PHYSICAL REVIEW D VOLUME 11, NUMBER 9 1 MAY 1975

A model for the decay $X^0 \rightarrow 2\pi\gamma$ based on finite dispersion relations*

R. H. Graham and Toaning Ng
Department of Physics, University of Toronto, Toronto, Canada
(Received 18 July 1974)

The amplitude for the decay $X^0 \rightarrow 2\pi\gamma$ is constructed using a finite dispersion relation. The decay width and distribution are found to be consistent with recent experiment. A finite-energy sum rule analysis indicates that the coupling of the X^0 to the photon and the ρ meson, $g_{\rho X^0\gamma}$, has its quark-model value. A similar study is made of $\eta \rightarrow 2\pi\gamma$ from which it is concluded that $g_{\rho\eta\gamma}$ probably also has its quark-model value.

I. INTRODUCTION

The spin and parity of the $X^0(958)$ have not yet been determined, although it almost certainly has either $J^P = 0^-$ or $J^P = 2^-$. Support for the former assignment comes from analyses[1,2] of the distribution of events in the $X^0 \rightarrow \eta\pi^+\pi^-$ and $X^0 \rightarrow \pi^+\pi^-\gamma$ Dalitz plots. However, $J^P = 2^-$ is favored over $J^P = 0^-$ to explain possible anisotropies[3] in production and decay correlations of the X^0. We have previously reported[4] the results of a finite dispersion relation (FDR) model for $X^0 \rightarrow \eta\pi\pi$, based on the pseudoscalar assignment for X^0, from which both the width and the small slope parameter of the Dalitz plot can be obtained.[5,6] We wish to present here a similar FDR model for the decay $X^0 \rightarrow 2\pi\gamma$, with X^0 again taken to be pseudoscalar, which we feel lends more weight to the $J^P = 0^-$ possibility.

The FDR approach is based on the application of Cauchy's theorem, over a finite contour, to two-body scattering amplitudes. This leads to a finite-contour dispersion relation and to families of finite-energy sum rules (FESR's). The FESR's can be used to correlate the Regge asymptotic form of the amplitude (assumed valid at intermediate energies) with the resonance structure, in order that the dispersion relation may represent a consistent model for the amplitude.

The use of FDR in conjunction with FESR's has covered a fairly broad range of applications. In addition to its natural employment in the study of scattering amplitudes,[7,8] it has provided an interesting means of describing three-body decays,[4,9-11] in which the decay amplitude is obtained from the corresponding two-body scattering amplitude by crossing.

The FDR model considered here predicts a width for $X^0 \rightarrow 2\pi\gamma$ in the range $0.045 \lesssim \Gamma(X^0 \rightarrow 2\pi\gamma) \lesssim 0.143$ MeV. This is consistent with the new upper bound[6] of $\Gamma(X^0 \rightarrow \text{all}) \lesssim 0.8$ MeV, if the branching ratio is taken[12] to be ~0.25. Not too surprisingly, the decay $X^0 \rightarrow 2\pi\gamma$ is found to proceed mainly through $X^0 \rightarrow \rho\gamma$ in accord with a recent experimental determination.[13]

The model predicts (from a FESR) a value for the coupling of the X^0 to the photon and the ρ meson close to its value in the quark model, when X^0-η mixing is taken into account. This is in marked contrast to the conclusions drawn by Lassila and Young[11] from a FDR model of $\eta \rightarrow 2\pi\gamma$. We reexamine their analysis and find it likely that the FDR models of $X^0 \rightarrow 2\pi\gamma$ and $\eta \rightarrow 2\pi\gamma$ are compatible with each other and with quark-model couplings.

Finally, we will discuss the extended vector-meson dominance model for radiative decays proposed by Bramôn and Greco,[14] in which several of our results have previously been obtained.

In Sec. II we give a detailed description of the $X^0 \rightarrow 2\pi\gamma$ model. Its most important parameters will be discussed and evaluated in Sec. III. Our results for $X^0 \rightarrow 2\pi\gamma$ will be given in Sec. IV, together with a reexamination of the FDR model of Lassila and Young for the decay $\eta \rightarrow 2\pi\gamma$. A brief summary and our conclusions are presented in Sec. V.

II. FORM OF THE DECAY AMPLITUDE

In the usual fashion[15] we begin by considering the process

$$X^0(p) + \pi(-q_1) \rightarrow \pi(q_2) + \gamma(k), \tag{2.1}$$

where the particle momenta, indicated in parenthesis, satisfy

$$p = q_1 + q_2 + k\,.$$

The scattering amplitude M for process (2.1) can be written as

$$M = \epsilon_{\mu\nu\rho\sigma}\epsilon^\mu q_1^\nu q_2^\rho k^\sigma A(\nu, t)\,, \tag{2.2}$$

where ϵ^μ is the polarization of the photon. The relevant kinematic variables are defined as

$$\nu = \tfrac{1}{2}(s-u)\,,$$
$$s = (p-q_1)^2 = (k+q_2)^2\,,$$
$$t = (p-k)^2 = (q_1+q_2)^2\,,$$
$$u = (p-q_2)^2 = (k+q_1)^2$$

and satisfy

$$s+t+u = m_X^2 + 2m_\pi^2\,.$$

The invariant amplitude $A(\nu, t)$ is even under crossing

$$A(\nu, t) = A(-\nu, t) \tag{2.3}$$

and is assumed to satisfy a fixed-t dispersion relation in ν. Using Cauchy's theorem for the contour shown in Fig. 1 and Eq. (2.3), we obtain the FDR

$$\begin{aligned} A(\nu, t) &= \frac{1}{2\pi i}\oint d\nu' \,\frac{A(\nu', t)}{\nu'-\nu} \\ &= \frac{2}{\pi}\int_0^N d\nu'\nu' \,\frac{\mathrm{Im}A(\nu', t)}{\nu'^2-\nu^2} \\ &\quad + \frac{1}{2\pi i}\int_{C_N} d\nu' \,\frac{A(\nu', t)}{\nu'-\nu} \end{aligned} \tag{2.4}$$

The first term on the right-hand side of Eq. (2.4) arises from the cuts along the Reν axis, while the second term is the contribution from the semicircular contours of radius $|\nu| = N$, lying in the upper and lower ν plane, which are denoted collectively by C_N in Fig. 1. N is given by

$$N = s_{\max} - (\tfrac{1}{2}m_X^2 + m_\pi^2) + \tfrac{1}{2}t\,, \tag{2.5}$$

where $s_{\max}$ will be chosen below.

Since we will be interested in $A(\nu, t)$ for values of $\nu < 1$ GeV^2, the radius of the circular contour would not have to be larger than that. As a practical matter it can not be much larger because of the limited experimental information on the πX^0 interaction. Thus we will assume with Lassila and Young[11] that the contour includes the A_2 meson and excludes any other important πX^0 effect. (Note that the possible $J^P = 0^+$ meson at 960 MeV cannot couple to $\pi\gamma$ because of parity conservation.) The A_2 is assumed to saturate the first integral on the right-hand side of Eq. (2.4). We will take[4,11] $s_{\max}$ to be the average of m_A^2 and $\frac{7}{3}m_A^2$, the latter corresponding to the position of a possible $J^P = 4^+$ recurrance of the A_2. Thus

$$s_{\max} = \tfrac{5}{3}m_A^2 = 2.86\ \mathrm{GeV}^2\,. \tag{2.6}$$

To determine the second integral on the right-hand side of Eq. (2.4) we will use the Regge asymptotic form for $A(\nu, t)$ on the circular contour C_N. That this may be a very good approximation is suggested by the success of the previous applications, and can also be supported on the basis of

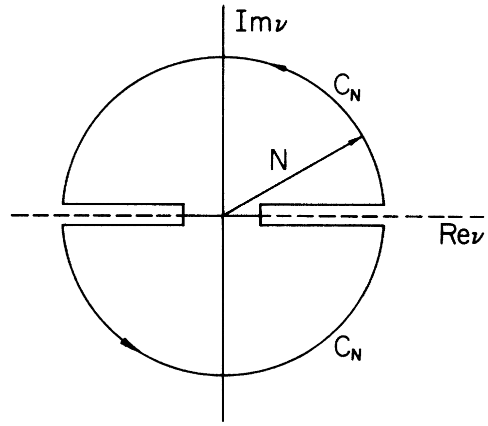

FIG. 1. Contour in complex ν plane over which the FDR and FESR are evaluated. The semicircular portions of the contour are denoted collectively by C_N.

duality.[16] The dominant Regge singularity in the t channel is the ρ trajectory. We will assume that the asymptotic behavior of $A(\nu, t)$ is governed entirely by Regge ρ exchange.

The amplitude $A(\nu, t)$, obtained in this way from the FDR, is assumed to describe the decay

$$X^0(p) \to \pi(q_1) + \pi(q_2) + \gamma(k) \tag{2.7}$$

throughout the allowable phase space.

In accordance with the above discussion we will denote the first and second terms on the right-hand side of Eq. (2.4) by A_{Res} and A_{Reg}, respectively, and so we have

$$A(\nu, t) = A_{\mathrm{Res}}(\nu, t) + A_{\mathrm{Reg}}(\nu, t)\,. \tag{2.8}$$

We will first deal with the determination of A_{Res}.

The A_2 contribution to A_{Res} depends on the couplings of the A_2 meson to πX^0 and to $\pi\gamma$. We define these couplings in the momentum representation to be

$$T_{A_2\to X^0\pi} = g_{A_2X^0\pi}h^{\rho\alpha}(Q)(p+q_1)_\rho(p+q_1)_\alpha\,, \tag{2.9}$$

$$\begin{aligned} T_{A_2\to\pi\gamma} &= g_{A_2\pi\gamma}\epsilon^\mu h^{\nu\lambda}(Q) \\ &\quad\times[(k-q_2)_\nu\epsilon_{\mu\lambda\sigma\tau}k^\sigma q_2^\tau \\ &\quad + (k-q_2)_\lambda\epsilon_{\mu\nu\sigma\tau}k^\sigma q_2^\tau]\,, \end{aligned} \tag{2.10}$$

where $h^{\mu\nu}(Q)$ is the polarization tensor of the A_2 meson, which carries momentum $Q = p - q_1 = q_2 + k$. Using these couplings we find

$$A_{\mathrm{Res}}(\nu, t) = -4g_{A_2\pi\gamma}g_{A_2X\pi}\left[\frac{2t+B}{\nu - \frac{1}{2}t + C} + (\nu \to -\nu)\right]\,, \tag{2.11}$$

with

$$B = m_A^2 - m_X^2 - 2m_\pi^2 - m_\pi^2(m_X^2 - m_\pi^2)/m_A^2 \tag{2.12}$$

and

$$C = \tfrac{1}{2}m_X^2 + m_\pi^2 - m_A^2 . \tag{2.13}$$

The quantity $|g_{A_2\pi\gamma}g_{A_2X\pi}|$ will be estimated in the next section.

We come now to the determination of A_{Reg}. The asymptotic form for $A(\nu, t)$ is assumed to be[17]

$$A(\nu,t) \underset{\nu\to\infty}{\sim} \frac{\pi\beta(t)}{\Gamma(\alpha(t))\sin\pi\alpha(t)}\left[\nu^{\alpha(t)-1} + (-\nu)^{\alpha(t)-1}\right], \tag{2.14}$$

where $\alpha(t)$ is the ρ Regge trajectory. We take

$$\alpha(t) = 0.5 + \frac{t}{2m_\rho^2} + i\,\frac{\Gamma_{\rho\pi\pi}}{2m_\rho}\,\frac{(t-4m_\pi^2)^{1/2}}{(m_\rho^2-4m_\pi^2)^{1/2}}, \tag{2.15}$$

with[12]

$$\Gamma_{\rho\pi\pi} \equiv \Gamma(\rho\to 2\pi) = 0.145 \text{ GeV} \tag{2.16}$$

An imaginary part has been included in the ρ trajectory, because the ρ pole in Eq. (2.14) lies in the allowed region of phase space for $X^0 \to 2\pi\gamma$.

When the form of $A(\nu, t)$ given in Eq. (2.14) is substituted in the integral along C_N in Eq. (2.4), we obtain

$$A_{\text{Reg}}(\nu,t) = \frac{-2\beta(t)}{\Gamma(\alpha(t))}\sum_{n=0}^{\infty}\frac{N^{\alpha(t)-1}}{\alpha(t)-2n-1}\left(\frac{\nu}{N}\right)^{2n}, \tag{2.17}$$

where N is given by Eq. (2.5). We will discuss several different determinations of $\beta(t)$ in the next section.

III. DETERMINATION OF THE PARAMETERS

To fix the magnitude of $g_{A_2\pi\gamma}g_{A_2X\pi}$ we first determine $g_{A_2\pi\gamma}$ by using vector-meson dominance (VMD) to relate[11] it to $g_{A_2\pi\rho}$. The partial width[12] $\Gamma(A_2\to\rho\pi) = 64$ MeV results in

$$|g_{A_2\pi\gamma}| = 0.253 \text{ GeV}^{-2}, \tag{3.1}$$

corresponding to

$$\Gamma(A_2\to\pi\gamma) = 0.9 \text{ MeV}.$$

An analysis by Eisenberg *et al.*[18] of $\gamma + p \to n + A_2^+$ gives $\Gamma(A_2\to\pi\gamma) \cong 0.5 \pm 0.5$ MeV.

For $g_{A_2X\pi}$ we use the model of Glashow and Socolow.[19] The magnitude of this coupling depends on the mixing between the X^0 and η mesons. The physical X^0 and η states are expressed in terms of pure SU(3) octet and singlet states in the usual manner:

$$\begin{aligned} |X^0\rangle &= \sin\theta_P|\eta_8\rangle + \cos\theta_P|\eta_1\rangle, \\ |\eta\rangle &= \cos\theta_P|\eta_8\rangle - \sin\theta_P|\eta_1\rangle . \end{aligned} \tag{3.2}$$

A quadratic Gell-Mann–Okubo mass formula for the pseudoscalar mesons leads to $\theta_P \cong \pm 11°$ while a linear mass formula implies $\theta_P \cong \pm 24°$. There are phenomenological studies which support both possibilities.[20]

The results of the present investigation indicate that θ_P lies in the range

$$-11° \gtrsim \theta_P \gtrsim -23°;$$

we will list results for the extremes of this range in the following.

It follows from the model of Glashow and Socolow[19] that

$$|g_{A_2X\pi}| = \begin{cases} 3.5 \text{ GeV}^{-1} & (\theta_P = -11°), \\ 2.63 \text{ GeV}^{-1} & (\theta_P = -23°) \end{cases} \tag{3.3}$$

so that

$$\Gamma(A_2\to X\pi) = \begin{cases} 1.0 \text{ MeV} & (\theta_P = -11°), \\ 0.58 \text{ MeV} & (\theta_P = -23°). \end{cases} \tag{3.4}$$

Neither of these values is incompatible with experiment.[12] From Eqs. (3.1) and (3.3) we get

$$|g_{A_2\pi\gamma}g_{A_2X\pi}| = \begin{cases} 0.87 \text{ GeV}^{-3} & (\theta_P = -11°), \\ 0.66 \text{ GeV}^{-3} & (\theta_P = -23°). \end{cases} \tag{3.5}$$

We turn now to the determination of $\beta(t)$ in Eq. (2.17). This will be attempted in two ways. The first method involves the use of the FESR

$$\frac{2}{\pi}\int_0^N d\nu\,\nu\,\text{Im}A(\nu,t) = \frac{1}{2\pi i}\int_{C_N} d\nu\,\nu A(\nu,t) \tag{3.6}$$

over the contour of Fig. 1. Using the A_2 contribution to evaluate the left-hand side and Eq. (2.14) in the right-hand side one finds

$$\beta_{\text{FESR}}(t) = 4g_{A_2\pi\gamma}g_{A_2X\pi}(2t+B)(\tfrac{1}{2}t - C) \times \frac{\Gamma(\alpha(t))[\alpha(t)+1]}{N^{\alpha(t)+1}}, \tag{3.7}$$

where B and C are defined in Eqs. (2.12) and (2.13), respectively, and N is given in Eq. (2.5). $\beta_{\text{FESR}}(t)$ changes by only ~10% in the decay region, and at $t=0$ we find (taking $g_{A_2\pi\gamma}g_{A_2X\pi} > 0$) that

$$\beta_{\text{FESR}}(0) = \begin{cases} 2.65 \text{ GeV}^{-3} & (\theta_P = -11°), \\ 2.01 \text{ GeV}^{-3} & (\theta_P = -23°). \end{cases} \tag{3.8}$$

The real part of $\beta_{\text{FESR}}(t)$ (for $\theta_P = -11°$) is shown in Fig. 2 for values of t in the decay region. We might expect the FESR to be less reliable than the FDR of Eq. (2.4) since the asymptotic form of $A(\nu, t)$ [Eq. (2.14)] used along the contour C_N is emphasized more in the FESR than in the FDR.

Nevertheless, we believe that the FESR is probably valid to within ~20%.

A second method of determining β depends on the comparison of $A_{\rm Reg}(\nu, t)$ near $t=m_\rho^2$ with the ρ pole contribution to $A(\nu, t)$ arising from a dispersion relation in t:

$$A(\nu, t) \underset{t\approx m_\rho^2}{\sim} \frac{2g_{\rho\pi\pi}g_{\rho X\gamma}}{m_\rho^2 - t + i\Gamma_{\rho\pi\pi}m_\rho} \quad . \tag{3.9}$$

The couplings have been defined by

$$T_{\rho\to\pi\pi} = g_{\rho\pi\pi}\epsilon^\mu_{(\rho)}(q_1 - q_2)_\mu \tag{3.10}$$

and

$$T_{\rho\to X\gamma} = g_{\rho X\gamma}\epsilon_{\lambda\mu\nu\sigma}\epsilon^\lambda_{(\rho)}\epsilon^\mu k^\nu p^\sigma , \tag{3.11}$$

where $\epsilon^\mu_{(\rho)}$ is the polarization of the ρ. The requirement that $A_{\rm Reg}(\nu, t)$ in Eq. (2.17) should agree with Eq. (3.9) for $t\approx m_\rho^2$ leads to the condition

$$\beta(m_\rho^2) = g_{\rho\pi\pi}g_{\rho X\gamma}/2m_\rho^2 . \tag{3.12}$$

From Eq. (2.16) one finds

$$\frac{g_{\rho\pi\pi}^2}{4\pi} = 2.81 . \tag{3.13}$$

Our knowledge of $g_{\rho X\gamma}$ is less certain, however. Since we expect the FESR to be reasonably reliable, then we can use its predicted values of

$$\beta_{\rm FESR}(m_\rho^2) = \begin{cases} 2.99\ {\rm GeV}^{-3} & (\theta_P = -11°) , \\ 2.27\ {\rm GeV}^{-3} & (\theta_P = -23°) \end{cases} \tag{3.14}$$

as a guide to both the magnitude and the sign of $g_{\rho\pi\pi}g_{\rho X\gamma}$. Equations (3.12)–(3.14) imply that

$$g_{\rho X\gamma} = \begin{cases} 0.58\ {\rm GeV}^{-1} & (\theta_P = -11°) , \\ 0.45\ {\rm GeV}^{-1} & (\theta_P = -23°) \end{cases} \tag{3.15}$$

if we take $g_{\rho\pi\pi} > 0$. In the following we will take all coupling constants whose signs are arbitrary to be positive.

We can gain some insight into the possible significance of these values of $g_{\rho X\gamma}$ by using SU(3) to relate it to other couplings of the photon to vector and pseudoscalar mesons. From Eq. (3.2) we have

$$g_{\rho X\gamma} = \sin\theta_P\, g_{\rho\eta_8\gamma} + \cos\theta_P\, g_{\rho\eta_1\gamma} . \tag{3.16}$$

The assumption that the $I=0$ vector mesons, ω and ϕ, mix ideally[21] (and that the electromagnetic current is pure octet) leads to the relations

$$g_{\rho\eta_8\gamma} = \frac{1}{\sqrt{3}} g_{\omega\pi\gamma} , \qquad g_{\rho\eta_1\gamma} = 3g_{\omega\eta_1\gamma} . \tag{3.17}$$

If we now write[11]

$$g_{\omega\eta_1\gamma} = r\,\tfrac{1}{3}(\tfrac{2}{3})^{1/2} g_{\omega\pi\gamma} , \tag{3.18}$$

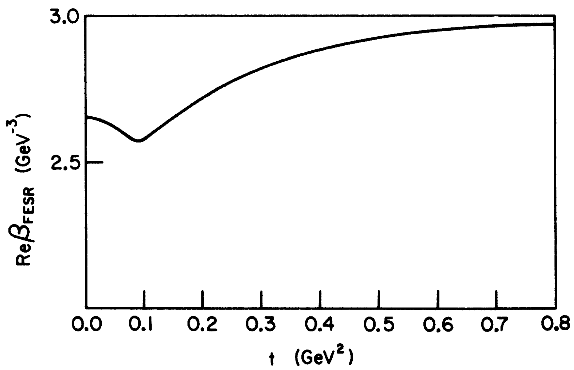

FIG. 2. The real part of $\beta_{\rm FESR}(t)$, obtained from Eq. (3.7), is plotted against t.

we have from Eqs. (3.17)–(3.19)

$$g_{\rho X\gamma} = \frac{1}{\sqrt{3}}(\sin\theta_P + r\sqrt{2}\cos\theta_P)\,g_{\omega\pi\gamma} . \tag{3.19}$$

The choice of $r=1$ in Eq. (3.19) corresponds to the quark-model result.[22]

The most direct way of determining $g_{\omega\pi\gamma}$ is evidently from the partial width of the decay $\omega\to\pi\gamma$. Taking[12]

$$\Gamma(\omega\to\pi\gamma) = 1\ {\rm MeV} \tag{3.20}$$

one finds

$$g_{\omega\pi\gamma} = 0.872\ {\rm GeV}^{-1} . \tag{3.21}$$

In order to obtain a value for $g_{\rho X\gamma}$ from Eq. (3.19) we will take the quark model seriously and set $r=1$. Then from Eqs. (3.19) and (3.21)

$$g_{\rho X\gamma} = \begin{cases} 0.60\ {\rm GeV}^{-1} & (\theta_P = -11°) , \\ 0.46\ {\rm GeV}^{-1} & (\theta_P = -23°) , \end{cases} \tag{3.22}$$

which lead, via Eqs. (3.12) and (3.13), to

$$\beta_Q = \begin{cases} 3.06\ {\rm GeV}^{-3} & (\theta_P = -11°) , \\ 2.32\ {\rm GeV}^{-3} & (\theta_P = -23°) , \end{cases} \tag{3.23}$$

where we use the subscript Q to denote the quark-model value of $\beta(m_\rho^2)$. These values of β_Q compare favorably with the corresponding predictions of the FESR in Eq. (3.14).

The quark-model value $r=1$, suggested by our treatment of $X^0\to 2\pi\gamma$, differs considerably from the much larger values $1.4 \lesssim r \lesssim 3.2$ found by Lassila and Young[11] in the analogous model for $\eta\to 2\pi\gamma$. A possible resolution of this apparent lack of consistency between the two models will be put forward in the next section.

IV. RESULTS

We will now discuss the predictions of our model for the decay $X^0 \to 2\pi\gamma$. We will also reexamine the results of Lassila and Young in an attempt to reconcile their analysis with ours. Of particular interest is the question of the compatibility of the quark model with the FDR approach to $X^0 \to 2\pi\gamma$ and $\eta \to 2\pi\gamma$.

Let us first choose for the Regge residue $\beta(t)$ the FESR value given by Eq. (3.7). This gives

$$\Gamma(X^0 \to 2\pi\gamma) = \begin{cases} 0.135 \text{ MeV} \quad (\theta_P = -11^\circ), \\ 0.078 \text{ MeV} \quad (\theta_P = -23^\circ). \end{cases} \tag{4.1}$$

An indication of the sensitivity of these results (and the ones to follow) to small changes in s_{max} and the ρ trajectory parameters may be obtained from Table I.

The $X^0 \to 2\pi\gamma$ branching ratio is found experimentally to be[12]

$$\frac{\Gamma(X^0 \to 2\pi\gamma)}{\Gamma(X^0 \to \text{all})} = 0.262 \pm 0.035 . \tag{4.2}$$

The above prediction for $\Gamma(X^0 \to 2\pi\gamma)$ then implies the values for the total X^0 width, $\Gamma(X^0 \to \text{all})$, of

$$\Gamma(X^0 \to \text{all}) = \begin{cases} 0.51 \text{ MeV} \quad (\theta_P = -11^\circ), \\ 0.30 \text{ MeV} \quad (\theta_P = -23^\circ), \end{cases} \tag{4.3}$$

which are consistent with the recent experimental upper bound[6] of $\Gamma(X^0 \to \text{all}) \lesssim 0.8$ MeV.

As was pointed out in Sec. III, the FESR relation in Eq. (3.7) may only be good to ~20%. We are therefore led to consider the attractive alternative choices given in Eq. (3.23) for $\beta(t)$; $\beta_{FESR}(m_\rho^2)$ and β_Q differ by only ~5%. In order to proceed further we assume that the slow variation of $\beta_{FESR}(t)$ with t in the decay region reflects the behavior of the actual residue function. We thus take $\beta(t) = \beta_Q$ throughout the decay region.[23] Using Eq. (3.23) for β_Q we find

$$\Gamma(X^0 \to 2\pi\gamma) = \begin{cases} 0.143 \text{ MeV} \quad (\theta_P = -11^\circ), \\ 0.082 \text{ MeV} \quad (\theta_P = -23^\circ), \end{cases} \tag{4.4}$$

which implies by Eq. (4.2) that

$$\Gamma(X^0 \to \text{all}) = \begin{cases} 0.545 \text{ MeV} \quad (\theta_P = -11^\circ), \\ 0.313 \text{ MeV} \quad (\theta_P = -23^\circ). \end{cases} \tag{4.5}$$

Now, it is found[13] experimentally that almost all of the $X^0 \to 2\pi\gamma$ decays proceed through $X^0 \to \rho\gamma$. We also find this to be true in our model, if we use the same criteria as in Ref. 13. In Fig. 3 we show the decay distribution as a function of dipion mass predicted by the model, compared with some recent experimental data.[13] Although there are no errors quoted for these data, the difference in the two curves probably indicates that the model for the imaginary part of $\alpha(t)$ [see Eq. (2.15)] is only approximately correct. A moderately larger ρ width would lessen the discrepancy without doing serious damage to the agreement between Eqs. (3.14) and (3.23).

Motivated by the experimental results, we take

TABLE I. Sensitivity of $\Gamma(X^0 \to 2\pi\gamma)$ to modest changes in s_{max} and in the ρ Regge trajectory which we shall write as $\alpha(t) = \alpha_0 + \alpha' t + i\alpha' m_\rho \Gamma_{\rho\pi\pi}(t - 4m_\pi^2)^{1/2}/(m_\rho^2 - 4m_\pi^2)^{1/2}$. The trajectory used in our calculations [Eq. (2.15)] has $\alpha_0 = 0.5$ and $\alpha' = 1/2m_\rho^2$. The value of $\Gamma(X^0 \to 2\pi\gamma)$ shown in a particular row of the table corresponds to a change in only the parameter listed at the left of that row, all other parameters having the values used in the calculations, namely α_0 and α' as given above, $s_{max} = 2.86$ GeV^2, and $\Gamma_{\rho\pi\pi} = 0.145$ GeV. We have taken $\theta_P = -11°$ for the sake of illustration.

Parameter	Value	$\Gamma(X^0 \to 2\pi\gamma)$ (MeV)
s_{max}	2.5 GeV^2	0.228
	3.0 GeV^2	0.113
$\Gamma_{\rho\pi\pi}$	0.125 GeV	0.164
	0.135 GeV	0.145
α_0	0.52	0.145
α'	0.82 GeV^2	
α_0	0.54	0.155
α'	0.80 GeV^2	

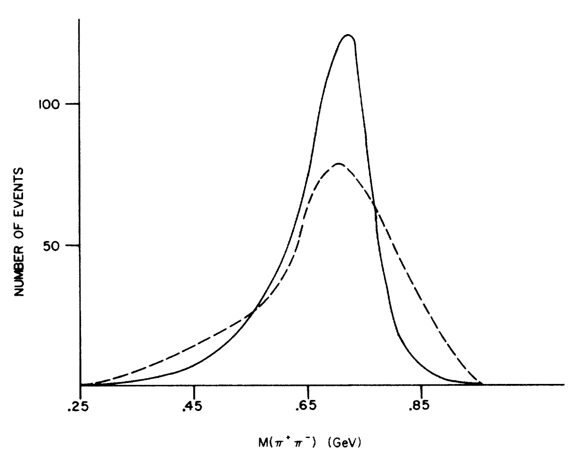

FIG. 3. The distribution of $X^0 \to 2\pi\gamma$ decay events versus the $\pi^+\pi^-$ invariant mass. The dashed line represents the experimental results of Ref. 13. The solid line is our prediction.

Eq. (4.4) to mean that

$$\Gamma(X^0 \to \rho\gamma) \approx \begin{cases} 0.143 \text{ MeV} & (\theta_P = -11^\circ)\,, \\ 0.082 \text{ MeV} & (\theta_P = -23^\circ)\,. \end{cases} \tag{4.6}$$

At this point a consistency check can be made; if one calculates the $X^0 \to \rho\gamma$ width using the quark-model couplings of Eq. (3.22), one obtains

$$\Gamma(X^0 \to \rho\gamma) = \begin{cases} 0.150 \text{ MeV} & (\theta_P = -11^\circ)\,, \\ 0.087 \text{ MeV} & (\theta_P = -23^\circ)\,. \end{cases} \tag{4.7}$$

These values are in close agreement[24] with their counterparts in Eq. (4.6).

We will turn now to the question of the consistency between the FDR analyses of $X^0 \to 2\pi\gamma$ and $\eta \to 2\pi\gamma$. In the FDR model for the latter process the determination of the residue function, which we shall denote by $\beta'(t)$, from the FESR analogous to our Eq. (3.6) yields[25]

$$\beta'_{\text{FESR}}(0) = 3.94 \text{ GeV}^{-3}\,, \tag{4.8}$$

$$\beta'_{\text{FESR}}(m_\rho^2) = 2.97 \text{ GeV}^{-3}\,, \tag{4.9}$$

The t dependence of the real part of $\beta'_{\text{FESR}}(t)$ is shown in Fig. 4 for $0 \leq t \lesssim m_\rho^2$. Using $\beta'_{\text{FESR}}(t)$ one finds

$$\Gamma(\eta \to 2\pi\gamma) = 0.146 \text{ keV}, \tag{4.10}$$

which is quite consistent with the experimental result[12] of

$$\Gamma_{\text{exp}}(\eta \to 2\pi\gamma) = 0.131 \pm 0.033 \text{ keV}\,. \tag{4.11}$$

The arguments of the previous section, however, require that

$$\beta'(m_\rho^2) = g_{\rho\pi\pi}\, g_{\rho\eta\gamma}/2m_\rho^2\,, \tag{4.12}$$

with

$$g_{\rho\eta\gamma} = \frac{1}{\sqrt{3}}\,(\cos\theta_P - r\sqrt{2}\,\sin\theta_P)\, g_{\omega\pi\gamma}\,. \tag{4.13}$$

If we set $r = 1$ and use Eq. (3.21) for $g_{\omega\pi\gamma}$ we find from Eqs. (4.12) and (4.13) that the quark-model value of $\beta'(m_\rho^2)$ is

$$\beta'_Q = \begin{cases} 3.23 \text{ GeV}^{-3} & (\theta_P = -11^\circ), \\ 3.76 \text{ GeV}^{-3} & (\theta_P = -23^\circ). \end{cases} \tag{4.14}$$

Comparing these values with the FESR results in Eqs. (4.8) and (4.9) we note that for $\theta_P = -11^\circ$ the agreement between β'_Q and $\beta'_{\text{FESR}}(m_\rho^2)$ is fairly good; there is a ~23% discrepancy if $\theta_P = -23^\circ$. If we ignore the t dependence of $\beta'(t)$ implied by the FESR and take $\beta'(t) = \beta'_Q$ throughout the decay region, we find

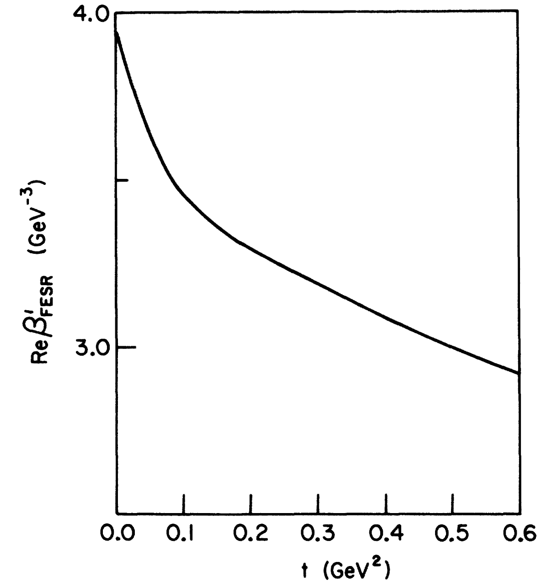

FIG. 4. The real part of $\beta'_{\text{FESR}}(t)$, obtained from the FESR for $\pi\eta \to \pi\gamma$ corresponding to Eq. (3.7) is plotted versus t.

$$\Gamma(\eta \to 2\pi\gamma) = \begin{cases} 0.138 \text{ keV} & (\theta_P = -11^\circ), \\ 0.167 \text{ keV} & (\theta_P = -23^\circ). \end{cases} \tag{4.15}$$

On the other hand, if $\beta'(t)$ is chosen to have the t dependence suggested by the FESR and $\beta'(m_\rho^2)$ is set equal to β'_Q, then one finds

$$\Gamma(\eta \to 2\pi\gamma) = \begin{cases} 0.160 \text{ keV} & (\theta_P = -11^\circ), \\ 0.175 \text{ keV} & (\theta_P = -23^\circ). \end{cases} \tag{4.16}$$

Lassila and Young[11] assume that the true residue function $\beta'(t)$ is approximately constant in the region $0 \lesssim t \lesssim m_\rho^2$ and that its value in this region is reliably obtained from the FESR at $t = 0$. Thus they take[26] $\beta'(t) \approx \beta'_{\text{FESR}}(0)$ and, consequently, have

$$\beta'_{\text{FESR}}(0) = g_{\rho\pi\pi}\, g_{\rho\eta\gamma}/2m_\rho^2 \tag{4.17}$$

instead of Eq. (4.12).

If Eq. (4.17) is used to determine r, then one finds, from Eqs. (3.21), (4.8), and (4.13), that

$$r = \begin{cases} 2.20 & (\theta_P = -11^\circ), \\ 1.10 & (\theta_P = -23^\circ). \end{cases} \tag{4.18}$$

It is apparent that the assumption embodied in Eq. (4.17) is quite compatible with quark-model couplings, if $\theta_P = -23^\circ$. (Lassila and Young considered only the case in which $\theta_P = -11^\circ$).

For $\theta_P = -11^\circ$ the value of r given in Eq. (4.18) falls within the range of values which can be obtained by applying VMD to the reactions $\pi^0 \to 2\gamma$ and $\eta \to 2\gamma$. If the $P \to 2\gamma$ coupling $g_{P\gamma\gamma}$ (where

$P=\pi^0$, η, or X^0) is defined by

$$T_{P\to 2\gamma} = g_{P\gamma\gamma}\epsilon_{\lambda\mu\nu\rho}\epsilon_1^\lambda k_1^\mu \epsilon_2^\nu k_2^\rho ,$$

with ϵ_i and k_i the photon polarizations and momenta, respectively, then the VMD hypothesis implies that[27]

$$g_{\pi^0\gamma\gamma} = \frac{2e}{3f_\rho} g_{\omega\pi\gamma} , \qquad (4.19a)$$

$$g_{\eta\gamma\gamma} = \frac{1}{\sqrt{3}} g_{\pi^0\gamma\gamma}\cos\theta_P(1-2\sqrt{2}r\tan\theta_P), \qquad (4.19b)$$

$$g_{X^0\gamma\gamma} = \frac{1}{\sqrt{3}} g_{\pi^0\gamma\gamma}\cos\theta_P(\tan\theta_P+2\sqrt{2}r), \qquad (4.19c)$$

where e is the proton charge, f_ρ is the ρ-photon coupling,[28] and r is defined in Eq. (3.18).

Using Eq. (4.19b) one finds,[11] with $\theta_P=-11°$ and $\Gamma(\eta\to 2\gamma)=1.0\pm 0.22$ keV,[12] that $r=1.94\pm 0.55$ or 2.71 ± 0.55 depending on whether one uses $\Gamma(\pi^0\to 2\gamma) = 11.2\pm 1.2$ eV (Ref. 29) or 7.74 ± 0.93 eV (Ref. 12), respectively. The former value of $\Gamma(\pi^0\to 2\gamma)$ is probably too high, so that VMD seems to imply $r \gtrsim 2.2$. However, not only is Eq. (4.19b) fairly insensitive to r, but Eqs. (4.19a)–(4.19c) are only approximate at best.[30]

A comparison of Eqs. (3.8) and (3.14) with Eqs. (4.8) and (4.9) reveals that, in contrast to the situation in the $\eta\to 2\pi\gamma$ model, the ambiguities of where ($t=0$ or $t=m_\rho^2$) to relate $\beta_{FESR}(t)$ to the residue of the ρ pole and whether to take $\theta_P=-11°$ or $-23°$ do not appreciably affect the results for $X^0\to 2\pi\gamma$. Since $\beta_{FESR}(t)$ is possibly overestimated,[31] the $X^0\to 2\pi\gamma$ analysis gives an upper bound of $r\lesssim 1.0$. The $\eta\to 2\pi\gamma$ model yields $r\gtrsim 1.0$. Thus, to within the ~20% accuracy expected in these calculations, the two models are compatible only if $r\approx 1.0$. This, in turn, requires that, in the models for $X^0\to 2\pi\gamma$ and $\eta\to 2\pi\gamma$, we have a choice of two possibilities. Either[32]

Case 1: $\theta_P=-11°$, $\beta(t)=\beta_{FESR}(t)$, $\beta'(t)=\beta'_{FESR}(t)$, with $\beta_{FESR}(t)$ $[\beta'_{FESR}(t)]$ normalized to $\beta_Q(-11°)$ $[\beta'_Q(-11°)]$ at $t=m_\rho^2$; or

Case 2: $\theta_P=-23°$, $\beta(t)=\beta_{FESR}(t)$, $\beta'(t)=\beta'_Q(-23°)$, with $\beta_{FESR}(t)$ normalized to $\beta_Q(-23°)$ at $t=m_\rho^2$.

In Table II we give predictions, based on the above two cases combined with VMD [Eqs. (4.19a)–(4.19c)], for the decays $X^0\to 2\pi\gamma$, $X^0\to 2\gamma$, $\eta\to 2\pi\gamma$, $\eta\to 2\gamma$, $\pi^0\to 2\gamma$, and $\omega\to\pi\gamma$. There are two sets of results displayed in Table II; the first is based on the use of the mean values $f_\rho=5.24$ and $\Gamma(\pi^0\to 2\gamma) = 7.75$ eV as inputs, while the second set is based on the extreme values of $f_\rho=5.6$ and $\Gamma(\pi^0\to 2\gamma) = 8.6$ eV. For consistency the values of $\beta_Q(\theta_P)$ and $\beta'_Q(\theta_P)$ are determined [using Eqs. (3.19) and (4.13), respectively] from the predicted values of $\Gamma(\omega\to\pi\gamma)$ listed in the table.

V. SUMMARY AND DISCUSSION

Let us briefly review the main features of the above analysis. With the assumption that the $X^0(958)$ is a pseudoscalar meson, we have used a FDR to determine the amplitude for the process $X^0\to 2\pi\gamma$. For the representative values of the

TABLE II. Predictions for mesonic radiative decays based on vector-meson dominance and the quark model for the two cases described in Sec. IV. In case 1 $\theta_P=-11°$, $\beta(t)=\beta_{FESR}(t)$, $\beta'(t)=\beta'_{FESR}(t)$, with $\beta_{FESR}(t)$ $[\beta'_{FESR}(t)]$ normalized to $\beta_Q(-11°)$ $[\beta'_Q(-11°)]$ at $t=m_\rho^2$. In case 2 $\theta_P=-23°$, $\beta(t)=\beta_{FESR}(t)$, $\beta'(t)=\beta'_Q(-23°)$, with $\beta_{FESR}(t)$ normalized at $t=m_\rho^2$ to $\beta_Q(-23°)$. The input values of f_ρ and $\Gamma(\pi^0\to 2\gamma)$ are underlined. The predicted values of $\Gamma(\omega\to\pi\gamma)$ are used to determine $\beta_Q(\theta_P)$ and $\beta'_Q(\theta_P)$. The experimental values are taken from Ref. 12.

Quantity	Case 1	Case 2	Case 1	Case 2	Experimental value
$\Gamma(\pi^0\to\gamma\gamma)$	<u>7.75 eV</u>	<u>7.75 eV</u>	<u>8.6 eV</u>	<u>8.6 eV</u>	7.75 ± 0.92 eV
f_ρ	<u>5.24</u>	<u>5.24</u>	<u>5.6</u>	<u>5.6</u>	$5.24^{+0.36}_{-0.24}$
$\Gamma(X^0\to 2\pi\gamma)$	81 keV	45 keV	100 keV	58 keV	<0.25 MeV
$\Gamma(X^0\to\gamma\gamma)$	5.8 keV	4.2 keV	6.5 keV	4.7 keV	
$\frac{\Gamma(X^0\to\gamma\gamma)}{\Gamma(X^0\to 2\pi\gamma)}$	0.071	0.093	0.065	0.08	$0.073^{+0.024}_{-0.019}$
$\Gamma(\eta\to 2\pi\gamma)$	0.109 keV	0.133 keV	0.126 keV	0.157 keV	0.131 ± 0.033 keV
$\Gamma(\eta\to\gamma\gamma)$	0.38 keV	0.67 keV	0.42 keV	0.74 keV	1.0 ± 0.22 keV
$\frac{\Gamma(\eta\to 2\pi\gamma)}{\Gamma(\eta\to\gamma\gamma)}$	0.28	0.19	0.3	0.2	$0.131^{+0.079}_{-0.051}$
$\Gamma(\omega\to\pi\gamma)$	0.52 MeV	0.52 MeV	0.661 MeV	0.661 MeV	1.0 ± 0.2 MeV

X^0-η^0 mixing angle θ_P of $-11°$ and $-23°$ the model yields predictions of the width and decay distribution which are consistent with recent measurements. Furthermore, results based on a FESR strongly suggest that the $X^0 \to \rho\gamma$ coupling $g_{\rho X \gamma}$ has the value it would take in the quark model.

Additional support for this result follows from our reexamination of the FDR model for $\eta \to 2\pi\gamma$ first studied by Lassila and Young.[11] From the latter model we find it likely that the coupling $g_{\rho\eta\gamma}$ also has its quark-model value. This conclusion holds for either $\theta_P = -11°$ or $\theta_P = -23°$ providing the ρ Regge residue function associated with the asymptotic form of the $\pi\eta \to \pi\gamma$ amplitude is appropriately chosen. It should be a constant if $\theta_P = -23°$ and have the t dependence of the FESR if $\theta_P = -11°$. Lassila and Young, having made different assumptions about the residue function and mixing angle, arrived at a larger value of $g_{\rho\eta\gamma}$. Nevertheless, as we stressed above, we feel that the quark-model values of these couplings are definitely preferred. Our reasons are as follows.

From the assumption of SU(3) symmetry and "ideal" mixing[21] of the ω and ϕ vector mesons, $g_{\rho X \gamma}$ and $g_{\rho\eta\gamma}$ can be expressed in terms of θ_P and a parameter r which takes the value 1.0 in the quark model. The $X^0 \to 2\pi\gamma$ analysis implies $r \lesssim 1.0$, while from our study of $\eta \to 2\pi\gamma$ we find $r \gtrsim 1.0$. Since the $X^0 \to 2\pi\gamma$ calculations are not terribly sensitive to changes in the structure of the residue function, and because they are more sensitive to r, we are encouraged to believe the former bound on r. In order for the $X^0 \to 2\pi\gamma$ and $\eta \to 2\pi\gamma$ models to be compatible, r must be very close to 1.0.

Before concluding, a few comments are in order about the related work of Bramòn and Greco[14] who have constructed an "extended" VMD model to describe two- and three-body radiative decays of the vector and pseudoscalar mesons. In their scheme there is, in addition to the established nonet of vector mesons, a higher-lying nonet as well, and all coupling constants are taken from the quark model. Their models for $X^0 \to 2\pi\gamma$ and $\eta \to 2\pi\gamma$, like the ones studied here, consist of a resonance (A_2) term and a Regge term. In spite of the similarities between their model and ours, and the rough numerical agreement of their respective predictions for $\Gamma(X^0 \to 2\pi\gamma)$ and $\Gamma(\eta \to 2\pi\gamma)$ with ours, there are several important differences in the two approaches.

In the first place, although they claim to be using a FDR model to describe these decays, in fact there seems to be a basic inconsistency in their choice of resonance contribution to the decay amplitude.[33] This has little effect on the structure of the $X^0 \to 2\pi\gamma$ amplitude, which is dominated by the Regge (ρ) contribution, but can be quite important for $\eta \to 2\pi\gamma$. Secondly, their choice of a fairly low cutoff on the FESR for[34] $\pi\eta \to \pi\gamma$, in order to obtain consistency, can lead to numerical differences between our models. Finally, the introduction by Bramòn and Greco of a second nonet of vector mesons makes still more difficult a direct comparison of our work with theirs.

In closing, we wish to emphasize that if the $X^0(958)$ should prove to have $J^P = 0^-$, there seem to be no theoretical problems with the description of its three-body decays through finite dispersion relations. In the present investigation of $X^0 \to 2\pi\gamma$ and our previous study[4,5] of $X^0 \to \eta\pi\pi$, both based on the pseudoscalar assignment, we have obtained results consistent with experiment.

---

*Work supported in part by the National Research Council of Canada.

[1]A. Rittenberg, Ph.D. thesis, UCRL Report No. UCRL-18863, 1969 (unpublished).

[2]J. S. Danburg *et al.*, Phys. Rev. D 8, 3744 (1973).

[3]G. R. Kalbfleisch *et al.*, Phys. Rev. Lett. 31, 333 (1973).

[4]R. H. Graham and Toaning Ng, Phys. Rev. D 8, 2957 (1973).

[5]Although the width of $X^0 \to \eta\pi\pi$ was predicted in Ref. 4 to be 2.85 MeV, a small change of several of the model parameters, within their allowed ranges, could give a width consistent with the new upper bound reported by Duane *et al.* (Ref. 6). This could be achieved without significantly changing the slope prediction.

[6]A. Duane *et al.*, Phys. Rev. Lett. 32, 425 (1974).

[7]J. Engels, Nucl. Phys. B51, 269 (1973).

[8]J. Baacke and J. Engels, Nucl. Phys. B51, 434 (1973).

[9]R. Aviv and Nussinov, Phys. Rev. D 2, 209 (1970).

[10]G. J. Gounaris and A. Verganelakis, Nucl. Phys. B34, 418 (1971).

[11]B.-L. Young and K. E. Lassila, Phys. Rev. Lett. 28, 1491 (1972); Phys. Rev. D 7, 2174 (1973).

[12]Particle Data Group, Phys. Lett. 50B, 1 (1974).

[13]S. M. Jacobs *et al.*, Phys. Rev. D 8, 18 (1973).

[14]A. Bramòn and M. Greco, Nuovo Cimento 14A, 323 (1973).

[15]The analysis of this section is similar to that in the earlier applications of the FDR technique to three-body decays (Refs. 4, 9, and 11). Initially, our treatment will, quite naturally, parallel that of Lassila and Young (Ref. 11).

[16]For a review of duality ideas, see, e.g., J. D. Jackson, Rev. Mod. Phys. 42, 12 (1970).

[17]This is the form used in the earlier applications.

$\Gamma(\alpha(t))$ provides the usual ghost-eliminating mechanism. The scale factor is 1 GeV^2.

[18]Y. Eisenberg *et al.*, Phys. Rev. D 5, 15 (1972).

[19]S. L. Glashow and R. H. Socolow, Phys. Rev. Lett. 15, 329 (1965). As pointed out in Ref. 4, the analysis of Glashow and Socolow seems still to be valid when account is taken of the latest data (Ref. 12).

[20]See for example the discussion by F. J. Gilman, in *Experimental Meson Spectroscopy—1972*, proceedings of the Third International Conference, Philadelphia, edited by Kwan-Wu Lai and Arthur H. Rosenfeld (A.I.P., New York, 1972), p. 460.

[21]S. Okubo, Phys. Lett. 5, 165 (1963); J. J. Sakurai, Phys. Rev. 132, 434 (1963).

[22]See, e.g., B. T. Feld, *Models of Elementary Particles* (Blaisdell, Waltham, Mass., 1969), Chap. 15.

[23]Bramòn and Greco (Ref. 14) postulate an $X^0 \to 2\pi\gamma$ decay amplitude which is equivalent to our A_{Reg} with $\beta(t)$ taken to be a constant proportional to $g_{\rho\pi\pi}g_{\rho X\gamma}$; $g_{\rho X\gamma}$ is then obtained from a modified quark model. Their prediction for $\Gamma(X^0 \to 2\pi\gamma)$ is quite close to ours in spite of differences in the values of the parameters ($s_{\max}$, $g_{\rho X\gamma}$, etc.) and their neglect of the A_2 contribution.

[24]The actual agreement may not be quite so close. The assumption underlying Eq. (4.6) is that, in our model, all pion pairs are emitted in a "ρ" state; in fact, A_{Reg}, which contains the ρ pole, accounts for 80% of $\Gamma(X^0 \to 2\pi\gamma)$. However, one might expect, on the basis of duality, that A_{Res} would average out to provide some additional ρ contribution.

[25]Our model for $\eta \to 2\pi\gamma$ differs from that of Lassila and Young (Ref. 11) only in the choice of ρ trajectory; the real part of our trajectory [Eq. (2.15)] is equal to their trajectory. The inclusion of an imaginary part affects $\Gamma(\eta \to 2\pi\gamma)$ negligibly, but increases $\beta'_{\text{FESR}}(0)$ by $\sim 10\%$. Note that the sign of $\beta'_{\text{FESR}}(t)$ is not arbitrary; we have previously taken $g_{A_2X\pi}g_{A_2\pi\gamma} > 0$, and in the scheme of Glashow and Socolow (Ref. 19), which we have adopted, $g_{A_2\eta\pi}$ and $g_{A_2X\pi}$ have the same sign.

[26]In arriving at Eq. (4.10), however, they have apparently made use of the full FESR t dependence, $\beta'_{\text{FESR}}(t)$, in the decay region; taking $\beta'(t) = \beta'_{\text{FESR}}(0)$ leads to $\Gamma(\eta \to 2\pi\gamma) = 0.161$ keV (with a real ρ trajectory).

[27]M. Gell-Mann, D. Sharp, and W. D. Wagner, Phys. Rev. Lett. 8, 261 (1962); R. H. Dalitz and D. G. Sutherland, Nuovo Cimento 37, 1777 (1965).

[28]This is defined by

$$(2k_0)^{1/2}\langle 0|J_\mu^{\text{em}}(0)|\rho(k,\epsilon)\rangle = \frac{m_\rho^2}{f_\rho}\epsilon_\mu(k),$$

where $J_\mu^{\text{em}}(x)$ is the electromagnetic current, and k and ϵ are the momentum and polarization of the ρ, respectively.

[29]G. Bellettini *et al.*, Nuovo Cimento 66A, 243 (1969).

[30]Indeed, if Eq. (4.19a) is used to predict $\Gamma(\omega \to \pi\gamma)$ from $\Gamma(\pi^0 \to 2\gamma)$, it is found that $\Gamma(\omega \to \pi\gamma) = 0.82$ MeV or 0.63 MeV for $\Gamma(\pi^0 \to 2\gamma) = 11.2$ eV or 7.7 eV, respectively, where f_ρ is taken to be 5.24 from the $\rho \to e^+e^-$ rate (Ref. 12). This should be compared with the experimental value (Ref. 12) of $\Gamma(\omega \to \pi\gamma) = 1.0 \pm 0.2$ MeV. The agreement is worse when f_ρ is taken from the $\rho \to \mu^+\mu^-$ rate (Ref. 12), and in both cases the higher value of $\Gamma(\pi^0 \to 2\gamma) = 11.2$ eV gives a better result than does the preferred lower value.

[31]From Ref. 12 we find that the preferred value of $\Gamma(A_2 \to X\pi)$ is less than 1 MeV and so our input of $g_{A_2X\pi}$, especially for $\theta_P = -11°$, could be too high.

[32]Case 2 is superficially similar to the model of Bramòn and Greco (Ref. 14) for $\eta \to 2\pi\gamma$. We will discuss the relationships of their model to ours more fully in Sec. V.

[33]Instead of using a form like our Eq. (2.11) for the A_2 contribution to their decay amplitude, Bramòn and Greco employ the field-theoretic form, which has additional s dependence in the numerator. This certainly might lead to a possible *ad hoc* model for these decays, but not to an FDR model. They use as a consistency check on their $\pi\eta \to \pi\gamma$ amplitude the same FESR we have used here [Eq. (3.7) with the substitution $X^0 \to \eta$]; however, the probably *more reliable* finite contour dispersion relation over the same contour leads to our model and not to theirs.

[34]We should mention, in passing, that they did not investigate the FESR for $\pi X^0 \to \pi\gamma$.