Moving daughter Regge trajectories in the Van Hove model with SL(2, C)⊗SL(2, C) symmetry

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A Van-Hove-type model with SL(2, C)⊗SL(2,C) symmetry is constructed for the elastic scattering of pseudoscalar bosons. Reggeization in the complex plane of the principal quantum number of the little group O(4) is performed. Two Lorentz pole trajectories with opposite "Lorentz signatures" and opposite intrinsic parities are shown to give rise to moving Regge daughters without the explicit introduction of unitarity. This result is independent of the choice of 4-momentum squared or the mass squared in the numerator of the spin-1 propagator. The results are the same in the equal-mass case as well.

I. INTRODUCTION

Several authors have considered the Van Hove model for the scattering of spinless bosons.[1-3] For example, Sugar and Sullivan[2] obtain the contribution of a spin-J particle pole in the t channel as

$$T(J)=g_J^{\,2}(2J+1)\frac{\tilde{q}^{2J}P_J(\tilde{z})}{m^2(J)-t}.$$

On account of the choice of $m^2(J)$ in the numerator of the spin-1 propagator,[2] $\tilde{q}$ and $\tilde{z}\equiv\cos\tilde{\theta}$ occur instead of the c.m. momentum q and the cosine of the scattering angle $z\equiv\cos\theta$. Writing the expansion of $\tilde{q}^{2J}P_J(\tilde{z})$ in terms of $P_l(z)$, where $l=J,J-1,J-2,\ldots$, and then making a Sommerfeld-Watson (S-W) transform, they obtain fixed Regge daughters in addition to the moving Regge trajectory. For equal external masses, these daughters disappear. To make the daughters move they introduce unitarity by dressing the spin-J propagators. If the spin-1 propagator is written in the Landau gauge, i.e., its numerator part is taken as

$$\left(\delta_{\mu\nu}+\frac{f_\mu f_\nu}{-f_\lambda f_\lambda}\right), \tag{1.1}$$

where f_μ is the propagator momentum,[3,4] then $T(J)$ is simply given by[1]

$$T(J)=g_J^{\,2}(2J+1)\frac{q^{2J}P_J(z)}{m^2(J)-t}, \tag{1.2}$$

and *no daughters* are obtained in the usual construction of the Van Hove model.

In the present work we consider a Van Hove model based on the SL(2, C)⊗SL(2, C) symmetry, which gives moving Regge daughters when analytic continuation is performed in the complex n plane of the O(4) principal quantum number, without having to introduce unitarity. Also, this result does not depend upon the choice of $m^2(J)$ or of $-f_\lambda f_\lambda$ in the numerator (1.1). Further, the final result of the appearance of moving daughters in this field theoretic model is not affected if the incoming masses in the scattering are taken to be equal.

The plan of the paper is as follows. In Sec. II, some preliminary considerations related to this model are given. Section III is divided into two parts and deals with the actual construction of the Van Hove model and the consequent Reggeization in the n plane. The conclusion and discussion follow in Sec. IV, where a qualitative comparison of our result with that of Freedman and Wang[5] is also given.

II. PRELIMINARY CONSIDERATIONS

Consider the single-particle t-channel (direct) exchange diagram (Fig. 1) for the scattering of pseudoscalar bosons with momenta and masses given by

$$p(\mu)+q(m)\to p'(\mu)+q'(m).$$

Just as in the SL(2, C)-symmetric (Lorentz-invariant) case the exchange particle is labeled by the eigenvalue of the O(3) Casimir operator J^2, in the SL(2, C)⊗SL(2, C)-symmetric case the exchange particle will be labeled by the eigenvalues of the two O(4) Casimir operators.[5,6] Thus, in the scattering of spinless bosons considered here, the exchange particle belongs to the $D^{j,j}\sim D^{2j,0}$

$=D^{n,0}$ $(n=2j)$ representation[7] of O(4), where $n+1$ is the principal quantum number. In a previous work by one of us,[7] two effective interaction Lagrangians $\mathcal{L}^{(n)}(x)$ and $\mathcal{L}_5^{(n)}(x)$ were considered:

$$\mathcal{L}^{(n)}(x)=g_n^{(1)}(C^{-1}\gamma_{\mu_1})_{\alpha_1\beta_1}(C^{-1}\gamma_{\mu_2})_{\alpha_2\beta_2}\cdots(C^{-1}\gamma_{\mu_n})_{\alpha_n\beta_n}\times\psi^{j,j}_{\alpha_1\cdots\alpha_n,\beta_1\cdots\beta_n}(x)\varphi_1(x)\partial_{\mu_1}\cdots\partial_{\mu_n}\varphi_2(x)+\text{H.c.} \quad (2.1)$$

$\mathcal{L}_5^{(n)}(x)$ is obtained by replacing γ_{μ_1} by $\gamma_5\gamma_{\mu_1}$ in $\mathcal{L}^{(n)}(x)$. In Eq. (2.1), $\psi^{j,j}_{\alpha_1\cdots\alpha_n,\beta_1\cdots\beta_n}(x)$ is the field of the exchange boson which is completely symmetric in the α's and the β's, separately, and satisfies the Dirac equation in each of the indices. Further, $\varphi_1(x)$ and $\varphi_2(x)$ are the pseudoscalar fields with masses m and μ, respectively. The parity-conserving Lagrangians with coupling constants and masses $M_n^{(+)}$ and $M_n^{(-)}$, corresponding to even and odd intrinsic parity of the exchange particle, are given in Table I. Then the amplitudes for the above scattering process for even n $(T_e^{(n)})$ and for odd n $(T_0^{(n)})$ are given by[7]

$$T_e^{(n)}=\frac{g_n^{(1)2}p^{2n}}{t+(M_n^{(+)})^2}C_n^{(1)}(z)+\frac{g_n^{(2)2}p^{2(n-1)}}{t+(M_n^{(-)})^2}p_0^{\,2}\times\tfrac{1}{4}[n(n+1)-L^2(\partial)]C_{n-1}^{(1)}(z)\,, \quad (2.2)$$

$$T_0^{(n)}=\frac{g_n^{(3)2}p^{2n}}{t+(M_n^{(-)})^2}C_n^{(1)}(z)+\frac{g_n^{(4)2}p^{2(n-1)}}{t+(M_n^{(+)})^2}p_0^2\times\tfrac{1}{4}[n(n+1)-L^2(\partial)]C_{n-1}^{(1)}(z)\,, \quad (2.3)$$

where $C_n^{(1)}(z)$ is the Gegenbauer polynomial (note that $z=\cos\theta_t\sim -s$ for $s\to-\infty$), $L^2(\partial)$ $(\partial\equiv\partial/\partial z)$ is the Legendre operator with the property

$$L^2(\partial)P_l(z)=l(l+1)P_l(z)\,, \quad (2.4)$$

$p=|\vec{p}|$ is the t-channel c.m. momentum, and p_0 is the energy of the boson with mass μ. Now consider the O(4, 2) weight diagram[8] (Fig. 2), which is a plot of the principal quantum number $n'=n+1$

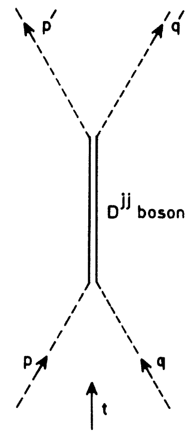

FIG. 1. Feynman diagram for $\pi_1\pi_2\to\pi_1\pi_2$ with the propagator corresponding to a boson belonging to the D^{jj} representation of O(4).

$=2j+1$ against the angular momentum l. We see that each O(4) state characterized by a given value of n contains angular momentum states $l=0,1,2,\ldots,n$. Since $C_n^{(1)}(z)$ can be[9] expanded in terms of $P_l(z)$ with $l=n, n-2, n-4,\ldots,0$ or 1, according to whether n is an even or odd integer, we notice from Eqs. (2.2) and (2.3) that if we had not used the additional Lagrangian $L_5^{(n)}(x)$ with the odd intrinsic parity of the $D^{j,j}$ particle, we would have missed the contribution to the amplitude from all the odd angular momentum states for the even n and from all the even angular momentum states for the odd n values. Now, summing up over all n, i.e., calculating

$$T=T_e+T_o\,, \quad (2.5)$$

where

$$T_e=\sum_{n\text{ even}}T_e^{(n)}\,,\qquad T_o=\sum_{n\text{ odd}}T_o^{(n)}\,, \quad (2.6)$$

we obtain the contributions of all the points of the O(4, 2) weight diagram. This is worked out in the next section.

TABLE I. Lagrangians for the pole diagram in Fig. 1.

No.	Lagrangian	Intrinsic parity P_n	Mass	Coupling constant
1	$\mathcal{L}^{(n,+)}(x)$, n even	+1	$M_n^{(+)}$	$g_n^{(1)}$
2	$\mathcal{L}_5^{(n,-)}(x)$, n even	−1	$M_n^{(-)}$	$g_n^{(2)}$
3	$\mathcal{L}^{(n,-)}(x)$, n odd	−1	$M_n^{(-)}$	$g_n^{(3)}$
4	$\mathcal{L}_5^{(n,+)}(x)$, n odd	+1	$M_n^{(+)}$	$g_n^{(4)}$

III. CALCULATION OF POLE CONTRIBUTIONS

A. Amplitude representation

Let us start with a given even integer n. The amplitude in this case is given by Eqs. (2.2) and (2.4). Now, as remarked earlier, $C_n^{(1)}(z)$ can be expressed as a linear sum of O(3) Legendre polynomials

$$C_n^{(1)}(z) = \sum_{r=0,2,4,\ldots,n} a_n^{(r)} P_{n-r}(z)\,, \qquad (3.1)$$

where

$$a_n^{(r)} = \frac{(2n-2r+1)}{(2n-r+1)!} 2^{2(n-r)} r! \left\{\frac{[(2n-r)/2]!}{(r/2)!}\right\}^2 . \qquad (3.1')$$

Since r is even, the above curly bracket always contains the ratio of two integers. Further, since n is even, $n-1$ is odd and hence, $C_{n-1}^{(1)}(z)$ can be expanded using Eq. (3.1), in terms of odd-degree Legendre polynomials. Thus,

$$C_{n-1}^{(1)}(z) = \sum_{r=0,2,\ldots,n-2} a_{n-1}^{(r)} P_{n-r-1}(z)\,, \qquad (3.2)$$

where $a_{n-1}^{(r)}$ can be calculated from (3.1′). Using Eqs. (3.2) and (2.4), we immediately obtain

$$L^2(\theta) C_{n-1}^{(1)}(z) = \sum_{r=0,2,\ldots,n-2} a_{n-1}^{(r)} (n-r-1)(n-r) \times P_{n-r-1}(z)\,. \qquad (3.3)$$

Then $T_e^{(n)}$ becomes, on substituting Eq. (3.3) into Eq. (2.2),

$$T_e^{(n)} = K_n^{(1)} \sum_{r=0,2,\ldots,n} a_n^{(r)} P_{n-r}(z) + K_n^{(2)} \sum_{r=0,2,\ldots,n-2} b_{n-1}^{(r)} P_{n-r-1}(z)\,, \qquad (3.4)$$

where

$$b_{n-1}^{(r)} = \tfrac{1}{4}[n(n+1) - (n-r-1)(n-r)] a_{n-1}^{(r)}\,, \qquad (3.4')$$

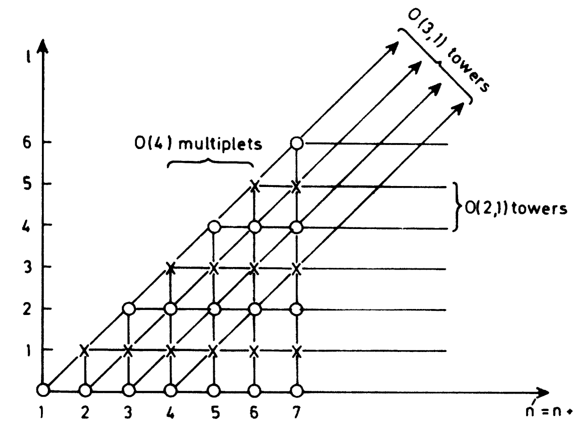

FIG. 2. Multiplets and towers of O(4,2).

$$K_n^{(1)} = \frac{g_n^{(1)2} p^{2n}}{t + (M_n^{(+)})^2}\,, \qquad (3.4'')$$

$$K_n^{(2)} = \frac{g_n^{(2)2} p^{2(n-1)}}{t + (M_n^{(-)})^2} p_0^2\,. \qquad (3.4''')$$

Therefore, summing over all even integers n,

$$T_e = \sum_{n=\text{even integers}} T_e^{(n)}\,, \qquad (3.5)$$

where $T_e^{(n)}$ is given by Eq. (3.4).

The above sum can be extended to include all integers as well by the formal introduction of the "signature factor" $\{[1+(-1)^n]/2\}$. Thus,

$$T_e = \sum_{n=\text{all integers}} T_e^{(n)} \left(\frac{1+(-1)^n}{2}\right), \qquad (3.6)$$

where n in $T_e^{(n)}$ can now take all integral values and hence, the above sum is over all integers. Thus, the final expression for T_e becomes

$$T_e = \sum_{n=\text{all integers}} \left\{ \left[\sum_{r=0,2,\ldots,n} K_n^{(1)} a_n^{(r)} P_{n-r}(z) + \sum_{r=0,2,\ldots,n-2} K_n^{(2)} b_{n-1}^{(r)} P_{n-r-1}(z) \right] \left(\frac{1+(-1)^n}{2}\right) \right\}. \qquad (3.7)$$

Note that the two sums over n and r appearing in Eq. (3.7) as a double sum for T_e must be dealt with as follows, separately. In the terms containing P_{n-r} (even-degree polynomials), the summation over n must start from r ranging over $r, r+1, \ldots, \infty$; for any allowed value of r, whereas, in the terms containing P_{n-r-1} (odd-degree polynomials), the summation over n must start from $r+1$ ranging over $r+1, r+2, \ldots, \infty$, for a given allowed value of r. To obtain an expansion in $P_n(z)$ (where n ranges over all integers) instead of $P_{n-r}(z)$ and $P_{n-r-1}(z)$ we make the change $n \to n+r$ in the first summation and $n \to n+r+1$ in the second summation, thereby obtaining

$$T_e = \sum_{n=0,1,2,\ldots,\infty} \left[\sum_{r=0,2,\ldots,n} K_{n+r}^{(1)} a_{n+r}^{(r)} \left(\frac{1+(-1)^{n+r}}{2}\right) P_n(z) + \sum_{r=0,2,\ldots,n-2} K_{n+r+1}^{(2)} b_{n+r}^{(r)} \left(\frac{1+(-1)^{n+r+1}}{2}\right) P_n(z) \right]. \qquad (3.8)$$

Notice that opposite signatures appear in the two sums over r in Eq. (3.8).

B. Sommerfeld-Watson transform in the n plane

Equation (3.8) is now ready for the Sommerfeld transform. We take the two types of rth terms for the required continuation in n, and note that for any (even) r in the above equation $(-1)^r = +1$ and $(-1)^{r+1} = -1$. Then for Re $n > N$, we can perform the continuation[10] by replacing the sum over n by the integral in the usual way:

$$\sum_{n=\text{all integers}} \left[K^{(1)}_{n+r} a^{(r)}_{n+r} \left(\frac{1+(-1)^n}{2}\right) P_n(z) + K^{(2)}_{n+r+1} b^{(r)}_{n+r} \left(\frac{1-(-1)^n}{2}\right) P_n(z)\right]$$
$$= \left(-\frac{i}{2}\right) \int_C dn \left[K^{(1)}(n+r) a^{(r)}(n+r) \left(\frac{1+e^{-i\pi n}}{2}\right)\left(\frac{P_n(-z)}{\sin \pi n}\right) + K^{(2)}(n+r+1) b^{(r)}(n+r) \left(\frac{1-e^{-i\pi n}}{2}\right)\left(\frac{P_n(-z)}{\sin \pi n}\right)\right]. \tag{3.9}$$

The continued functions above can be obtained from (3.4′)–(3.4‴). Consider, now, a "Lorentz trajectory" $n = \alpha(t)$ in the n plane which passes through n for $t = -M^2(n)$. Then near $n = \alpha(t)$

$$\frac{1}{t + M^2(n)} \simeq \frac{-[d\alpha(t)/dt]_{t=-M^2(n)}}{n - \alpha(t)},$$

which gives a pole at $n = \alpha(t)$ in the n plane. Thus, the factors

$$\frac{1}{t + [M^{(+)}(n+r)]^2} \cong \frac{-[d\alpha^+(t)/dt]\,|_{t=-M^{(+)2}(n+r)\equiv t_+}}{n + r - \alpha^+(t)}, \tag{3.10}$$

and

$$\frac{1}{t + [M^{(-)}(n+r+1)]^2} \simeq \frac{-[d\alpha^-(t)/dt]\,|_{t=-M^{(-)2}(n+r+1)\equiv t_-}}{n + r + 1 - \alpha^-(t)}, \tag{3.11}$$

where $\alpha^+(t)$ $[\alpha^-(t)]$ are trajectories corresponding to even intrinsic parity (odd intrinsic parity) and even "Lorentz signature" (odd "Lorentz signature"), give rise to two sets of Regge pole families for various values of r ($r = 0, 2, 4, \cdots$). The above factors (3.10) and (3.11) occur in the integral in Eq. (3.9) through the continued function $K^{(1)}(n+r)$ and $K^{(2)}(n+r+1)$ [see Eqs. (3.4″) and (3.4‴)]. If the couplings $g^{(1)}(n)$ and $g^{(2)}(n)$ are sufficiently smooth, one can deform the n-plane contour C (enclosing the positive real axis) and shift it to a line parallel to a certain vertical line in the left half plane (Re$n > N$) picking up contributions over the sets of Regge daughters $\alpha^+(t) - r$ ($r = 0, 2, 4, \cdots$) and $\alpha^-(t) - r - 1$ ($r = 0, 2, 4, \cdots$) ($r = 0$ gives parents) as

$$-\pi\left\{[g^{(1)}(\alpha^+(t))]^2 [a^{(r)}(\alpha^+(t))] \left(\frac{d\alpha^+(t)}{dt}\right)_{t=t_+} \left(\frac{1+e^{-i\pi(\alpha^+(t)-r)}}{2}\right) p^{2\alpha^+(t)} \frac{P_{\alpha^+(t)-r}(-z)}{\sin[\pi(\alpha^+(t)-r)]}\right.$$
$$\left. + [g^{(2)}(\alpha^-(t))]^2 (b^{(r)}(\alpha^-(t)-1)) p_0^{\,2} \left(\frac{d\alpha^-(t)}{dt}\right)_{t=t_-} \left(\frac{1-e^{-i\pi(\alpha^-(t)-r-1)}}{2}\right) p^{2(\alpha^-(t)-1)} \frac{P_{\alpha^-(t)-r-1}(-z)}{\sin[\pi(\alpha^-(t)-r-1)]}\right\}.$$

We may rewrite the above rth-Regge daughters' contributions by introducing the energy-dependent couplings (form-factor type). This is done here to emphasize the asymptotic behavior, as usual, in the dimensionless variable z which occurs as an argument of the Legendre functions. Thus, the above becomes

$$-\pi\left\{[\lambda_r^{(1)}(p;\alpha^+(t))]^2 [a^{(r)}(\alpha^+(t))] \left(\frac{d\alpha^+(t)}{dt}\right)_{t_+} \left(\frac{1+e^{-i\pi(\alpha^+(t)-r)}}{2}\right) p^{2(\alpha^+(t)-r)} \frac{P_{\alpha^+(t)-r}(-z)}{\sin[\pi(\alpha^+(t)-r)]}\right.$$
$$\left. + [\lambda_r^{(2)}(p;\alpha^-(t))]^2 [b^{(r)}(\alpha^-(t)-1)] p_0^{\,2} \left(\frac{d\alpha^-(t)}{dt}\right)_{t_-} \left(\frac{1-e^{-i\pi(\alpha^-(t)-r-1)}}{2}\right) p^{2(\alpha^-(t)-r-1)} \frac{P_{\alpha^-(t)-r-1}(-z)}{\sin[\pi(\alpha^-(t)-r-1)]}\right\}, \tag{3.12}$$

where

$$\lambda_r^{(1)}(p;\alpha^+(t)) = p^r g^{(1)}(\alpha^+(t)),$$
$$\lambda_r^{(2)}(p;\alpha^-(t)) = p^r g^{(2)}(\alpha^-(t)) \tag{3.12′}$$
$$(r = 0, 2, 4, \cdots).$$

Carrying out the same analysis as above for odd integers n, we obtain the rth Regge daughters' contributions to T_o as

$$-\pi\left\{[\lambda_r^{(3)}(p;\alpha^-(t))]^2[a^{(r)}(\alpha^-(t))]\left(\frac{d\alpha^-(t)}{dt}\right)_{t_-}\left(\frac{1-e^{-i\pi(\alpha^-(t)-r)}}{2}\right)p^{2(\alpha^-(t)-r)}\frac{P_{\alpha^-(t)-r}(-z)}{\sin[\pi(\alpha^-(t)-r)]}\right.$$
$$\left.+[\lambda_r^{(4)}(p;\alpha^+(t))]^2[b^{(r)}(\alpha^+(t)-1)]p_0^2\left(\frac{d\alpha^+(t)}{dt}\right)_{t_+}\left(\frac{1+e^{-i\pi(\alpha^+(t)-r-1)}}{2}\right)p^{2(\alpha^+(t)-r-1)}\frac{P_{\alpha^+(t)-r-1}(-z)}{\sin[\pi(\alpha^+(t)-r-1)]}\right\},\qquad(3.13)$$

where

$$\lambda_r^{(3)}(p;\alpha^-(t))=p^r g^{(3)}(\alpha^-(t)),$$
$$\lambda_r^{(4)}(p;\alpha^+(t))=p^r g^{(4)}(\alpha^+(t))\qquad(3.13')$$
$$(r=0,2,4,\cdots).$$

Then, substituting the rth Regge daughters' contributions (3.12) and (3.13) to T_e and T_o, respectively, in Eqs. (2.5) and (2.6), we can obtain the net contribution of the rth Regge daughters to the total amplitude T.

IV. CONCLUSION AND DISCUSSION

We have seen that in (3.12) and (3.13) we obtain evenly spaced in angular momentum *moving Regge daughters* contributing to the amplitude *without having* to introduce unitarity. The above results are obtained in the Landau gauge in which the numerator part of the spin-1 propagator is given by (1.1). If now we take the usual propagator in which M_n^2 occurs instead of $-f_\lambda f_\lambda$ in (1.1), we have to replace p, p_0 and $z=\cos\theta_t$ by $\tilde{p}, \tilde{p}_0$ and $\tilde{z}=\cos\tilde{\theta}_t$ in Eqs. (2.2)–(2.4), where

$$\tilde{p}^2=-\mu^2+\frac{tp^2}{M_n^2}+\frac{t\mu^2}{M_n^2}=\tilde{p}'^2\qquad(4.1)$$

and

$$\tilde{p}^2\tilde{z}=p^2z+p_0^2\left(\frac{t}{M_n^2}-1\right).\qquad(4.2)$$

Using Eqs. (4.1) and (4.2), and the expansion of the Gegenbauer polynomials[9] in powers of the argument, we obtain in the same way as Sugar and Sullivan have done for the Legendre polynomials[2]

$$\tilde{p}^{2n}C_n^{(1)}(z)=p^{2n}C_n^{(1)}(z)+\text{lower-degree polynomials}.\qquad(4.3)$$

Thus, the transformed quantities $\tilde{p}, \tilde{p}_0$, and $\tilde{z}$ when substituted in Eqs. (2.2)–(2.4) will again result in leading terms like $p^{2n}C_n^{(1)}(z)$ and $p^{2(n-1)}C_{n-1}^{(1)}(z)$ because of Eq. (4.3). In addition to these terms there will be lower-degree polynomials which may be ignored because we are interested in the leading singularities in the n plane. These polynomials, thus retained in the amplitude, will again lead us, after summing over n and on carrying out the S-W transform, to four sets of moving Regge daughters as before. Hence, the result is independent of the choice of the spin-1 propagator used. This result is essentially a consequence of the invariance of the Lagrangian under $SL(2,C)\otimes SL(2,C)$ whose little group is O(4). Further, our model is consistent with the predictions of the O(4)-symmetric theory of Freedman and Wang.[5] In their work O(4) symmetry is shown to hold in pairwise equal mass scattering in the *forward direction*. In particular, for the case of $N\bar{N}\to N\bar{N}$ (equal mass) they obtain

$$n=j+k,\quad k=0,2,4,\ldots,\qquad(4.4)$$

where n is the principal quantum number and j and k are the ordinary angular momenta. Now consider two Lorentz poles $\alpha^\pm$ in the n plane corresponding to opposite "Lorentz signatures" at $t=0$. These poles give rise to two evenly spaced families of fixed daughters in the j plane because of Eqs. (4.4):

$$j=\alpha^\pm(0)-k,\quad k=0,2,4,\ldots.\qquad(4.5)$$

In our model where O(4) symmetry is built in for all t, we have considered the expansion of resulting polynomials $C_n^{(1)}(z)$ and $C_{n-1}^{(1)}(z)$ in the calculation for the amplitude $T^{(n)}$ [where $(n+1)$ is the principal quantum number and n is even, say] in terms of the even and odd degree Legendre polynomials as given in Eqs. (3.1) and (3.2). In these expansions, we have implicitly used the following transformations which connect the O(3) quantum number l (or what is the same here, the angular momentum j), with the O(4) quantum number n:

$$l\equiv j=n-r,\quad r=0,2,4,\ldots,n$$

and (4.6)

$$l\equiv j=n-r-1,\quad r=0,2,4,\ldots,n-2.$$

Now, in our model for spinless bosons where conservation of parity is also taken into account, the two trajectories $\alpha^+(t)$ and $\alpha^-(t)$ (having even intrinsic parity and even "Lorentz signature," and odd intrinsic parity and odd "Lorentz signature," respectively) automatically give rise to two sets of moving Regge daughters because of Eqs. (4.6):

$$j=\alpha^+(t)-r \text{ and } j=\alpha^-(t)-r-1 \qquad (r=0,2,4,\ldots). \qquad (4.7)$$

Similarly, as seen before, two more sets of daughters contribute to the odd-integral n part of T. Further, the Regge daughters in a family have exactly the same signature $(-1)^j$ as the "Lorentz signature" of the parent trajectory wherein $r=0$. Although, strictly speaking, a comparison of Eqs. (4.5) and (4.7) cannot be made because the two results relate to two different processes, we may conclude that our model is consistent with the predictions of the theory of Freedman and Wang,[5] at least qualitatively.

Finally, in our model the calculation of the ratio of the residues of Regge daughters having the same intrinsic parity, Lorentz signature (say positive) and coupling can be explicitly *done for all t*. For example, the required residue ratio for the rth and $(r+2)$nd daughters (successive) can be written using (3.12) as

$$R_r^+(t)=\left(\frac{\lambda_r^{(1)}(p;\alpha^+(t))}{\lambda_{r+2}^{(1)}(p;\alpha^-(t))}\right)^2\frac{a^{(r)}(\alpha^+(t))}{a^{(r+2)}(\alpha^+(t))}$$
$$=\frac{1}{(p^2)^2}\left(\frac{2\alpha^+(t)-2r+1}{2\alpha^+(t)-2r-3}\right)\frac{(2\alpha^+(t)-r)}{(2\alpha^+(t)-r+1)}\left(\frac{r+2}{r+1}\right). \qquad (4.8)$$

Similar results can be derived for the corresponding ratios of odd parity and odd "Lorentz signature" daughter residues.

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[10]For reasons of simplicity, the analytic continuation in this model is based on the S-W representation and not on the Froissart-Gribov F-G representation which takes care of the analyticity of the amplitude. The F-G representation would in the usual way result in two scattering functions interpolating between even n and odd n in a unique manner. However, we would like to emphasize the fact that the total amplitude in the Van Hove model in the framework of the complex Lorentz symmetry [$SL(2,C)\otimes SL(2,C)\cong$ complex Lorentz symmetry] naturally leads to incorporation of the "signature" without recourse to the explicit assumption of the existence of exchange forces.