μ is a solution of

$$\mu^2 - \left(\frac{\omega_c \sin\theta}{n_0\omega}\right)^2 \mu - \left(\frac{\omega_c\cos\theta}{\omega}\right)^2 = 0. \tag{A2}$$

Here θ is the angle between the propagation vector and the external magnetic field, ω_c is the cyclotron frequency, and

$$n_0^2 = 1 - \frac{\omega_p^2}{\omega^2}\,. \tag{A3}$$

The motion of the ions together with all dissipative phenomena have been neglected. The solutions of (A2) can be written as

$$\mu = \sigma\frac{\omega_c}{\omega}\cos\theta, \tag{A4}$$

where

$$\sigma = \epsilon \pm (1+\epsilon^2)^{1/2}$$

and

$$\epsilon = \frac{\omega_c}{2n_0^2\omega}\,\frac{\sin^2\theta}{\cos\theta}\,,$$

so that in general there exists double refraction. The electric field has a component along the direction of propagation which vanishes in the absence of the external magnetic field. Let $E_\parallel = \vec{E}\cdot\hat{e}_\parallel$ and $E_\perp = \vec{E}\cdot\hat{e}_\perp$ ($\hat{e}_\perp = \hat{K}\times\hat{e}_\parallel$) be the electric field components orthogonal to $\vec{K}$ such that $\hat{e}_\parallel$ is parallel to the plane of the external magnetic field and $\vec{K}$. Then it can be shown that

$$E_\perp = i\sigma E_\parallel. \tag{A5}$$

Consider now the quasilongitudinal approximation where $|\epsilon| \ll 1$. Then to first order in the external field $\sigma = \pm 1$, and thus $\sigma = 1$ corresponds to a nearly right-circularly-polarized (extraordinary) wave and $\sigma = -1$ to a nearly left-circularly-polarized (ordinary) wave. This approximation is valid in most astrophysical circumstances where the propagation of high-frequency waves ($\omega \gg \omega_p$, $\omega \gg \omega_c$) is considered.[20] Then the dispersion relation can be written as

$$\omega^2 \simeq \omega_p^2 + K^2 + \sigma\omega^{-1}\omega_p^2\omega_c\cos\theta, \tag{A6}$$

where the presence of the helicity–magnetic-field coupling is evident. It is clear that $\theta = \pi/2$ is not included in $|\epsilon| \ll 1$, and in fact for propagation perpendicular to the magnetic field the double refraction is of second order in the external field. For propagation in the x_2-x_3 plane with the external field in the x_3 direction (A6) can be written as

$$\omega^2 \simeq \omega_p^2 + (K_2^\pm)^2 + (K_3^\pm)^2 \pm \omega^{-2}\omega_p^2\omega_c K_3^\pm, \tag{A7}$$

so that, for incident circularly polarized waves traveling in the same direction, $K_2^+ = K_2^-$ and

$$K_3^- - K_3^+ = \omega^{-2}\omega_p^2\omega_c\,. \tag{A8}$$

The double refraction angle δ is then given by

$$\delta \simeq \omega^{-3}\omega_p^2\omega_c\sin\theta. \tag{A9}$$

In general, both double refraction and Faraday rotation occur; the latter effect can be readily computed from (A8).

APPENDIX B: ROTATION OF THE PLANE OF POLARIZATION

In this appendix an approximate expression is derived for the geometrical-optics effect of the rotation of the plane of linear polarization in the gravitational field of a rotating body. This problem has been discussed in the literature by Skrotskii,[21] Balazs,[22] Plebanski,[4] and (more recently) Godfrey.[12] Dehnen[23] has considered this effect in the field of a rotating homogeneous ring and Brans[24] has studied the influence of an anisotropic cosmological field on the rotation of the plane of polarization of the radiation propagating through the universe.

Skrotskii has used the method developed by Rytov[25] to discuss the geometrical-optics approximation in a curved spacetime. In the field of a slowly rotating body the null geodesic equation amounts to

$$\vec{\nabla}S - \vec{G} = N\hat{e}\,, \tag{B1}$$

where S is the phase function and $\hat{e}$ is a unit vector. Then the angle Δ' between the electric field and the osculating plane of the curve with tangent vector $\hat{e}$ is given by

$$(\hat{e}\cdot\vec{\nabla})\,\Delta' = -\tau + \tfrac{1}{2}\hat{e}\cdot(\vec{\nabla}\times\hat{e}). \tag{B2}$$

The rate of rotation of the osculating plane is given by the torsion of the path τ; therefore, if Δ is the angle between the electric field and an initially specified osculating plane (containing the initial electric field) then

$$(\hat{e}\cdot\vec{\nabla})\,\Delta = \tfrac{1}{2}\hat{e}\cdot(\vec{\nabla}\times\hat{e}). \tag{B3}$$

It is important to recognize that in general Δ is not the angle of rotation of the plane of polarization, since the unit vector tangent to the path of the photon is not $\hat{e}$ but rather $\hat{l}$ given by $\vec{\nabla}S = |\vec{\nabla}S|\hat{l}$. It turns out, however, that to first order in the angular momentum of the body Δ does give the rotation angle for the plane of polarization and (B3) implies that

$$(\hat{l}\cdot\vec{\nabla})\,\Delta = N^{-1}\,\hat{l}\cdot\vec{\Omega}_D, \qquad \text{(B4)}$$

where $\vec{\Omega}_D = -\frac{1}{2}\vec{\nabla}\times\vec{G}$. Equation (B4) has a simple physical interpretation in terms of the dragging of the inertial frames.[12] This equation can be used to show that for propagation along the rotation axis exactly the same result as (23) is obtained, whereas for waves traveling from $-\infty$ to $+\infty$ parallel to the rotation axis at a large distance from the body or propagating in the equatorial plane the net result is zero. It should be remarked that Δ is not the angle calculated by Skrotskii and this probably explains the discrepancy between his results and those presented here.

The expression obtained by Balazs[22] for the rotation of the plane of polarization of waves traveling parallel to and at some distance from the rotation axis is in conflict with the result presented above. Balazs has given both a qualitative argument for the existence of an effect and a quantitative estimate of it. It can be shown, however, that his qualitative remarks make an incorrect use of the principle of equivalence and that the calculations are based on an invalid approximation procedure.

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[1] M. Harwit, R. V. E. Lovelace, B. K. Dennison, D. L. Jauncey, and J. Broderick, Nature 249, 230 (1974).

[2] B. Mashhoon, Nature 250, 316 (1974).

[3] Greek indices run from 0 to 3, and Latin indices from 1 to 3. In the Cartesian coordinate system under consideration $[t = x^0,\ (\vec{x})_i = x^i]$, $-ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. $\epsilon^{\mu\nu\rho\sigma}$ is the alternating symbol with $\epsilon^{0123} = 1$. Gravitational units in which $G = c = 1$ are used throughout.

[4] J. Plebanski, Phys. Rev. 118, 1396 (1960). See also A. M. Volkov, A. A. Izmest'ev, and G. V. Skrotskii, Zh. Eksp. Teor. Fiz. 59, 1254 (1970) [Sov. Phys.—JETP 32, 686 (1971)] and the references cited therein.

[5] B. Mashhoon, Phys. Rev. D 7, 2807 (1973).

[6] B. Mashhoon, Phys. Rev. D 10, 1059 (1974).

[7] B. Mashhoon, Ann. Phys. (N.Y.) 89, 254 (1975).

[8] H. Weyl, *Raum, Zeit, Materie*, fifth edition (Springer, Berlin, 1923), p. 258.

[9] If the propagation vectors of the two orthogonal circular components are not constrained to be nearly equal then the two dispersion relations in (10) can be made identical with $K_3^+ = -K_3^-$ and $n^+ = n^-$. Moreover, with $K_2^+ = K_2^-$ and $\psi_3^+ = -\psi_3^-$ one obtains $\psi_1^+ = \psi_1^-$ and $\psi_2^+ = \psi_2^-$. Thus, in the limited region of space under consideration, an LCP solution can be obtained from an RCP wave propagating in a mirror-image direction with respect to the x_1-x_2 plane, and vice versa. This is also valid for electromagnetic waves propagating in the Gödel universe.

[10] The angular resolution attainable at a wavelength of a few centimeters (~3–4 cm) with ground-based observations using very long baseline interferometry is $\sim 3 \times 10^{-4}$ arc sec.

[11] Compare this with the discussion of the gravitational deflection of linearly polarized radiation in Ref. 1.

[12] B. B. Godfrey, Phys. Rev. D 1, 2721 (1970).

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[14] Alternatively, (26) can be the metric of a universe with $\Lambda = 0$ and matter of density ρ, $\Omega^2 = 8\pi\rho$, and pressure $p = \rho$. The possibility of such an asymptotic equation of state has been pointed out by Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. 41, 1609 (1961) [Sov. Phys.—JETP 14, 1143 (1962)].

[15] This can also be seen from the fact (see Sec. I) that the concept of helicity is well defined for very-high-frequency waves with $\Omega/\omega \ll 1$.

[16] Consider the hypothetical situation where electromagnetic rays are arranged to follow a closed null path. The time coordinate along this curve is circular; therefore, the rays go forward in time along part of the curve and backward in time along the rest of the curve. This is also expected of high-frequency electromagnetic waves with $\omega \gg \Omega$.

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[18] This conclusion may have to be revised if some unexpected anisotropy is detected in the microwave background.

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