# Contribution of SL(2, $C$)⊗SL(2, $C$) poles to scalar bosons and $\pi N$ scattering amplitudes

Arif-uz-Zaman*
*International Centre for Theoretical Physics, Trieste, Italy*
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An SL(2, $C$)⊗SL(2, $C$)-symmetric model for calculating direct-channel pole diagrams for $\pi_1\pi_2\to\pi_1\pi_2$ and $\pi N\to\pi N$ is given. The angular dependence of the scattering amplitudes comes out in the form of O(4) rotation matrices. In order to include all angular momentum states of O(4) multiplets, intermediate particles of even and odd intrinsic parities have been introduced. Certain expressions of O(4) rotation matrices and their connections with the Gegenbauer polynomials are given in the Appendix.

## INTRODUCTION

Shortly after the demonstration by Freedman and Wang[1] of the O(4) symmetry of the scattering amplitude at the energy $t=0$, propagators showing O(4) symmetry in the direct-channel pole diagrams were investigated.[2] Later, Iwasaki[3] and Harnad[4] gave a prescription for writing down spin-$j$ boson propagators and used it to calculate the direct-channel pole diagram for the scattering of spinless bosons of equal masses. The scattering amplitude thus obtained exhibited O(4) symmetry (at $t=0$), in the sense that there the angular dependence was given by the Gegenbauer polynomial $C_{2j}^{(1)}(\theta)$.

In the present work we consider a model which gives O(4)-symmetric scattering amplitudes using the direct-channel pole diagrams with no restriction on the energy $t$. If we consider the O(4, 2) weight diagram (shown in Fig. 2) which is a plot of the principal quantum number $n'$ against the angular momentum $l$, we see that each O(4) state characterized by a particular value of $n'$ contains angular momentum states $l=0,1,2,\ldots,n'-1$. Our object is to obtain O(4) symmetric amplitudes for direct channel pole diagrams which will depend on $n'$ and contain contributions from all the angular momentum states mentioned above. A summation over $n'$ would then include contributions from all the points in the weight diagram, and a Van Hove model can then be made based on the Reggeization in the complex $n'$ plane. Two scattering processes have been considered:

(a) $\pi_1+\pi_2\to\pi_1+\pi_2$,

(b) $\pi+N\to\pi+N$.

In (a) the masses of the two pions may be different. The intermediate particle belongs to a representation of SL(2, $C$)⊗SL(2, $C$) and is represented by a generalized Wigner-Bargmann[5] field

$$\psi^{j_1,j_2}_{\alpha_1\alpha_2\cdots\alpha_{2j_1},\beta_1\beta_2\cdots\beta_{2j_2}}(x).$$

This multispinor field is completely symmetric in the indices $\alpha_1\alpha_2,\ldots,\alpha_{2j}$, and $\beta_1\beta_2,\ldots,\beta_{2j}$, separately, and obeys the Dirac equation in all the indices. In the momentum representation the spinors are characterized by the two eigenvalue numbers $j_1$ and $j_2$ of the Casimir operators $\vec{J}^{(1)2}$ and $\vec{J}^{(2)2}$ of O(4) and the two eigenvalues $m_1$ and $m_2$ of $J_3^{(1)}$ and $J_3^{(2)}$, respectively. $\vec{J}^{(1)}$ and $\vec{J}^{(2)}$, as is well known,[6] are related to the generator $J_{\mu\nu}$ of O(4) by $\vec{J}^{(1)}=\frac{1}{2}(\vec{L}+\vec{A})$ and $\vec{J}^{(2)}=\frac{1}{2}(\vec{L}-\vec{A})$ with $L_i=\frac{1}{2}\epsilon_{ijk}J_{jk}$ and $A_i=J_{i4}$. We have defined the O(4) rotation matrix $d^{j_1j_2}_{l'm';lm}(\theta)$ as the matrix element of $e^{-iA_2\theta}$ in the $|j_1j_2,lm\rangle$ basis, $l(l+1)$ and $m$ are the eigenvalues of $\vec{L}^2$ and $L_3$, respectively. This is different from the rotation matrix defined by Biedenharn[6] and Freedman and Wang[1] which is the matrix element of $e^{-iA_3\theta}$ in the $|j_1j_2,lm\rangle$ basis and is diagonal in $m$.

For the $\pi\pi$ scattering we write an effective Lagrangian

$$g(C^{-1}\gamma_{\mu_1})_{\alpha_1\beta_1}(C^{-1}\gamma_{\mu_2})_{\alpha_2\beta_2}\cdots(C^{-1}\gamma_{\mu_{2j}})_{\alpha_{2j}\beta_{2j}} \times\psi^{j,j}_{\alpha_1\cdots\alpha_{2j},\beta_1\cdots\beta_{2j}}\varphi(x)\partial_{\mu_1}\cdots\partial_{\mu_{2j}}\varphi(x)$$

and calculate the direct-channel pole diagram shown in Fig. 1(a). We find the angular dependence of the amplitude is given by $d^{jj}_{00,00}(\theta)=C^{(1)}_{2j}(\theta)$.[7] The expansion of $C^{(1)}_{2j}(\theta)$ in terms of the Legendre polynomials[4] $P_l(\cos\theta)$ contains only even or odd $l$ values according to whether $2j=n$ is even or odd, respectively. To include the missing angular momentum states occurring in the $n'=2j+=n+1$ state as shown in Fig. 2, we must introduce another Lagrangian which is obtained by replacing one $(C^{-1}\gamma_{\mu_i})_{\alpha_i\beta_i}$ factor in the Lagrangian mentioned above by $(C^{-1}\gamma_5\gamma_{\mu i})_{\alpha_i\beta_i}$. We find that this new Lagrangian gives an angular dependence in the form $d^{jj}_{10;\,10}(\theta)$ whose expansion in $P_l(\cos\theta)$ contains only odd or even $l$ values according to whether $n=2j$ is even or odd, respectively. Assuming space reflection invariance of the effective Lagrangians we introduce two $\psi^{jj}(x)$ fields describing bosons of even and odd intrinsic parities and write down four parity invariance effective Lagrangians, two for

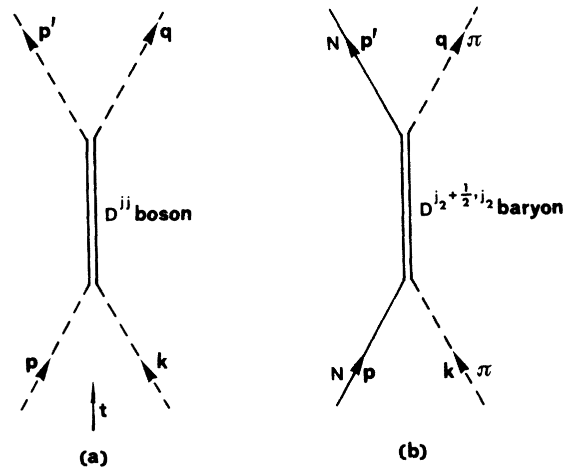


FIG. 1. (a) Feynman diagram for $\pi\pi\to\pi\pi$ with the propagator corresponding to a boson belonging to the $D^{jj}$ representation of O(4). (b) Feynman diagram for $\pi N\to\pi N$ with the propagator corresponding to a baryon belonging to the $D^{j_2+1/2,j_2}$ representation of O(4).

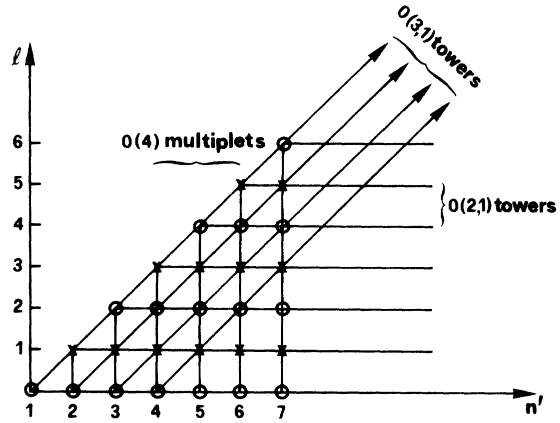


FIG. 2. Multiplets and towers of O(4, 2).

even $n$ and two for odd $n$. For any given $n$ all the states $l=0,1,2,\ldots,$ $n=n'-1$ then contribute to the scattering amplitude. A summation over $n'$ (or $n$) would then include the contribution of all the states of the O(4, 2) spectrum.[8,9]

The case of $\pi N\to\pi N$ has been treated in a similar fashion. The helicity amplitudes contain $d^{j_2+\frac{1}{2},j_2}_{\frac{1}{2},\mp\frac{1}{2};\frac{1}{2}\frac{1}{2}}(\theta)$ rotation matrices. An ambiguity coming from off-mass-shell continuation of the propagator momentum has been discussed. As in the case of spin-$j$ poles,[10] the highest order term in $\cos\theta$ is shown to be free of the ambiguity in the present case also.

The scattering amplitudes obtained in the text are in the form of certain summations involving the Clebsch-Gordan (C.G.) coefficients and the O(3) rotation matrices $d^l_{m_1m_2}(\theta)$. In Appendix A these summations have been related to the O(4) rotation matrices and these in turn have been expressed in terms of the Gegenbauer polynomials and their derivatives.

## I. THE SCALAR BOSON SCATTERING VIA $D^{jj}$ BARYON POLES

In order to construct Lagrangians whose contribution to the pole diagrams exhibits O(4) symmetry in the scattering amplitudes, we introduce a field $\psi^{j_1,j_2}$ which obeys a generalized Wigner-Bargmann[5] equation and belongs to a direct-product representation of two ordinary Wigner-Bargmann representations. The field

$$\psi^{j_1,j_2}_{\alpha_1\alpha_2\cdots\alpha_{2j_1},\beta_1\beta_2\cdots\beta_{2j_2}}(x)$$

is completely symmetric in all the indices $\alpha_1\alpha_2,\ldots,\alpha_{2j_1}$ and $\beta_1\beta_2,\ldots,\beta_{2j_2}$, separately, and obeys the Dirac equation in each of the indices $\alpha_i$ and $\beta_i$. In the momentum space the positive energy multispinor is given by[11]

$$U^{j_1m_1,j_2m_2}_{\alpha_1\cdots\alpha_{2j_1},\beta_1\cdots\beta_{2j_2}}(f)$$
$$=[L(f)\times L(f)\times\cdots\times L(f)\times\cdots\times L(f)]_{\alpha_1\cdots\alpha_{2j_1},\beta_1\cdots\beta_{2j_2};\lambda_1\cdots\lambda_{2j_1},\tau_1\cdots\tau_{2j_2}}U^{j_1m_1,j_2m_2}_{\lambda_1\cdots\lambda_{2j_1},\tau_1\cdots\tau_{2j_2}}(0).\quad(1.1\text{a})$$

$L(f)$ are the well-known Lorentz boost operators in the Dirac representation

$$L(f)=e^{(1/2)\gamma_5\vec{\sigma}\cdot\hat{f}\tanh^{-1}(|\vec{f}|/f_0)}.\quad(1.1\text{b})$$

The momentum $f$ is on the mass shell

$$-f_\mu f_\mu=f_0^2-\vec{f}^2=M^2_{j_1j_2},\quad(1.1\text{c})$$

where $M_{j_1j_2}$ is the mass of the particle associated with the field $\psi^{j_1,j_2}$. The rest spinors $U^{j_1m_1,j_2m_2}(0)$ in (1.1a) are now written as a Kronecker product of two completely symmetric Wigner-Bargmann rest spinors:

$$U^{j_1m_1,j_2m_2}(0)=U^{j_1m_1}\times U^{j_2m_2}.\quad(1.2\text{a})$$

$U^{jm}$ form a completely symmetric and orthonormal set defined by[11]

$$U^{jm}=\frac{1}{({}^{2j}C_{j-m})^{1/2}}\sum_P u^1\times u^2\times\cdots u^1.\quad(1.2\text{b})$$

$u^1$ and $u^2$ are spin up and down Dirac spinors satisfying $\gamma_4u^{1,2}=u^{1,2}$, $\sigma_3u^1=u^1$, $\sigma_3u^2=-u^2$. $\sum_P$ stands for the sum over all distinguishable permutations of $u^1$ and $u^2$ in (1.2b). If $n_1$ and $n_2$ are the numbers of $u^1$ and $u^2$, respectively, in each term of $U^{jm}$, then

$$n_1+n_2=2j\,,\qquad n_1-n_2=2m\,, \tag{1.3}$$

where $j$ and $m$ are, respectively, the spin and the third component of spin associated with the multispinor $U^{jm}$. The number of distinguishable permutations are

$$^{n_1+n_2}C_{n_2}={}^{2j}C_{j-m}={}^{2j}C_{j+m}\,. \tag{1.4}$$

Hence, the normalization factor $^{2j}C_{j-m}$ has been introduced in the denominator of the right-hand side of (1.2b), giving

$$\overline{U}^{jm'}U^{jm}=\delta_{mm'}\,. \tag{1.5}$$

The spin operator in the $U^{j_1m_1}$ space is

$$\vec{\mathrm{j}}^{(1)}=\tfrac{1}{2}(\vec{\sigma}\times 1\times 1\times\cdots\times 1+1\times\vec{\sigma}\times 1\times\cdots\times 1+\cdots+1\times 1\times 1\times\cdots\times\vec{\sigma})\,, \tag{1.6}$$

and $U^{j_1m_1}$ is an eigenvector of $(\vec{\mathrm{j}}^{(1)})^2$ and $j_3^{(1)}$ belonging to the eigenvalues $j_1(j_1+1)$ and $m_1$, respectively. Similar considerations hold for $U^{j_2m_2}$. It follows now that $U^{j_1m_1,j_2m_2}(0)$ are the eigenvectors of

$$\vec{\mathrm{J}}^{(1)2}=\vec{\mathrm{j}}^{(1)2}\times I\,,\qquad \vec{\mathrm{J}}^{(2)2}=I\times\vec{\mathrm{j}}^{(2)2}\,, \tag{1.7}$$

$$J_3^{(1)}=j_3^{(1)}\times I\,,\qquad J_3^{(2)}=I\times j_3^{(2)}\,. \tag{1.8}$$

$U^{j_1m_1,j_2m_2}(0)$ therefore represent the basis vectors of the $(j_1j_2)$ representation of[6] O(4). Further relevant discussion of these representations is given in Appendix A.

It follows from the construction given above that

$$U^{j_1m_1,j_2m_2}_{\alpha_1\cdots\alpha_{2j_1},\beta_1\cdots\beta_{2j_2}}(f)$$

are completely symmetric in the $\alpha$'s and $\beta$'s separately. The adjoint wave function is given by

$$\overline{U}^{j_1m_1,j_2m_2}(f)=U^{\dagger j_1m_1,j_2m_2}(f)\,\gamma_4\times\gamma_4\times\cdots\times\gamma_4\,. \tag{1.9}$$

The negative-energy wave function $V^{j_1m_1,j_2m_2}(f)$ is defined[11] in a similar way, the only difference being that $U^{j_1,-m_1}$ and $U^{j_2,-m_2}$ are multiplied by $2j_1$- and $2j_2$-fold Kronecker products of the charge-conjugation matrix $C^{-1}$. The field $\psi^{j_1,j_2}(x)$ is now constructed in the usual way[12]:

$$\psi^{j_1j_2}(x)=\frac{1}{(2\pi)^{3/2}}\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}\int\left(\frac{M_{j_1j_2}}{f_0}\right)^{1/2}\left[U^{j_1m_1,j_2m_2}a^{m_1m_2}_{j_1j_2}(f)e^{if\cdot x}+V^{j_1m_1,j_2m_2}b^{\dagger m_1m_2}_{j_1j_2}e^{-if\cdot x}\right]d^3f\,. \tag{1.10}$$

Let us now consider the scattering of two spinless bosons (e.g., two pions) via a direct-channel pole (diagram 1a) in which the intermediate particle is represented by $\psi^{j,j}(x)$ field. Let $p,k$ be the momenta of the ingoing particles and $p',q$ the momenta of the outgoing particles. For the present we shall suppose that all the external particles have the same mass $\mu$. An effective Lagrangian which will contribute to the process shown in Fig. 1(a) is

$$\mathcal{L}^{(n)}(x)=g_n^{(1)}(C^{-1}\gamma_{\mu_1})_{\alpha_1\beta_1}(C^{-1}\gamma_{\mu_2})_{\alpha_2\beta_2}\cdots(C^{-1}\gamma_{\mu_n})_{\alpha_n\beta_n}\psi^{jj}_{\alpha_1\alpha_2\cdots\alpha_n,\beta_1\beta_2\cdots\beta_n}(x)\varphi(x)\overleftrightarrow{\partial}_{\mu_1}\overleftrightarrow{\partial}_{\mu_2}\cdots\overleftrightarrow{\partial}_{\mu_n}\varphi(x)+\text{H.c.}\,, \tag{1.11a}$$

where

$$n=2j\,,\ \text{an integer}\geq 1\,. \tag{1.11b}$$

Let us examine the space reflection invariance of this Lagrangian. Under the parity operation the pion field $\varphi(x)$ transforms as

$$\mathcal{P}^{-1}\varphi(x)\mathcal{P}=-\varphi(x')\,, \tag{1.12a}$$

where

$$\vec{\mathrm{x}}'=-\vec{\mathrm{x}}\,,\qquad x_4'=x_4\,. \tag{1.12b}$$

Also $\mathcal{P}$ transforms $\psi^{jj}(x)$ in the following way:

$$\mathcal{P}^{-1}\psi^{j,j}_{\alpha_1\cdots\alpha_n,\beta_1\cdots\beta_n}\mathcal{P}=P_n(\gamma_4)_{\alpha_1\alpha_1'}\cdots(\gamma_\mu)_{\alpha_n\alpha_n'}(\gamma_4)_{\beta_1\beta_1'}\cdots(\gamma_4)_{\beta_n\beta_n'}\times\psi^{jj}_{\alpha_1'\cdots\alpha_n',\beta_1'\cdots\beta_n'}\,; \tag{1.13}$$

$P_n=P_{jj}=\pm 1$ is the intrinsic parity of the boson field $\psi^{jj}$. In calculating $\mathcal{P}^{-1}\mathcal{L}^{(n)}(x)\mathcal{P}$ we notice that each $C^{-1}\gamma_\mu(\partial/\partial x_\mu)$ will combine with the two $\gamma_4$ matrices, giving

$$\gamma_4^T C^{-1}\gamma_\mu\frac{\partial}{\partial x_\mu}\gamma_4=-C^{-1}\gamma_\mu\frac{\partial}{\partial x_\mu'}\,. \tag{1.14}$$

Since there are $n$ such factors and there are two pion fields,

$$\mathcal{P}^{-1}\mathcal{L}^{(n)}(x)\mathcal{P}=P_n(-1)^n\mathcal{L}^{(n)}(x')\,. \tag{1.15}$$

Hence, for $n$ even and $P_n=+1$ the Lagrangian (1.11) is parity invariant. For $n$ even and $P_n=-1$, $\mathcal{L}^{(n)}(x)$ is not parity invariant, and as we are assuming parity invariance $\mathcal{L}^{(n)}(x)$ for $P_n=-1$, $n$ even cannot be used. In this case we introduce a $\gamma_5$ with any one of the $(C^{-1}\gamma_{\mu_i})_{\alpha_i\beta_i}$ occurring in (1.11) and call the Lagrangian $\mathcal{L}_5^{(n)}(x)$. The symmetry of $\psi^{jj}$ in the $\alpha$'s and the $\beta$'s separately allows us to replace $C^{-1}\gamma_{\mu_1}$ by $C^{-1}\gamma_5\gamma_{\mu_1}$ to obtain $\mathcal{L}_5^{(n)}(x)$. Then

$$\mathcal{P}^{-1}\mathcal{L}_5^{(n)}(x)\mathcal{P} = P_n(-1)^{n-1}\mathcal{L}_5^{(n)}(x') \tag{1.16a}$$

$$= \mathcal{L}_5^{(n)}(x') \text{ for } n \text{ even, } P_n = -1. \tag{1.16b}$$

For $n$ odd and $P_n = -1$, Eq. (1.15) shows that $\mathcal{L}^{(n)}(x)$ is parity invariant. For $n$ odd and $P_n = -1$, (1.15) shows that $\mathcal{L}^{(n)}(x)$ is invariant. For $n$ odd and $P_n = +1$, (1.16a) shows that $\mathcal{L}_5^{(n)}(x)$ is invariant.

The masses of the intermediate particles denoted by $M_n^{(+)}$ and $M_n^{(-)}$ corresponding to the two intrinsic parity values $P_n = +1$ and $P_n = -1$, respectively, will, in general, be different. The coupling constants of the four parity-invariant Lagrangians will also, in general, be different. In Table I the coupling constants and the masses are given for the four Lagrangians.

TABLE I. Lagrangians for the pole diagram in Fig. 1(a).

| No. | Lagrangian | Intrinsic parity $P_n$ | Mass | Coupling constant |
|---|---|---|---|---|
| 1 | $\mathcal{L}^{(n,+)}(x)$, $n$ even | +1 | $M_n^{(+)}$ | $g_n^{(1)}$ |
| 2 | $\mathcal{L}_5^{(n,-)}(x)$, $n$ even | −1 | $M_n^{(-)}$ | $g_n^{(2)}$ |
| 3 | $\mathcal{L}^{(n,-)}(x)$, $n$ odd | −1 | $M_n^{(-)}$ | $g_n^{(3)}$ |
| 4 | $\mathcal{L}_5^{(n,+)}(x)$, $n$ odd | +1 | $M_n^{(+)}$ | $g_n^{(4)}$ |

We now calculate the $t$-channel pole diagram 1a via the Lagrangians $\mathcal{L}^{(n)}(x)$ and $\mathcal{L}_5^{(n)}(x)$. The correct masses and coupling constants will be put in later on. The $T$-matrix element for the pole diagram 1a is given by[10, 13]

$$T_A^{(n)} = \frac{g_n^2}{t+M_n^2}(C^{-1}\Gamma_\mu P'_\mu)_{\alpha_1\beta_1}(C^{-1}\gamma\cdot P')_{\alpha_2\beta_2}\cdots(C^{-1}\gamma\cdot P')_{\alpha_n\beta_n}$$
$$\times\sum_{m_1, m_2=-j}^{j} U^{jm_1, jm_2}_{\alpha_1\cdots\alpha_n,\beta_1\cdots\beta_n}(f)\,\bar{U}^{jm_1, jm_2}_{\alpha'_1\cdots\alpha'_n,\beta'_1\cdots\beta'_n}(f)(\Gamma_\nu P_\nu C)_{\alpha'_1\beta'_1}(\gamma\cdot PC)_{\alpha'_2\beta'_2}\cdots(\gamma\cdot PC)_{\alpha'_n\beta'_n}. \tag{1.17}$$

$\Gamma_\mu = \gamma_\mu$ for $\mathcal{L}^{(n)}(x)$ and $\gamma_5\gamma_\mu$ for $\mathcal{L}_5^{(n)}(x)$, respectively, and

$$P' = p' - q,$$
$$P = p - k, \tag{1.18}$$
$$t = -(p+k)^2.$$

The matrix element will be evaluated in the center-of-mass frame. The vector $\vec{p}$ is along the $x_3$ axis and the scattering is supposed to take place in the $x_1x_3$ plane with $\vec{p}'$ making an angle $\theta$ with the $x_3$ axis. We now take the Lorentz transformations $L(f)$ out of $U^{jm_1, jm_2}(f)$ in (1.17) and combine them with the $C^{-1}\gamma\cdot P'$ factors contracted with the indices of $U^{jm_1, jm_2}$. $(C^{-1}\gamma\cdot P')$ will combine with $L_{\alpha_i\lambda_i}(f)\,L_{\beta_i\tau_i}(f)$ giving[10]

$$[L^T(f)C^{-1}\gamma\cdot P' L(f)]_{\lambda_i\tau_i} = (C^{-1}\gamma\cdot\tilde{P}')_{\lambda_i\tau_i} \tag{1.19a}$$
$$= (C^{-1}\vec{\gamma}\cdot\vec{\tilde{P}}' + C^{-1}\gamma_4\tilde{P}')_{\lambda_i\tau_i} \tag{1.19b}$$
$$= (i\gamma_4 i\sigma_2\vec{\sigma}\cdot\vec{\tilde{P}}' + \gamma_5 i\sigma_2\gamma_4\tilde{P}'_4)_{\lambda_i\tau_i}. \tag{1.19c}$$

$\tilde{P}'$ in the above equation is the Lorentz-transformed vector

$$\tilde{P}'_\nu = P'_\mu a_{\mu\nu}(f). \tag{1.20}$$

In the metric we are using, the Lorentz boosts $a_{\mu\nu}(f)$ are given in Ref. 10. The second term in (1.19c) when contracted between the positive energy rest spinors in $U^{jm_1, jm_2}(0)$ vanishes and $\gamma_4$ in the first term gets absorbed in the rest spinors. Now, writing the Dirac matrix $i\sigma_2$ as $c^{-1}$, we can replace $i\gamma_4 i\sigma_2\vec{\sigma}\cdot\vec{\tilde{P}}'$ by $c^{-1}\vec{\sigma}\cdot\vec{\tilde{P}}'$. Further, $\gamma\cdot PC$ factors contracted with $\bar{U}^{jm_1, jm_2}(f)$ in (1.17) may also be treated in a similar fashion and then the matrix element $T_A^{(n)}$ is written in the form

$$T_A^{(n)} = \frac{g_n^2}{t+M_n^2}\sum_{m_1 m_2}\{[(c^{-1}\vec{\sigma}\cdot\vec{\tilde{P}}')_{\lambda_1\tau_1}(c^{-1}\vec{\sigma}\cdot\vec{\tilde{P}}')_{\lambda_2\tau_2}\cdots(c^{-1}\vec{\sigma}\cdot\vec{\tilde{P}}')_{\lambda_n\tau_n}U^{jm_1, jm_2}_{\lambda_1\cdots\lambda_n,\tau_1\cdots\tau_n}(0)]$$
$$\times[U^{jm_1, jm_2}_{\lambda'_1\cdots\lambda'_n,\tau'_1\cdots\tau'_n}(0)(\vec{\sigma}\cdot\vec{\tilde{P}}c)_{\lambda'_1\tau'_1}(\vec{\sigma}\cdot\vec{\tilde{P}}c)_{\lambda'_2\tau'_2}\cdots(\vec{\sigma}\cdot\vec{\tilde{P}}c)_{\lambda'_n\tau'_n}]\}, \tag{1.21}$$

The factor within the first square brackets in (1.21) is denoted by $X^{jm_1,jm_2}$ and, using (1.2a), can be written in the Kronecker product form as

$$X^{jm_1,jm_2}=U^{Tjm_1}c^{-1}\vec{\sigma}\cdot\vec{\tilde{P}}'\times c^{-1}\vec{\sigma}\cdot\vec{\tilde{P}}'\times\cdots\times c^{-1}\vec{\sigma}\cdot\vec{\tilde{P}}'U^{jm_2}. \tag{1.22}$$

$\vec{\tilde{P}}'$ also lies in the $x_1x_3$ plane and makes an angle $\varphi'$ with the $x_3$ axis.[10] Using the rotation matrix

$$R(\varphi')=e^{-(i/2)\sigma_2\varphi'}, \tag{1.23a}$$

we have

$$c^{-1}\vec{\sigma}\cdot\vec{\tilde{P}}'=c^{-1}R(\varphi')\sigma_3R^{\dagger}(\varphi')\tilde{P}'. \tag{1.23b}$$

Then, using the properties of the Kronecker products, (1.22) can be recast in the form

$$X^{jm_1,jm_2}=(\tilde{P}')^nU^{Tjm_1}c^{-1}\times c^{-1}\times\cdots\times c^{-1}\cdot R(\varphi')\times R(\varphi')\times\cdots\times R(\varphi')$$
$$\times\sigma_3\times\sigma_3\times\cdots\times\sigma_3\cdot R^{\dagger}(\varphi')\times R^{\dagger}(\varphi')\times\cdots R^{\dagger}(\varphi')\cdot U^{jm_2}. \tag{1.24}$$

$U^{Tjm_1}$ is obtained by replacing $u^1$ and $u^2$ in (1.2b) by their transposes $u^{1T}$ and $u^{2T}$, respectively. The matrix $c^{-1}=i\sigma_2$ acting from the right on $u^{1T}$ and $u^{2T}$ not only changes them into $u^{2T}$ and $-u^{1T}$, respectively, but also alters their transformation properties under rotations, i.e., $u^Tc^{-1}$ transforms as $\bar{u}=u^{*T}$. We write,

$$u^{1T}c^{-1}=\bar{u}^2,\qquad u^{2T}c^{-1}=-\bar{u}^1, \tag{1.25}$$

and, as $U^{Tjm_1}$ contains $j-m_1$ factors $u^{2T}$ which get changed into $-\bar{u}^1$,

$$U^{Tjm_1}c^{-1}\times c^{-1}\times\cdots\times c^{-1}=(-1)^{j-m_1}\bar{U}^{j,-m_1}. \tag{1.26}$$

Now, using the rotation properties of $U^{jm_2}$ and $U^{j,-m_1}$ we obtain

$$X^{jm_1,jm_2}=\sum_{\sigma_1\sigma_2}(\tilde{P}')^n(-1)^{j-m_1}d^j_{-m_1,\sigma_1}(\varphi')\,d^j_{m_2,\sigma_2}(\varphi')\times\bar{U}^{j\sigma_1}\sigma_3\times\sigma_3\times\cdots\times\sigma_3U^{j\sigma_2}. \tag{1.27}$$

Equation (1.2b) shows that the Kronecker product of $\sigma_3$ matrices in (1.27) acting on $U^{j\sigma_1}$ will give $U^{j\sigma_1}$ multiplied by $(-1)^{j-\sigma_1}$, as there are $j-\sigma_1$ factors of $u^2$ in $U^{j\sigma_1}$. Hence,

$$X^{jm_1,jm_2}=\sum_{\sigma_1\sigma_2}(\tilde{P}')^n(-1)^{j-m_1+j-\sigma_1}\times d^j_{-m_1,\sigma_1}(\varphi')\,d^j_{m_2,\sigma_2}(\varphi')\delta_{\sigma_1\sigma_2}. \tag{1.28}$$

Using the property (A11b) of the O(3) rotation matrices we get

$$X^{jm_1,jm_2}=\sum_{\sigma_1}(\tilde{P}')^n(-1)^{2j}\,d^j_{m_1,-\sigma_1}(\varphi')\,d^j_{m_2\sigma_1}(\varphi'). \tag{1.29}$$

The two rotation matrices can be combined into one by using the well-known formula (given in Appendix A). This gives

$$X^{jm_1,jm_2}=\sum_{\sigma_1=-j}^{j}\sum_{l}(\tilde{P}')^n(-1)^{2j}C^{j\,j\,l}_{m_1\,m_2\,m}C^{j\,j\,l}_{-\sigma_1\,\sigma_1\,0}\,d^l_{m,0}(\varphi'). \tag{1.30}$$

Similarly, we can show that the expression within the second square bracket in (1.21) (denoted by $Y^{jm_1,jm_2}$) can be reduced to a similar form,

$$Y^{jm_1,jm_2}=\sum_{\sigma_2=-j}^{j}\sum_{l}C^{j\,j\,l'}_{m_1\,m_2\,m}C^{j\,j\,l'}_{-\sigma_2\,\sigma_2\,0}\,d^{l'}_{m,0}(\varphi'), \tag{1.31}$$

where $\varphi$ is the angle made by $\vec{\tilde{P}}$ with the $x_3$ axis, $\tilde{P}$ being the Lorentz transformed vector

$$\tilde{P}_\nu=P_\mu a_{\mu\nu}(f). \tag{1.32}$$

Substituting these expressions for $X^{jm_1,jm_2}$ and $Y^{jm_1,jm_2}$ in (1.21) we obtain

$$T_A^{(n)}=\frac{g_n^2\tilde{P}'^n\tilde{P}^n}{t+M_n^2}\sum_{m_1,m_2}\sum_{\sigma_1,l}\sum_{\sigma_2,l}C^{j\,j\,l}_{m_1\,m_2\,m}C^{j\,j\,l'}_{m_1\,m_2\,m}C^{j\,j\,l}_{\sigma_1\,-\sigma_1\,0}C^{j\,j\,l'}_{\sigma_2\,-\sigma_2\,0}\,d^l_{m,0}(\varphi')\,d^{l'}_{m,0}(\varphi). \tag{1.33}$$

The summations over $m_1$ and $m_2$ can be performed using the well-known orthogonality properties of the Clebsch-Gordan coefficients, and then, using the result

$$d^l_{m,0}(\varphi)=d^l_{0,m}(-\varphi), \tag{1.34}$$

we obtain a rather compact form for $T_A^{(n)}$:

$$T_A^{(n)}=\frac{g_n^2\tilde{P}'^n\tilde{P}^n}{t+M_n^2}\sum_{l}\sum_{\sigma_1\sigma_2}C^{j\,j\,l}_{\sigma_1\,-\sigma_1\,0}C^{j\,j\,l}_{\sigma_2\,-\sigma_2\,0}\,d^l_{00}(\tilde{\theta}), \tag{1.35}$$

where

$$\tilde{\theta}=\varphi'-\varphi \tag{1.36}$$

is the angle between $\vec{\tilde{P}}'$ and $\vec{\tilde{P}}$. Now, as mentioned in a previous work,[10] the magnitudes $\tilde{P}'$, $\tilde{P}$ and the angle $\tilde{\theta}$ are given by

$$\vec{\tilde{P}}^2 = P_\mu P_\nu\left(\delta_{\mu\nu} + \frac{f_\mu f_\nu}{M_n^2}\right), \tag{1.37a}$$

$$\vec{\tilde{P}}'^2 = P'_\mu P'_\nu\left(\delta_{\mu\nu} + \frac{f_\mu f_\nu}{M_n^2}\right), \tag{1.37b}$$

$$\tilde{P}\tilde{P}'\cos\tilde{\theta} = P_\mu P'_\nu\left(\delta_{\mu\nu} + \frac{f_\mu f_\nu}{M_n^2}\right). \tag{1.37c}$$

In the center-of-mass frame $\vec{f} = \vec{p} + \vec{k} = 0$ and for the equal masses $P_0 = P'_0 = (p_0 - k_0) = 0$. It follows then from (1.37) that

$$\tilde{P} = \tilde{P}' = 2p, \tag{1.38}$$

$$\cos\tilde{\theta} = \cos\theta. \tag{1.39}$$

In the case of elastic scattering and unequal masses of the ingoing particles, we remove the arrows over the derivatives in the Lagrangian (1.11a) and also replace $M_n^2$ in Eqs. (1.37a)–(1.37c) by $-f_\mu f_\mu$. $(\delta_{\mu\nu} + f_\mu f_\nu / M_n^2)$ is the numerator of the spin-one propagator, and the replacement $M_n^2 \to -f_\mu f_\mu$ means that we are now using the Landau gauge[14] which removes the spin-zero part of the propagator. $P'$ and $P$ will now be given by $P' = p'$, $P = p$, and

$$\tilde{P}' = \tilde{P} = p, \quad \cos\tilde{\theta} = \cos\theta. \tag{1.40}$$

When we use the Lagrangian $\mathfrak{L}_5^{(n)}(x)$, $\Gamma_\mu$ in (1.17) is $\gamma_5\gamma_\mu$ and then, as far as this factor is concerned, it gives an additional $\gamma_5$ matrix in each term of Eqs. (1.19). Then, the first term on the right-hand side of (1.19c) involving $\sigma \cdot \tilde{P}'$ gives a vanishing contribution and the contribution of the second term will be just $c^{-1}P'_4$. In the equation corresponding to (1.22), the first factor in the Kronecker product will not be $c^{-1}\vec{\sigma}\cdot\vec{\tilde{P}}'$ but only $c^{-1}P'_4$, the other factors will remain the same, and if $X_5^{jm_1, jm_2}$ corresponds to $X^{jm_1, jm_2}$, then instead of (1.24) we will have

$$X_5^{j_1 m_1, j_2 m_2} = (\tilde{P}')^{n-1}\tilde{P}'_4 U^{Tjm_1} c^{-1}\times c^{-1}\times\cdots\times c^{-1} \times R(\varphi')\times R(\varphi')\times\cdots\times R(\varphi')\times 1\times\sigma_3\times\sigma_3\times\cdots \times\sigma_3\cdot R^\dagger(\varphi')\times R^\dagger(\varphi')\times\cdots\times R^\dagger(\varphi')U^{jm_2}. \tag{1.41}$$

Treating this equation in the same way as (1.24) we obtain

$$X_5^{jm_1, jm_2} = \sum_{\sigma_1\sigma_2} (\tilde{P}')^{n-1}P'_4 d^j_{-m_1, \sigma_1}(\varphi') d^j_{m_2\sigma_2}(\varphi')(-1)^{j-m_1} \times[\bar{U}^{j\sigma_1} 1\times\sigma_3\times\sigma_3\times\cdots\times\sigma_3 U^{j\sigma_2}]. \tag{1.42}$$

To proceed further it becomes necessary to break up $U^{j\sigma_2}$ in the following way. From (1.2b) it follows that

$$U^{j\sigma_2} = \frac{1}{({}^{2j}C_{j-\sigma_2})^{1/2}}\left(u^1\times\sum_P u^1\times u^2\times\cdots\times u^1 + u^2\times\sum_P u^1\times u^2\times\cdots\times u^1\right). \tag{1.43}$$

The first permutation sum contains ${}^{2j-1}C_{j-\sigma_2}$ terms and, comparing with (1.2b) and (1.4), it is easily seen that it is the normalized spinor $U^{j-\frac{1}{2}, \sigma_2-\frac{1}{2}}$ multiplied by $({}^{2j-1}C_{j-\sigma_2})^{1/2}$. Similar considerations show that the second permutation in (1.43) is $U^{j-\frac{1}{2}, \sigma_2+\frac{1}{2}}({}^{2j-1}C_{j+\sigma_2})^{1/2}$. Simplifying these combination factors we find that

$$U^{j\sigma_2} = u^1\times U^{j-\frac{1}{2}, \sigma_2-\frac{1}{2}}\left(\frac{j+\sigma_2}{2j}\right)^{1/2} + u^2\times U^{j-\frac{1}{2}, \sigma_2+\frac{1}{2}}\left(\frac{j-\sigma_2}{2j}\right)^{1/2}. \tag{1.44}$$

Now $1\times\sigma_3\times\sigma_3\times\cdots\times\sigma_3$ operating on $u'\times U^{j-\frac{1}{2}, \sigma_2-\frac{1}{2}}$ gives $u'\times U^{j-\frac{1}{2}, \sigma_2-\frac{1}{2}}(-1)^{j-\sigma_2}$, and operating on $u^2\times U^{j-\frac{1}{2}, \sigma_2+\frac{1}{2}}$ gives $u^2\times U^{j-\frac{1}{2}, \sigma_2+\frac{1}{2}}\times(-1)^{j-\sigma_2-1}$. The multispinor $U^{j\sigma_1}$ may also be broken in the same form as (1.44) and the quantity within the square brackets in (1.42) is easily calculated, resulting in

$$X_5^{jm_1, jm_2} = \sum_{\sigma_1} \frac{(\tilde{P}')^{n-1}\tilde{P}'_4}{j}\sigma_1 d^j_{-m_1, \sigma_1}(\varphi') d^j_{m_2, \sigma_1}(\varphi')(-1)^{2j-m_1-\sigma_1} \tag{1.45a}$$

$$= (-1)^{2j}\frac{(\tilde{P}')^{n-1}P'_4}{j}\sum_l\sum_{\sigma_1}[C^{j\,j\,l}_{m_1 m_2 m} C^{j\,j\,l}_{-\sigma_1\sigma_1 0}\sigma_1 d^l_{m,0}(\varphi')] \tag{1.45b}$$

as before. The factor corresponding to $Y^{jm_1, jm_2}$ is denoted by $Y_5^{jm_1, jm_2}$ and is calculated in the same way as $X_5^{jm_1, jm_2}$. The result is

$$Y_5^{jm_1, jm_2} = (-1)^{2j}\frac{\tilde{P}^{n-1}\tilde{P}_4}{j}\sum_{l'\sigma_2} C^{j\,j\,l'}_{m_1 m_2 m} C^{j\,j\,l'}_{\sigma_2 -\sigma_2 0}\sigma_2 d^{l'}_{m,0}(\varphi). \tag{1.46}$$

The contribution of $\mathfrak{L}_5^{(n)}(x)$ to the diagram (1a) can now easily be calculated. The result is

$$T_{A\cdot 5}^{(n)}=\sum_{m_1}\sum_{m_2} X_5^{jm_1,jm_2} Y_5^{jm_1,jm_2}\frac{g_n'^2}{t+M_n'^2}$$
$$=\frac{1}{j^2}\frac{g_n^2}{t+M_n'^2}(\tilde{P}\tilde{P}')^{n-1}\tilde{P}_0'\tilde{P}_0\sum_l\sum_{\sigma_1\sigma_2}C^{j\,j\,l}_{\sigma_1\,-\sigma_1\,0}C^{j\,j\,l}_{\sigma_2\,-\sigma_2\,0}\,\sigma_1\sigma_2\,d^l_{00}(\tilde{\theta})\,. \tag{1.47}$$

Now since we take $\tilde{P}'=\tilde{p}'=p$, $\tilde{P}=\tilde{p}=p$, it is easy to find $\tilde{P}_0=\tilde{p}_0$ and $\tilde{P}_0'=\tilde{p}_0'$ in terms of $p$ and the mass $\mu$ of the particle to which $p_\mu'$ and $p_\mu$ correspond. From Lorentz invariance

$$\tilde{p}_\nu\tilde{p}_\nu=p_\nu p_\nu=-\mu^2\,,$$
$$\tilde{p}_\nu'\tilde{p}_\nu'=p_\nu' p_\nu'=-\mu^2\,. \tag{1.48a}$$

Hence,

$$\tilde{P}_0=\tilde{p}_0=(p^2+\mu^2)^{1/2}=p_0\,,$$
$$\tilde{P}'=\tilde{p}_0'=(p^2+\mu^2)^{1/2}=p_0\,. \tag{1.48b}$$

In Appendix A summations over $l$, $\sigma_1$, and $\sigma_2$ occurring in (1.36) and (1.47) have been evaluated in terms of the rotation matrices of O(4). Using these results we are now able to write down the contributions to the pole diagram 1(a) by the four Lagrangians given in Table I,

$$T_A^{(n)}(n\text{ even})=\frac{g_n^{(1)2}p^{2n}}{t+(M_n^+)^2}d^{n/2,n/2}_{00;00}(\theta)+\frac{g_n^{(2)2}}{t+(M_n^-)^2}p^{2(n-1)}p_0^{\,2}\,\frac{n}{6}\left(\frac{n}{2}+1\right)(n+1)d^{n/2,n/2}_{10;10}(\theta)\,, \tag{1.49a}$$

$$T_A^{(n)}(n\text{ odd})=\frac{g_n^{(3)2}}{t+(M_n^-)^2}p^{2n}d^{n/2,n/2}_{00;00}+\frac{(g_n^{(4)})^2}{t+(M_n^+)^2}p^{2(n-1)}p_0^{\,2}\,\frac{n}{6}\left(\frac{n}{2}+1\right)(n+1)d^{n/2,n/2}_{10;10}(\theta)\,. \tag{1.49b}$$

The two O(4) rotation matrices occurring in the above equation have been shown to be connected with the Gegenbauer polynomial $C_n^{(1)}(\cos\theta)$ by the following equations:

$$d^{n/2,n/2}_{00,00}(\theta)=C_n^{(1)}(\cos\theta)\,, \tag{1.50a}$$

$$\frac{n}{6}\left(\frac{n}{2}+1\right)(n+1)d^{n/2,n/2}_{10;10}(\theta)=\tfrac{1}{4}[n(n+1)-L^2(\partial)]C^{(1)}_{n-1}(\theta)\,, \tag{1.50b}$$

where $L^2(\partial)$ is the Legendre operator with the property $L^2(\partial)P_l(\cos\theta)=l(l+1)P_l(\cos)$. Now consider the O(4, 2) weight diagram[8] (shown in Fig. 2), particularly the O(4) multiplet in it. The "principal quantum number" is $n'=n+1=2j+1$. Any state belonging to the $D^{j,j}\equiv D^{2j,0}\equiv D^{n'-1,0}$ representation[1,6] of O(4) can be expanded in the form

$$|jm_1,jm_2\rangle=\sum_{l=0}^{n'-1}C^{j\,j\,l}_{m_1\,m_2\,m}|jjlm\rangle\,, \tag{1.51}$$

where $l(l+1)$ is the eigenvalue of $\vec{L}^2$, $\vec{L}$ being the O(3) angular momentum operator given in terms of $\vec{j}^{(1)}$ and $\vec{J}^{(2)}$ (mentioned earlier) by $\vec{L}=\vec{j}^{(1)}\times 1+1\times\vec{j}^{(2)}$. As shown in Fig. 2 and by (1.17), any state characterized by $D^{j,j}\equiv D^{n'-1,0}$ contains angular momentum states $l=0,1,2\cdots,n'-1=n$. For even $n=2j$ the expansion of the Gegenbauer polynomial $C_n^{(1)}(\cos\theta)$ in terms of the Legendre polynomials $P_l(\cos\theta)$ contains[4] only the even values of $l$, i.e., $l=0,2,4,\ldots,n-2$, $n=n'-1$, and $C^{(1)}_{n-1}$ contains only the odd $l$ values, i.e., $l=1,3,\ldots,n-3$, $n-1=n'-2$. Hence, the $(g_n^{(1)})^2$ square term in (1.49a) gives the contribution of all the even angular momentum states and the $g_n^{(2)}$ term in (1.49a) gives the contribution of all the odd angular momentum states. Therefore, for even $n$ values the introduction of the Lagrangian $\mathcal{L}_5^{(n,-)}(x)$ is necessary for obtaining the contributions to odd angular momentum states. Similarly, the Lagrangian $\mathcal{L}_5^{(n,+)}(x)$ is necessary to obtain nonvanishing contributions to even angular momentum state. Hence, if we sum up over all $n$, $\sum_n T_A^{(n)}$ given by Eqs. (1.49) will contain the contribution from all the angular momentum states of all the O(4) multiplets shown in Fig. 2. In a future paper we shall consider the Reggeization of the amplitude $T_A=\sum_n T_A^{(n)}$ in the principal quantum number $n'=n+1$.

## II. PION-NUCLEON SCATTERING VIA $D^{j_2+1/2,j_2}$ BARYON POLES

For the pion-nucleon scattering diagram given in Fig. 1(b) we write an effective Lagrangian

$$\mathcal{L}^{j_1,j_2}(x)=g_{j_1j_2}\overline{\psi}_\alpha(x)(c^{-1}\gamma_{\mu_1})_{\alpha_1\beta_1}\cdots(c^{-1}\gamma_{\mu_s})_{\alpha_s\beta_s}\times\psi^{j_1j_2}_{\alpha\alpha_1\alpha_2\cdots\alpha_s,\beta_1\beta_2\cdots\beta_s}(x)\partial_{\mu_1}\cdots\partial_{\mu_s}\varphi(x)+\text{H.c.}\,, \tag{2.1a}$$

where the integer $s$ is related to $j_1$ and $j_2$ by

$$j_2=\tfrac{1}{2}s\ , \tag{2.1b}$$

$$j_1=j_2+\tfrac{1}{2}=\tfrac{1}{2}s+\tfrac{1}{2}\ , \tag{2.1c}$$

and $\psi_\alpha(x)$, $\varphi(x)$ stand for the nucleon and pion fields, respectively. If $P_{j_1j_2}$ is the parity of the baryon field $\psi^{j_1j_2}$ relative to the nucleon field $\psi_\alpha(x)$, then it can be shown as before that

$$\mathcal{P}^{-1}\mathcal{L}^{j_1j_2}(x)\mathcal{P}=-P_{j_1j_2}(-1)^{2j_2}\mathcal{L}^{j_1j_2}(x'). \tag{2.2}$$

For $s=2j_2$ odd and $P_{j_1j_2}=+1$, $\mathcal{L}^{j_1j_2}(x)$ is parity invariant, and for $s$ odd $P_{j_1j_2}=-1$; the Lagrangian $\mathcal{L}_5^{j_1j_2}(x)$ obtained by replacing $\bar\psi(x)$ in (2.1a) by $\bar\psi(x)\gamma_5$ will be parity invariant. Similar considerations hold for the case when $s$ is even. The coupling constants, parities, and the masses of the $D^{j_2+1/2,j_2}$ baryons are given in Table II.

TABLE II. Lagrangians for the pole diagram in Fig. 1(b).

| No. | Lagrangian | Parity $P_{j_1j_2}$ | Mass | Coupling constant |
|---|---|---|---|---|
| 1 | $\mathcal{L}^{j_1j_2}(x)s=2j_2$ odd | +1 | $M_s^{(+)}$ | $g_s^{(1)}$ |
| 2 | $\mathcal{L}_5^{j_1j_2}(x)s=2j$ odd | −1 | $M_s^{(-)}$ | $g_s^{(2)}$ |
| 3 | $\mathcal{L}^{j_1j_2}(x)s=2j$ even | −1 | $M_s^{(-)}$ | $g_s^{(3)}$ |
| 4 | $\mathcal{L}_5^{j_1j_2}(x)s=2j$ even | +1 | $M_s^{(+)}$ | $g_s^{(4)}$ |

Let us first calculate diagram 1b via the Lagrangian $\mathcal{L}^{j_1j_2}(x)$. As for the scalar boson case, the matrix element $T^{j_1j_2}$ is given by[10]

$$T_B^{j_1j_2}=\sum_{m_1=-j_1}^{j_1}\sum_{m_2=-j_2}^{j_2}[\bar u_\alpha(p')(c^{-1}\gamma\cdot q)_{\alpha_1\beta_1}\cdots(c^{-1}\gamma\cdot q)_{\alpha_s\beta_s}U^{j_1m_1,j_2m_2}_{\alpha\alpha_1\cdots\alpha_s,\beta_1\cdots\beta_s}(f)]$$
$$\times[\bar U^{j_1m_1,j_2m_2}_{\alpha'\alpha_1'\cdots\alpha_s',\beta_1'\cdots\beta_s'}(f)(\gamma\cdot kC)_{\alpha_1'\beta_1'}\cdots(\gamma\cdot kC)_{\alpha_s'\beta_s'}u_{\alpha'}(p)]\,\frac{g^2_{j_1j_2}}{t+M_s^2}\,. \tag{2.3}$$

$p$ and $p'$ are the momenta of the initial and final nucleons and $k,q$ are those of the initial and final pions, respectively. As before, we shall calculate $T^{j_1j_2}$ by taking the Lorentz boosts $L(f)$ out of the multispinors and combining them with $C^{-1}\gamma\cdot q$ and $\gamma\cdot kC$ factors. Only one $L(f)$ from $U(f)$ and one from $\bar U(f)$ will not be combined with these but with $\bar u(p')$ and $u(p)$. The expression within the first square brackets in (2.3) denoted by $X^{j_1m_1,j_2m_2}$ is easily seen to reduce to the Kronecker product form,

$$X^{j_1m_1,j_2m}=q^s[\bar u(p')L(f)]_\lambda U^{Tj_1m_1}_{\lambda\cdots}c^{-1}\times c^{-1}\times\cdots\times c^{-1}$$
$$\times R(\varphi')\times R(\varphi')\times\cdots\times R(\varphi')\cdot\sigma_3\times\sigma_3\times\cdots\times\sigma_3\cdot R^\dagger(\varphi')R^\dagger(\varphi')\times\cdots\times R^\dagger(\varphi')U^{j_2m_2}. \tag{2.4}$$

The dots in $U^{Tj_1m_1}$ above stand for the indices on which the $s$-fold Kronecker products of the matrices operate. Now, writing

$$U^{Tj_1m_1}_{\lambda\cdots}\equiv U^{Tj_1m_1}_{K\cdots}[c^{-1}R(\varphi')]_{K\theta}[R^\dagger(\varphi')c]_{\theta\lambda}, \tag{2.5}$$

we have

$$U^{Tj_1m_1}_{\lambda\cdots}c^{-1}\times c^{-1}\times\cdots\times c^{-1}R(\varphi')\times R(\varphi')\times\cdots\times R(\varphi')$$
$$=(R^\dagger(\varphi')c)_{\theta\lambda}U^{Tj_1m_1}_{K\cdots}[c^{-1}\times c^{-1}\times\cdots\times c^{-1}R(\varphi')\times R(\varphi')\times\cdots\times R(\varphi')]_{K\cdots,\theta\cdots} \tag{2.6}$$
$$=(R^\dagger(\varphi')c)_{\theta\lambda}(-1)^{j_1-m_1}\sum_{\sigma_1}d^{j_1}_{-m,\sigma_1}(\varphi')\bar U^{j_1\sigma_1}_{\theta\cdots}\,. \tag{2.7}$$

$U^{j_1\sigma_1}$ is now broken up according to (1.44); then, substituting (2.7) in (2.4) and proceeding as before, we obtain after some calculation

$$X^{j_1m_1,j_2m_2}=\sum_{\sigma_1}q^s\bar u(p')L(f)[R^\dagger(\varphi')c]^T\bar u'^T\left(\frac{j_1+\sigma_1}{2j_1}\right)^{1/2}(-1)^{2j_2+1}d^{j_1}_{m_1,-\sigma_1}(\varphi')d^{j_2}_{m_2,\sigma_1-1/2}(\varphi')$$
$$+\sum_{\sigma_1}q^s\bar u(p')L(f)[R^\dagger(\varphi')c]^T\bar u^{2T}\left(\frac{j_1-\sigma_1}{2j_1}\right)^{1/2}(-1)^{2j_2}d^{j_1}_{m_1,-\sigma_1}(\varphi')d^{j_2}_{m_2,\sigma_1+1/2}(\varphi')\,. \tag{2.8}$$

We notice further that

$$[R^\dagger(\varphi')c]^T\bar u'^T=-R(\varphi')u^2, \tag{2.9a}$$

$$[R^\dagger(\varphi')c]^T\bar u^{2T}=R(\varphi')u^1 \tag{2.9b}$$

and combine the two $d^j$ matrices after changing $\sigma_1$ to $-\sigma_1$ in the summation over $\sigma_1$, obtaining

$$X^{j_1 m_1, j_2 m_2} = \frac{(-1)^{2j_2} q^s}{(2j_1)^{1/2}} \Big[ \bar{u}(p')L(f)R(\varphi')u^2 \sum_l \sum_{\sigma_1} (j_1 - \sigma_1)^{1/2} C^{j_1\, j_2\, l}_{m_1\, m_2\, m} C^{j_1\, j_2\, l}_{\sigma_1\, -\sigma_1 - \frac{1}{2}\, -\frac{1}{2}} d^l_{m, -\frac{1}{2}}(\varphi')$$
$$+ \bar{u}(p')L(f)R(\varphi')u^1 \sum_l \sum_{\sigma_1} (j_1 + \sigma_1)^{1/2} C^{j_1\, j_2\, l}_{m_1\, m_2\, m} C^{j_1\, j_2\, l}_{\sigma_1\, -\sigma_1 + \frac{1}{2}\, \frac{1}{2}} d^l_{m, \frac{1}{2}}(\varphi') \Big]. \tag{2.10}$$

The expression within the second square brackets in (2.3) is calculated in the same way and is given by

$$Y^{j_1 m_1, j_2 m_2} = \frac{(-1)^{2j_2} k^s}{(2j_1)^{1/2}} \Big[ \bar{u}^1 R^\dagger(\varphi) L^{-1}(f) u(p) \sum_{l'} \sum_{\sigma_2} (j_1 + \sigma_2)^{1/2} C^{j_1\, j_2\, l'}_{m_1\, m_2\, m} C^{j_1\, j_2\, l'}_{\sigma_2\, -\sigma_2 + \frac{1}{2}\, \frac{1}{2}} d^{l'}_{m, \frac{1}{2}}(\varphi)$$
$$+ \bar{u}^2 R^\dagger(\varphi) L^{-1}(f) u(p) \sum_{l'} \sum_{\sigma_2} (j_2 - \sigma_2)^{1/2} C^{j_1\, j_2\, l'}_{m_1\, m_2\, m} C^{j_1\, j_2\, l'}_{\sigma_2\, -\sigma_2 - \frac{1}{2}\, -\frac{1}{2}} d^{l'}_{m, -\frac{1}{2}}(\varphi) \Big]. \tag{2.11}$$

Substituting in (2.3) and proceeding as before, we obtained four terms for $T_B^{j_1 j_2}$:

$$T_B^{j_1 j_2} = g^2_{j_1 j_2} \frac{(-1)^{4j_1}}{2j_1} \frac{q^s k^s}{t + M_s^2} \Big( \bar{u}(p')L(f)R(\varphi')u^2 \bar{u}^1 R^\dagger(\varphi) L^{-1}(f) u(p)$$
$$\times \sum_l \sum_{\sigma_1 \sigma_2} \{[(j_1 - \sigma_1)(j_1 + \sigma_2)]^{1/2} C^{j_1\, j_2\, l}_{\sigma_1\, -\sigma_1 - \frac{1}{2}\, -\frac{1}{2}} C^{j_1\, j_2\, l}_{\sigma_2\, -\sigma_2 + \frac{1}{2}\, \frac{1}{2}} d^l_{\frac{1}{2}, -\frac{1}{2}}(\theta)\}$$
$$+ \bar{u}(p')L(f)R(\varphi')u^2 \bar{u}^2 R^\dagger(\varphi) L^{-1}(f) u(p)$$
$$\times \sum_l \sum_{\sigma_1 \sigma_2} \{[(j_1 - \sigma_1)(j_1 - \sigma_2)]^{1/2} C^{j_1\, j_2\, l}_{\sigma_1\, -\sigma_1 - \frac{1}{2}\, -\frac{1}{2}} C^{j_1\, j_2\, l}_{\sigma_2\, -\sigma_2 - \frac{1}{2}\, -\frac{1}{2}} d^l_{-\frac{1}{2}, -\frac{1}{2}}(\theta)\}$$
$$+ \bar{u}(p')L(f)R(\varphi')u^1 \bar{u}^1 R^\dagger(\varphi) L^{-1}(f) u(p)$$
$$\times \sum_l \sum_{\sigma_1 \sigma_2} \{[(j_1 + \sigma_1)(j_1 + \sigma_2)]^{1/2} C^{j_1\, j_2\, l}_{\sigma_1\, -\sigma_1 + \frac{1}{2}\, \frac{1}{2}} C^{j_1\, j_2\, l}_{\sigma_2\, -\sigma_2 + \frac{1}{2}\, \frac{1}{2}} d^l_{\frac{1}{2}, \frac{1}{2}}(\theta)\}$$
$$+ \bar{u}(p')L(f)R(\varphi')u^1 \bar{u}^2 R^\dagger(\varphi) L^{-1}(f) u(p)$$
$$\times \sum_l \sum_{\sigma_1 \sigma_2} \{[(j_1 + \sigma_1)(j_1 - \sigma_2)]^{1/2} C^{j_1\, j_2\, l}_{\sigma_1\, -\sigma_1 + \frac{1}{2}\, \frac{1}{2}} C^{j_1\, j_2\, l}_{\sigma_2\, -\sigma_2 - \frac{1}{2}\, -\frac{1}{2}} d^l_{-\frac{1}{2}, \frac{1}{2}}(\theta)\} \Big) \tag{2.12}$$

The expression on the right-hand side in the above equation can be simplified further by interchanging $\sigma_1 \leftrightarrow \sigma_2$ and by writing $d^l_{1/2,-1/2}(\theta) = -d^l_{-(1/2),(1/2)}(\theta)$ in the first term and combining it with the fourth term. Similarly, the second and third terms can be combined. The combination of the first and the fourth terms contains a common factor,

$$u^1\bar{u}^2 - u^2\bar{u}^1 = i\sigma_2(1 + \gamma_4), \tag{2.13a}$$

and that of the second and the third term contains a factor

$$u^1\bar{u}^1 + u^2\bar{u}^2 = (1 + \gamma_4). \tag{2.13b}$$

Thus, we are led to

$$T_B^{j_1 j_2} = \frac{g^2_{j_1 j_2}}{t + M_s^2} \frac{q^{2s}}{2j_1} \Big( \bar{u}(p')L(f)R(\theta) i\sigma_2 (1 + \gamma_4) L^{-1}(f) u(p)$$
$$\times \sum_l \sum_{\sigma_1 \sigma_2} \{[(j_1 + \sigma_2)(j_1 - \sigma_1)]^{1/2} C^{j_1\, j_2\, l}_{\sigma_2\, -\sigma_2 + \frac{1}{2}\, \frac{1}{2}} C^{j_1\, j_2\, l}_{\sigma_1\, -\sigma_1 - \frac{1}{2}\, -\frac{1}{2}} d^l_{-\frac{1}{2}, \frac{1}{2}}(\theta)\}$$
$$+ \bar{u}(p')L(f)R(\theta)(1 + \gamma_4) L^{-1}(f) u(p)$$
$$\times \sum_l \sum_{\sigma_1 \sigma_2} \{[(j_1 + \sigma_1)(j_1 + \sigma_2)]^{1/2} C^{j_1\, j_2\, l}_{\sigma_2\, -\sigma_2 + \frac{1}{2}\, \frac{1}{2}} C^{j_1\, j_2\, l}_{\sigma_1\, -\sigma_1 + \frac{1}{2}\, \frac{1}{2}} d^l_{\frac{1}{2}, \frac{1}{2}}(\theta)\} \Big). \tag{2.14}$$

The factors involving the Dirac spinors in the last equation have been discussed in a previous paper.[10] It is well known that

$$L(f)\frac{1}{2}(1 + \gamma_4)L^{-1}(f) = \frac{1}{2}\left(1 + \frac{\gamma_\mu f_\mu}{iM_s}\right) \equiv \Lambda^+(f), \tag{2.15}$$

and since the scattering takes place in $x_1x_3$ plane, $L(f)$ contains $\gamma_5\sigma_1$ and $\gamma_5\sigma_3$ which anticommute with $i\sigma_2$;

hence,

$$L(f)i\sigma_2 L^{-1}(f) = L^2(f)i\sigma_2 \tag{2.16a}$$

$$= \frac{f_0 + \gamma_5 \vec{\sigma}\cdot\vec{f}}{\sqrt{-f_\mu f_\mu}}\, i\sigma_2 \tag{2.16b}$$

$$= \frac{\gamma\cdot f}{iM_s}\,\gamma_4 i\sigma_2\,. \tag{2.16c}$$

For off-mass-shell continuations we shall use the form (2.16b) for $L(f)i\sigma_2 L^{-1}(f)$. Then, since $\vec{f} = \vec{p} + \vec{k} = 0$ and $f_0 = p_0 + k_0 = \sqrt{t}$ ,

$$L(f)i\sigma_2 L^{-1}(f) = i\sigma_2\,, \tag{2.17a}$$

$$L(f)R(\theta)L^{-1}(f) = R(\Theta)\,, \tag{2.17b}$$

$$\Lambda^+(f) = \frac{1}{2}\left(1 + \frac{\gamma_4\sqrt{t}}{M_s}\right). \tag{2.17c}$$

The summations over $l$, $\sigma_1$, and $\sigma_2$ occurring in (2.14) have been evaluated in the Appendix. These are just the O(4) rotation matrices $d^{j_2+\frac{1}{2},j_2}_{\frac{1}{2},\pm\frac{1}{2};\frac{1}{2},\frac{1}{2}}(\theta)$ multiplied by $2j_1(j_2+1)$. Using these relations

$$T_B^{j_1 j_2} = \frac{g^2_{j_1 j_2}}{t + M_s^2}\, q^{2s}(j_2+1)\Big[\bar{u}(p')R(\theta)i\sigma_2 \frac{1}{2}\left(1 + \frac{\gamma_4\sqrt{t}}{M_s}\right)u(p)d^{j_1 j_2}_{\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}}(\theta)$$
$$+ \bar{u}(p')R(\theta)\frac{1}{2}\left(1 + \frac{\gamma_4\sqrt{t}}{M_s}\right)u(p)d^{j_1 j_2}_{\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}}(\theta)\Big] \tag{2.18}$$

The direction of spin quantization in $u(p)$ and $\bar{u}(p')$ has been taken to be the $x_3$ axis. If we use the helicity states for these Dirac spinors, $\bar{u}(p')$ should be replaced by

$$\bar{u}^{(\lambda')}(p) = \bar{u}^{(\lambda')}(0)e^{-(1/2)\gamma_5\sigma_3 \tanh^{-1}p'/p_0'}R^\dagger(\theta)$$
$$\equiv \bar{u}^{\lambda}(p',3)R^\dagger(\theta)\,. \tag{2.19}$$

This $R^\dagger(\theta)$ will cancel the $R(\theta)$ occurring on the right of $\bar{u}(p')$ in (2.18) and then the angular dependence in (2.18) will be given purely by the O(4) rotation matrices. This is the advantage of using $(-f^2)^{1/2}$ rather than $M_s$ in the denominator of (2.16b). When we calculate diagram 1b using the Lagrangian $\mathcal{L}_5^{j_1 j_2}(x)$, then $\bar{u}(p')$ and $u(p)$ are replaced by $\bar{u}(p)\gamma_5$ and $\gamma_5 u(p)$, respectively. This would lead to changing the sign of $\gamma_4$ in $(1+\gamma_4\sqrt{t}/M_s)$ in (2.18). Now, using the helicity states, calculating the spinor parts, and applying the Lagrangians and masses according to Table II, we can easily write down the result for the helicity amplitudes $T_B^{j_2+1/2,j_2}(\lambda',\lambda,s \text{ odd})$ in the following form:

$$T_B^{j_2+1/2,j_2}(\tfrac{1}{2},\tfrac{1}{2},s\text{ odd}) = \left[\frac{g_s^{(1)2}}{t+(M_s^+)^2}\frac{1}{2}\left(1+\frac{\sqrt{t}}{M_s^+}\frac{p_0}{m}\right) + \frac{g^{(2)2}}{t+(M_s^-)^2}\frac{1}{2}\left(1-\frac{\sqrt{t}}{M_s^-}\frac{p_0}{m}\right)\right]\frac{s+2}{2}\,q^{2s}d^{j_2+1/2,j_2}_{\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2}}(\theta)\,, \tag{2.20a}$$

$$T_B^{j_2+1/2,j_2}(-\tfrac{1}{2},\tfrac{1}{2},s\text{ odd}) = \left[\frac{g_s^{(1)2}}{t+(M_s^-)^2}\frac{1}{2}\left(\frac{p_0}{m}+\frac{\sqrt{t}}{M_s^+}\right) + \frac{g_s^{(2)2}}{t+(M_s^-)^2}\frac{1}{2}\left(\frac{p_0}{m}-\frac{\sqrt{t}}{M_s^-}\right)\right]\frac{s+2}{2}\,q^{2s}d^{j_2+1/2,j_2}_{\frac{1}{2},-\frac{1}{2};\frac{1}{2},\frac{1}{2}}(\theta)\,. \tag{2.20b}$$

The amplitudes $T^{j_2+1/2,j_2}(+\frac{1}{2},\frac{1}{2})$ for $s$ even are obtained by replacing $M_s^\mp$ by $M_s^\pm$, $g_s^{(1)}$ by $g_s^{(3)}$, and $g_s^{(2)}$ by $g_s^{(4)}$ in the expressions given above for $T_B(+\frac{1}{2},\frac{1}{2},s \text{ odd})$. Results similar to the $\pi\pi$ scattering mentioned at the end of Sec. I are obtained at threshold for the following amplitudes:

$$T_B^{(\pm)j_2+1/2,j_2}(\tfrac{1}{2},\tfrac{1}{2},s) = (\sqrt{2}\cos\tfrac{1}{2}\theta)^{-1}T_B^{j_2+1/2,j_2}(\tfrac{1}{2},\tfrac{1}{2},s) \mp (-1)^{s+1}(\sqrt{2}\sin\theta)^{-1}T_B^{j_2+1/2,j_2}(-\tfrac{1}{2},\tfrac{1}{2},s)\,. \tag{2.21}$$

The amplitudes defined above are [apart from the factor $(-1)^{s+1} = (-1)^{2j_1}$ in the second term in (2.21)] just the parity-conserving amplitudes defined by Gell-Mann *et al.*[15] At the threshold, $p_0 = m$ and then (2.20), (2.21) together with (A43) and (A44) show that $T_B^{(+)j_1,j_2}(\frac{1}{2},\frac{1}{2},s \text{ odd})$ contains the factor $d/d\cos\theta\; C_{2j}^{(1)}(\cos\theta)$ which for $2j = s$ odd contains only even values of $l$ in its expansion in $P_l(\cos\theta)$. Similarly, on the threshold $T_B^{(-)j_1 j_2}(\frac{1}{2}\frac{1}{2}, s \text{ odd})$ contains the common factor $2(j+1)C_{2j}^{(1)}(\theta) + \cos\theta\; d/d\cos\theta\; C_{2j_2}^{(1)}(\cos\theta)$ which contains only odd values of $l$ in its expansion in $P_l(\cos\theta)$. Similar results hold for $T_B^+(\frac{1}{2},\frac{1}{2}s \text{ even})$ at the threshold.

If in (2.16b) $M_s$, rather than $(-f_\mu f_\mu)^{1/2}$, is used in the denominator, we obtain (2.16c). $\gamma\cdot f/(iM_s)$ commutes with $\gamma_4 i\sigma_2$ and is absorbed in $\Lambda^+(f)$. Proceeding in the same way as in Ref. 10 and using the results

(A43) and (A44) we arrive at

$$T_B^{j_1 j_2}=\frac{j_2+1}{4j_1}\, g^2_{j_1 j_2}\, \frac{1}{t+M_s^2}\, \bar{u}(p')\left[C^{(1)}_{2j_2}(\cos\theta)+\tfrac{1}{2} i\sigma_2\gamma_4 \sin\theta\, \frac{1}{j_2+1}\, \frac{d}{d\cos\theta}\, C^{(1)}_{2j_2}(\cos\theta)\right]\left[1+\frac{\gamma_4\sqrt{t}}{M_s}\right]u(p)\,. \tag{2.22}$$

The previous continuation using (2.16b) would lead to almost the same equation as (2.22), the only difference being that there would be no $\gamma_4$ occurring with $i\sigma_2$ in the second term on the right-hand side of (2.22). This means that these two continuations give the same result as far as the highest-order term in $\cos\theta$ is concerned.

Equation (49) of Ref. 10 gives the contribution of spin $j=l+\frac{1}{2}$ baryon pole to $\pi N\to\pi N$ and is similar to (2.22) given above. It contains $P_l(\cos\theta)$ instead of $C^{(1)}_{2j}(\cos\theta)$; otherwise, it is essentially the same as (2.22). Carlitz and Kislinger's result[16] was the $P_l(\cos\theta)$ term in Eq. (49) of Ref. 10. Equation (2.22) is thus a generalization to O(4) symmetry of their O(3) symmetric result.

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## APPENDIX A

In this appendix we derive certain results connected with the representation of the O(4) group, which have been used in the text. It is well known[6] that from the six generators

$$L_i=\tfrac{1}{2}\epsilon_{ijk}J_{jk}\,, \tag{A1}$$

$$A_i=J_{i4} \tag{A2}$$

of O(4), another set of generators $\vec{J}^{(1)}$ and $\vec{J}^{(2)}$ are formed by writing

$$\vec{J}^{(1)}=\tfrac{1}{2}(\vec{L}+\vec{A})\,,\qquad \vec{J}^{(2)}=\tfrac{1}{2}(\vec{L}-\vec{A})\,. \tag{A3}$$

$J_i^{(1)}$ amd $J_j^{(2)}$ commute with each other and individually they satisfy the commutation relations of O(3) generators. The simultaneous eigenvectors of $\vec{J}^{(1)2}$, $J_3$, $\vec{J}^{(2)2}$, and $J_3^{(2)}$ are

$$|j_1m_1\rangle\times|j_2m_2\rangle\equiv|j_1m_1,j_2m_2\rangle\,. \tag{A4}$$

As pointed out by Biedenharn[6] the simultaneous eigenvectors of $J^{(1)2}$, $\vec{J}^{(2)2}$, $\vec{L}^2$, and $L_3$ are obtained from the previous set of simultaneous eigenvectors with the help of Clebsch-Gordan coefficients,

$$|j_1j_2lm\rangle=\sum_{m_1,m_2} C^{j_1\,j_2\,l}_{m_1m_2m}\,|j_1m_1,j_2m_2\rangle\,, \tag{A5}$$

where $j_1(j_1+1)$, $j_2(j_2+1)$, $l(l+1)$, and $m$ are the eigenvalues of $\vec{J}^{(1)2}$, $\vec{J}^{(2)2}$, $\vec{L}^2$, and $L_3$, respectively. The $A^{j_1j_2}_{l'm';lm}(\theta)$ and $d^{j_1j_2}_{l'l,m}(\theta)$ rotation matrices of Biedenharn[6] and Freedman and Wang[1] are the matrix elements of $\exp(-iA_3\theta)=\exp(-i(J_3^{(1)}-J_3^{(2)})$ in the $|j_1j_2lm\rangle$ representations. From (A4) it is easily seen that these matrices are diagonal in $m'$, $m$. The matrices which we obtained in the text are, however, different from the rotation matrices mentioned above. These are special cases of the matrix elements of $\exp(-iA_2\theta)=\exp[-i(J_2^{(1)}-J_2^{(2)})\theta]$ in $|j_1j_2lm\rangle$ basis. We define, therefore, a matrix $d^{j_1j_2}_{l'm';lm}(\theta)$ by

$$d^{j_1j_2}_{l'm';lm}(\theta)=\langle j_1j_2l'm'|e^{-iA_2\theta}|j_1j_2lm\rangle \tag{A6}$$

$$=\sum_{m_1'm_2'}\sum_{m_1m_2}\{C^{j_1\,j_2\,l'}_{m_1'm_2'm'}C^{j_1\,j_2\,l}_{m_1m_2m}\langle j_1m_1',j_2m_2'|e^{-i(J_2^{(1)}-J_2^{(2)})\theta}|j_1m_1,j_2m_2\rangle\}\,. \tag{A7}$$

From (A4), (A7), and the definition of the O(3) rotation matrices $d^j_{m_1m_2}(\theta)$ we obtain

$$d^{j_1j_2}_{l'm';lm}(\theta)=\sum_{m_1'm_2'}\sum_{m_1m_2}\{C^{j_1\,j_2\,l'}_{m_1'm_2'm'}C^{j_1\,j_2\,l}_{m_1m_2m}d^{j_1}_{m_1'm_1}(\theta)d^{j_2}_{m_2,m_2'}(\theta)\}\,. \tag{A8}$$

We shall now calculate the O(4) rotation matrices $d^{jj}_{00,00}(\theta)$, $d^{jj}_{10;10}(\theta)$, and $d^{jj}_{\frac{1}{2},\pm\frac{1}{2};\frac{1}{2},\frac{1}{2}}(\theta)$ from (A8).

*The matrix element* $d^{jj}_{00;00}(\theta)$. Setting

$$j_1=j_2=j\,,\qquad l'=l=m'=m=0 \tag{A9}$$

in (A8),

$$d^{j\,j}_{00;\,00}(\theta)=\sum_{\sigma_1,\sigma_2} C^{j_1 j_2 0}_{\sigma_1 -\sigma_1 0}\, C^{j\ j\ 0}_{\sigma_1 -\sigma_2 0}\, d^j_{\sigma_1\sigma_2}(\theta)\, d^j_{-\sigma_2,-\sigma_1}(\theta)\,. \tag{A10}$$

From[17] $C^{jj0}_{\sigma-\sigma 0}=(-1)^{j-\sigma}$ and the symmetry properties of the $d^j$ matrices[17]

$$d^j_{m_1m_2}(\theta)=(-1)^{m_1-m_2}\, d^j_{m_2,m_1} \tag{A11a}$$

$$=(-1)^{m_1-m_2}\, d^j_{-m_1,-m_2} \tag{A11b}$$

$$=d^j_{-m_2,-m_1}(\theta)\,, \tag{A11c}$$

it follows at once that

$$d^{jj}_{00;00}(\theta)=\sum_{\sigma_1\sigma_2}(-1)^{2j-\sigma_1-\sigma_2}\, d^j_{\sigma_1\sigma_2}(\theta)\, d^j_{\sigma_2\sigma_1}(\theta)\,(-1)^{\sigma_2-\sigma_1}$$

$$=\sum_{\sigma_1}(-1)^{2(j-\sigma_1)}\, d^j_{\sigma_1\sigma_1}(2\theta)$$

$$=\sum_{\sigma_1} d^j_{\sigma_1\sigma_1}(2\theta)\quad . \tag{A12}$$

Also, using the symmetry property (A11c),

$$\sum_{\sigma} d^j_{\sigma_1\sigma_1}(2\theta)=\sum_{\sigma_1\sigma_2} d^j_{\sigma_1\sigma_2}(\theta)\, d^j_{-\sigma_1,-\sigma_2}(\theta)\quad . \tag{A13}$$

Now, using the formula for combining two $d^j$ matrices, which has been used quite often in this work[17], i.e.,

$$d^{j_1}_{m_1' m_1}(\theta)\, d^{j_2}_{m_2' m_2}(\theta)$$

$$=\sum_{l}\left[C^{j_1\, j_2\, l}_{m_1' m_2' m_1'+m_2'}\, C^{j_1\, j_2\, l}_{m_1\, m_2\, m_1+m_2}\, d^l_{m_1'+m_2',m_1+m_2}(\theta)\right]\,, \tag{A14}$$

together with (A12) and (A13), immediately gives

$$d^{jj}_{00,00}(\theta)=\sum_{l}\sum_{\sigma_1\sigma_2} C^{j\,j\,l}_{\sigma_1-\sigma_1 0}\, C^{j\,j\,l}_{\sigma_2-\sigma_2 0}\, d^l_{00}(\theta)\,. \tag{A15}$$

We will now show that the last expression is just the expansion of the Gegenbauer polynomials $C^{(1)}_{2j}(\theta)$ in terms of the Legendre polynomials $d^l_{00}(\theta) = P_l(\cos\theta)$. To this end we use the following formula for the Clebsch-Gordan coefficients[18]

$$C^{j_1\, j_2\, l}_{m_1\, m_2\, m}=\frac{(-1)^{-l+j_1+m_2}}{2^{l+j_1+j_2+1}}$$

$$\times\left[\frac{(2l+1)\,(l+m_1+m_2)!\,(j_1+j_2-l)!\,(j_1+j_2+l+1)!}{(j_1-m_1)!\,(j_1+m_1)!\,(j_2-m_2)!\,(j_2+m_2)!\,(l-m_1-m_2)!\,(l+j_1-j_2)!\,(l-j_1+j_2)!}\right]^{1/2}$$

$$\times\int_{-1}^{+1} dx(1-x)^{j_1-m_1}(1+x)^{j_2-m_2}\,\frac{d^{l-m_1-m_2}}{dx^{l-m_1-m_2}}\left[(1-x)^{l-j_1+j_2}(1+x)^{l+j_1-j_2}\right]\,. \tag{A16}$$

On calculating $\sum_\sigma C^{j\,j\,l}_{+\sigma-\sigma 0}$ from the above formula we obtain a binomial expansion giving $2^{2j}x^{2j}/(2j)!$, and further using the Rodrigues' formula for the Legendre polynomial we obtain

$$\sum_{\sigma} C^{j\,j\,l}_{\sigma-\sigma 0}=\frac{1}{2(2j)!}\left[(2l+1)\,(2j-l)!\,(2j+l+1)!\right]^{1/2}\int_{-1}^{+1} x^{2j}\,P_l(x)\,dx\,. \tag{A17}$$

The above integral is given by[19]

$$\int_{-1}^{+1} x^{2j}\,P_l(x)\,dx=0 \qquad \text{for } 2j-l \text{ odd or negative,}$$

$$=\frac{2^l(2j)!}{(2j+l+1)!}\,\frac{[(2j+l)/2]!}{[(2j-l)/2]!}\qquad \text{for } (2j-l) \text{ even and nonnegative.} \tag{A18}$$

Equations (A17) and (A18) give

$$\sum_{l}\sum_{\sigma_1}\sum_{\sigma_2} C^{j\,j\,l}_{\sigma_1-\sigma_1 0}\, C^{j\,j\,l}_{\sigma_2-\sigma_2 0}\, P_l(\cos\theta)=\sum_{l}\frac{(2l+1)\,2^{2l}\,(2j-l)!}{(2j+l+1)!}\left[\frac{[(2j+l)/2]!}{[(2j-l)/2]!}\right]^2 P_l(\cos\theta)\,, \tag{A19}$$

with

$l=0, 2, 4\cdots 2j$ for $2j$ even ,

$=1, 3, 5\cdots 2j$ for $2j$ odd .

The right-hand side of the previous equation is exactly the expansion of $C^{(1)}_{2j}(\theta)$ as mentioned by Harnad[4] which can be verified by expanding $C^{(1)}_{2j}(\cos\theta)$ in terms of $P_l(\cos\theta)$, calculating the expansion coefficients by using the following formulas[20]:

$$C_{2j}^{(1)}(Z)=\frac{\sin[(2j+1)\chi]}{\sin\chi}\ \text{with}\ Z=\cos\chi \tag{A20}$$

and

$$\int_0^{\pi}\sin[(2j+1)\chi]\,P_l(\cos\chi)\,d\chi=\left(\frac{2j+l}{2}\right)!\ \frac{\Gamma(\frac{1}{2}(2j-l)+\frac{1}{2})}{\Gamma(\frac{1}{2}(2j+l+2)+\frac{1}{2})}\quad \text{for } 2j+1>l \text{ and } 2j+l+1 \text{ odd},$$
$$=0 \quad \text{otherwise}, \tag{A21}$$

and simplifying the $\Gamma$ function by using

$$\Gamma(n+\tfrac{1}{2})=\frac{\pi^{1/2}2^{-2n}(2n)!}{(n)!}\ . \tag{A22}$$

Collecting the results

$$d_{00;00}^{jj}(\theta)=C_{2j}^{(1)}(\theta) \tag{A23a}$$

$$=\sum_{\sigma}d_{\sigma\sigma}^{j}(2\theta) \tag{A23b}$$

$$=\sum_{\sigma_1\sigma_2}C_{\sigma_1-\sigma_1 0}^{jjl}\,C_{\sigma_2-\sigma_2 0}^{jjl}\,P_l(\cos\theta)\ . \tag{A23c}$$

These together with expansion (A19) complete the formulas needed in this work. (A23b) is implied by Eq. (7) of Ref. 6.

*The matrix element* $d_{10;10}^{jj}(\theta)$. On setting

$$j_1=j_2=j\ ,$$
$$l'=l=1\ , \tag{A24}$$
$$m'=m=0\ ,$$

in (A8) and using[17]

$$C_{\sigma-\sigma 0}^{jj1}=\sqrt{3}(-1)^{j-\sigma}\frac{\sigma}{[(2j+1)(j+1)j]}\ , \tag{A25}$$

we obtain

$$d_{10;10}^{jj}(\theta)=\sum_{\sigma_1}\sum_{\sigma_2}\frac{3}{j(j+1)(2j+1)}\sigma_1\sigma_2\,d_{\sigma_1\sigma_2}^{j}(\theta)\,d_{\sigma_2\sigma_1}^{j}(\theta)\ . \tag{A26}$$

Writing $d_{\sigma_2\sigma_1}^{j}(\theta)=d_{-\sigma_1,-\sigma_2}^{j}(\theta)$ and using (A14), we obtain

$$d_{10;10}^{jj}(\theta)=\sum_{\sigma_1}\sum_{\sigma_2}\sum_{l}\frac{3}{j(j+1)(2j+1)}\sigma_1\,C_{\sigma_1-\sigma_1 0}^{jjl}\,\sigma_2\,C_{\sigma_2-\sigma_2 0}^{jjl}\,d_{00}^{l}(\theta)\ , \tag{A27}$$

which is the summation occurring in the text. Again we calculate $\sum_\sigma \sigma\, C_{\sigma-\sigma 0}^{jjl}$ using the integral formula (A16) and substitute the results in (A27). The result is

$$d_{10;10}^{jj}(\theta)=\frac{3}{4j(j+1)(2j+1)}\sum_{l}2^{l}(2l+1)\frac{(2j-1-l)!}{(2j-1+l+1)!}\left[\frac{[(2j-1+l)/2]!}{[(2j-1-l)/2]!}\right]^2(2j-l)(2j+l+1)\,P_l(\cos\theta) \tag{A28}$$

with $2j-l$ odd and nonnegative integer. Now

$$(2j-1)(2j+l+1)=2j(2j+1)-l(l+1)\ , \tag{A29}$$

and we know that[17]

$$l(l+1)\,P_l(x)=\left[-\frac{d}{dx}(1-x^2)\frac{d}{dx}\right]P_l(x) \tag{A30a}$$

$$\equiv \mathfrak{L}(\partial)\,P_l(x)\;. \tag{A30b}$$

Hence, using (A30) and comparing with the expansion (A19) of $C_{2j}^{(1)}(\cos\theta)$, we obtain the following form for the rotation matrix $d^{jj}_{10;10}(\theta)$ in terms of the Gegenbauer polynomial $C^{(1)}_{2j-1}(\theta)$:

$$d^{jj}_{10;10}(\theta)=\frac{3}{4}\,\frac{1}{j(j+1)(2j+1)}\,[2j(2j+1)-\mathfrak{L}(\partial)]\,C^{(1)}_{2j-1}(\cos\theta)\;. \tag{A31}$$

We have seen that $\sum_\sigma d^j_{\sigma\sigma}(2\theta)=C^{(1)}_{2j}(\theta)$. From the symmetry properties (A11) it follows that for odd $r$, $\sum_\sigma \sigma^r d^j_{\sigma\sigma}(\theta)=0$. It is interesting to note that $\sum_\sigma \sigma^2 d^j_{\sigma\sigma}(2\theta)$ is connected with $d^{jj}_{10;10}(\theta)$, and this can be shown as follows:

$$\begin{aligned}\sum_{\sigma_1\sigma_2}\sigma_1\sigma_2\,d^j_{\sigma_1\sigma_2}(\theta)\,d^j_{\sigma_2\sigma_1}(\theta)&=\sum_{\sigma_1\sigma_2}\langle j\sigma_1|e^{-iJ_2\theta}J_3|j\sigma_2\rangle\langle j\sigma_2|e^{-iJ_2\theta}J_3|j\sigma_1\rangle\\ &=\sum_{\sigma_1\sigma_2}\langle j\sigma_1|e^{-2iJ_2\theta}|j\sigma_2\rangle\langle j\sigma_2|(J_3\cos\theta-J_1\sin\theta)J_3|j\sigma_1\rangle\\ &=\sum_{\sigma_1}\sigma_1{}^2 d^j_{\sigma_1\sigma_1}(2\theta)\cos\theta-\sum_{\sigma_1\sigma_2}\sigma_1\sin\theta\,d^j_{\sigma_1\sigma_2}(\theta)\langle j\sigma_2|J_1|j\sigma_2\rangle\,.\end{aligned} \tag{A32}$$

The matrix element $\langle j\sigma_2|J_1|j\sigma_1\rangle$ is well known,[17] and using the formula[21]

$$[(j\pm\mu+1)(j\mp\mu)]^{1/2}d^j_{\lambda,\mu\pm1}=\left(\frac{-\lambda}{\sin\theta}+\mu\cot\theta\mp\frac{\partial}{\partial\theta}\right)d^j_{\lambda\mu}(\theta) \tag{A33}$$

and simplifying, we arrive at the result

$$\sum_{\sigma_1\sigma_2}\sigma_1\sigma_2\,d^j_{\sigma_1\sigma_2}(\theta)\,d^j_{\sigma_2\sigma_1}(\theta)=\frac{1}{\cos\theta}\sum_\sigma\sigma^2 d^j_{\sigma\sigma}(\theta)\,. \tag{A34}$$

Hence, from (A26)

$$\sum_\sigma\sigma^2 d^j_{\sigma\sigma}(2\theta)=\frac{(2j+1)(j+1)j}{3}\cos\theta\,d^{jj}_{10;\,10}(\theta)\,. \tag{A35}$$

The present method, however, becomes too complicated for computing

$$\sum_\gamma \sigma^{2r} d^j_{\sigma\sigma}(\theta),\;\; r>2\,.$$

*The matrix elements* $d^{j_2+\frac{1}{2},j_2}_{\frac{1}{2},\pm\frac{1}{2};\frac{1}{2},\frac{1}{2}}(\theta)$. In the formula (A8) we put

$$\begin{aligned}&j_1=j_2+\tfrac{1}{2},\;\; j_2 \text{ an integer}\geq 0\,,\\ &l'=l=\tfrac{1}{2}\,,\\ &m'=\mp\tfrac{1}{2},\;\; m=\tfrac{1}{2}\,,\end{aligned} \tag{A36}$$

and obtain

$$d^{j_1j_2}_{\frac{1}{2},\mp\frac{1}{2};\frac{1}{2}\frac{1}{2}}(\theta)=\sum_{\sigma_1\sigma_2}C^{j_1j_2\frac{1}{2}}_{\sigma_1\,-\sigma_1\mp\frac{1}{2}\,\mp\frac{1}{2}}\,C^{j_1j_2\frac{1}{2}}_{\sigma_2\,-\sigma_2+\frac{1}{2}\,\frac{1}{2}}\,d^{j_1}_{\sigma_1\sigma_2}(\theta)\,d^{j_2}_{-\sigma_2+\frac{1}{2};-\sigma_1\pm\frac{1}{2}}(\theta)\,. \tag{A37}$$

The Clebsch-Gordan coefficients occurring above are easily obtained from the table of Wigner's $3j$ symbols given in Ref. 17. The result is

$$C^{j_1j_2\frac{1}{2}}_{\sigma\,-\sigma\mp\frac{1}{2}\,\mp\frac{1}{2}}=\sqrt{2}\,(-1)^{j_2\mp\sigma-\frac{1}{2}}\left(\frac{j_1\mp\sigma}{(2j_2+2)(2j_2+1)}\right)^{1/2}\,. \tag{A38}$$

Substituting this in (A37), using the symmetry property (A11a) and remembering that $(2\sigma_2-1)$ is always even, we arrive at the result

$$d^{j_1 j_2}_{\frac{1}{2}, \mp\frac{1}{2}; \frac{1}{2}, \frac{1}{2}}(\theta) = \sum_{\sigma_1\sigma_2} \frac{(j_1 \mp \sigma_1)^{1/2}(j_1+\sigma_2)^{1/2}}{(j_2+1)(2j_2+1)} d^{j_1}_{\sigma_1\sigma_2}(\theta)\, d^{j_2}_{-\sigma_1\mp\frac{1}{2}, -\sigma_2+\frac{1}{2}}(\theta) \tag{A39}$$

$$= \sum_{\sigma_1\sigma_2} \frac{[(j_1\mp\sigma_1)(j_1+\sigma_2)]^{1/2}}{(j_2+1)(2j_2+1)} \sum_{l=1/2}^{2j_2+1/2} C^{j_1 j_2 l}_{\sigma_1 -\sigma_1\mp\frac{1}{2} \mp\frac{1}{2}} C^{j_1 j_2 l}_{\sigma_2 -\sigma_2+\frac{1}{2} \frac{1}{2}}\, d^{l}_{\mp\frac{1}{2}, \frac{1}{2}}(\theta). \tag{A40}$$

$d^{j_1 j_2}_{\frac{1}{2}, \pm\frac{1}{2}; \frac{1}{2}, \frac{1}{2}}(\theta)$ can also be expressed in terms of the polynomial $C^{(1)}_{2j}(\theta)$ and its first derivative. This is most easily done by using the following formula[21]:

$$(j_1+\sigma_2)^{1/2} d^{j_2+\frac{1}{2}}_{\sigma_1\sigma_2}(\theta) = (j_1+\sigma_1)^{1/2} d^{j_2}_{\sigma_1-\frac{1}{2}, \sigma_2-\frac{1}{2}}(\theta)\cos\tfrac{1}{2}\theta + (j_1-\sigma_1)^{1/2} d^{j_2}_{\sigma_1+\frac{1}{2}, \sigma_2-\frac{1}{2}}(\theta)\sin\tfrac{1}{2}\theta. \tag{A41}$$

Substituting this in (A39)

$$(j_2+1)(2j_2+1)\, d^{j_1 j_2}_{\frac{1}{2}\frac{1}{2}; \frac{1}{2}\frac{1}{2}}(\theta) = \sum_{\sigma_1\sigma_2} (j_1+\sigma_1)\, d^{j_2}_{\sigma_1-\frac{1}{2}, \sigma_2-\frac{1}{2}}(\theta)\, d^{j_2}_{-\sigma_1+\frac{1}{2}, -\sigma_2+\frac{1}{2}}(\theta)\cos\tfrac{1}{2}\theta$$

$$+ (j_1-\sigma_1)^{1/2}(j_1+\sigma_1)^{1/2} d^{j_2}_{\sigma_1+\frac{1}{2}, \sigma_2-\frac{1}{2}}(\theta)\, d^{j_2}_{-\sigma_1+\frac{1}{2}, -\sigma_2+\frac{1}{2}}(\theta)\sin\tfrac{1}{2}\theta. \tag{A42}$$

On setting $\sigma_1 - \frac{1}{2} = \sigma$, $\sigma_2 - \frac{1}{2} = \sigma'$ in the first term, and $\sigma_1 + \frac{1}{2} = \sigma$, $\sigma_2 - \frac{1}{2} = \sigma'$ in the second term on the right-hand side of (A42), and using (A33) for $d^{j}_{\sigma, \sigma-1}(2\theta)$, we obtain after some simple calculation

$$(2j_2+1)(j_2+1)\, d^{j_2+\frac{1}{2}, j_2}_{\frac{1}{2}\frac{1}{2}; \frac{1}{2}\frac{1}{2}}(\theta) = \cos\tfrac{1}{2}\theta\left[(j_2+1)C^{(1)}_{2j_2}(\cos\theta) - \frac{1-\cos\theta}{2}\frac{d}{d\cos\theta}C^{(1)}_{2j_2}(\cos\theta)\right]. \tag{A43}$$

Similarly, we obtain

$$(2j_2+1)(j_2+1)\, d^{j_2+\frac{1}{2}, j_2}_{\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}\frac{1}{2}}(\theta) = \sin\tfrac{1}{2}\theta\left[(j_2+1)C^{(1)}_{2j_2}(\cos\theta) + \frac{1+\cos\theta}{2}\frac{d}{d\cos\theta}C^{(1)}_{2j_2}(\cos\theta)\right]. \tag{A44}$$

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*On leave of absence from Department of Physics, University of Islamabad, Pakistan.


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