A practical relativistic theory of three-body scattering*

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A natural and unambiguous generalization of the boundary condition formalism previously proposed by this author is shown to result in a practical set of equations for describing relativistic three-particle scattering. These equations are exactly unitary, and can be readily generalized to the n-body case. Applied to the 0^- and 1^- states of the 3π system ($I = 0, 1, 2$), they yield the π and ω as natural consequences of the π-π s- and p-wave phases; there are no spurious predictions. The equations appear ideal for relativistic data analysis.

I. INTRODUCTION

One of the most striking unsolved problems of elementary particle physics is the surprisingly rich spectrum of meson and baryon resonances. Although unitary symmetry and Regge theory have been very useful concepts in classifying these "particles," there is no theory capable of providing a truly fundamental interpretation. In this circumstance, it is natural to ask whether a less ambitious approach incorporating a certain amount of empirical information might not prove highly useful. In particular, is there some set of input from which one can predict at least a subset of the hadron spectroscopy? An obvious choice for such input would be the observed one- and two-particle properties of those particles deemed "elementary" for this purpose (e.g., particles such as N and π which are stable under the strong interaction). That is, the input would consist of the masses and pairwise scattering data of the "elementary" particles. Of course, this program is not a new concept; for example, it was suggested by Chew in 1960 that the ω might exist as a natural consequence of the ρ.[1] The natural impediment has always been the absence of a calculable relativistic n-body scattering theory ($n \geq 3$).

The obvious analogy is to nuclear physics, in which interactions constructed empirically to produce N-N phase shifts and the deuteron are employed to calculate the properties of nuclei. In this case, for nuclei light enough for this program to be practicable, the results have been very impressive. Previous attempts to construct relativistic scattering theories have leaned heavily on the highly successful procedures employed in this nonrelativistic problem. Thus, in analogy to the rigorous equations developed by Faddeev,[2] a number of authors have proposed covariant theories for three-body scattering.[3] The basic ingredients of these treatments include (1) summing the two-particle graphs into (off-shell) two-body scattering amplitudes, (2) solving the disconnectedness problem according to the prescription of Faddeev, (3) introducing a separable approximation to the two-body amplitudes, and (4) eliminating the relative energies as variables via the technique of Blankenbecler and Sugar.[4] The result of this procedure is a set of covariant one-variable integral equations which can easily be solved on modern computers; solutions exactly satisfy (elastic) three-body unitarity.

However, despite the profusion of potentially interesting applications, the history of these equations has been short and inglorious. In large part this can be traced to the very disappointing results reported by Basdevant and Kreps for the 3π system.[5] They searched for 3π resonances (realizable in such formalisms as actual poles in the calculated amplitudes) in all isospin states with $J \leq 2$. Unfortunately, the theory was unable to produce the ω, A_1, or A_2, predicting instead a considerable number of spurious "resonances" in total conflict with experiment. Subsequent work by Mennessier, Pasquier, and Pasquier demonstrated that these results were to some extent dependent on the choice of a rather unrealistic ρ form factor, but they reported similar qualitative features.[6] Thus, although they found a specific choice of form factor (also unrealistic) which produced an "ω" of mass 850 MeV, the continued proliferation of spurious levels does not inspire much confidence in this prediction. Moreover, if one seeks to establish a natural connection between the ρ and the ω, a result depending crucially on the choice of an (unknown) form factor is not very informative. If the theory is to be predictive in the sense that the nonrelativistic theory is predictive, the results must be stable given any reasonable guess as to the off-shell characteristics.

There has been somewhat greater success with respect to meson-baryon calculations (where at least one particle is nonrelativistic). Thus Aaron, Amado, and Young[3] have reported three-body treatments of the $\pi N\,p$ and d waves. But in general the program outlined above has not proved viable. However, as shall be demonstrated below, equations constructed to imitate potential theory (a Hamiltonian approach) are not the only alternative one can employ. A completely distinct set of equations will be presented which follow in a natural way from simple physical considerations, and which can be applied in order to realize the same goals. In particular, calculations in the 3π system result in a plausible ω while producing *no* spurious effects.

The outline of this paper is as follows. Section II is devoted to a detailed exposition of our underlying philosophy and its realization for a system of three nonrelativistic particles. In Sec. III we develop the appropriate relativistic generalization, which is straightforward. Again a one-variable integral equation is obtained, but in this approach we are able to avoid making the more radical assumptions noted above. In particular, it is unnecessary to assume separable amplitudes or "dominance" by the resonance pole; this is crucial in channels such as the π-π s waves. The relation of the amplitudes generated by this formalism to the physical amplitudes is discussed in Sec. IV, which also deals with three-body unitarity. Applications to the 3π system are presented in Sec. V, and a general discussion of our results and their implications is contained in Sec. VI.

In view of the length of this paper, a balance had to be struck between presenting a special case of our equations having limited utility vs giving complicated (and totally opaque) formulas for the general problem. The solution adopted was to write the general equations in operator form; these are presented in the Appendix. This provides all the information necessary to employ the formalism, but leaves the rather standard algebraic manipulations to the reader. Since one will typically be concerned with relatively simple special cases this is not as callous as it sounds, and more specifics will be provided in subsequent articles. In any event, the quite similar equations developed for the nonrelativistic problem are also available for comparison.[7]

II. BOUNDARY-CONDITION APPROACH

If one takes the one- and two-particle properties as empirical input, it is clear that a three-body system plays a pivotal role in the type of program discussed above; it is the first level at which one hopes to make predictions. This is also the level at which one is first confronted with the major technical question: How is one to build in the $n<3$ properties as constraints on $n\geq 3$ systems? The classic response of nuclear physics to this question is the introduction of a potential, with parameters suitably adjusted to reproduce observables from solutions of the two-body Schrödinger equation. The prescription is then to use this potential in the n-body Schrödinger equation, with the implicit hope that specific three- and more-body effects are a small perturbation. That this works remarkably well is shown by the results of extensive trinucleon calculations during the past decade.

However, as this author has recently demonstrated, the potential prescription is not only nonunique, but is unnecessary in order to achieve these results.[8] Thus, the success of these calculations does not provide *a posteriori* justification of potential theory so much as it manifests the nontrivial fashion in which two-particle observables constrain a three-body system. The choice of a mechanism for building in these constraints is thus to some extent open, and one may exercise the resulting freedom in order to achieve certain desired properties. In the context of the relativistic three-body problem this is indeed fortunate, in view of the previously noted difficulties associated with the Hamiltonian approach. As we shall demonstrate below, the boundary-condition (BC) formalism developed by this author has an unambiguous relativistic generalization, and appears ideally suited for this purpose.

The philosophy of this appoach can be illustrated by first examining a nonrelativistic problem. Consider two spinless particles separated by a distance x, and described by a wave function $\psi^{>}(\vec{x})$ in the region $x>a$. Assuming that we are dealing with a particular partial wave, it is convenient to write $\psi_l^{>}(x)=\psi_l^{\text{ext}}(x)+\psi_l^{\text{pot}}(x)$, where

$$\begin{aligned}\psi_l^{\text{ext}}(x)&=j_l(\kappa x)+ie^{i\delta_l}\sin\delta_l\,h_l(\kappa x)\,,\\ \psi_l^{\text{pot}}(x)&=\chi_l(x,\kappa)t_l(\kappa)\,.\end{aligned}\tag{1}$$

Here κ is the c.m. momentum, and t_l is the two-particle t matrix, defined by

$$\begin{aligned}t_l(\kappa)&=-\frac{e^{i\delta_l}\sin\delta_l}{\pi m_r\kappa}\\ &\equiv\frac{N_l(\kappa)}{D_l(\kappa)}\,.\end{aligned}\tag{2}$$

For typical meson-theoretic potentials χ_l is a real function proportional to $\exp(-\mu x)$ for large x (μ is the mass of the lightest exchanged particle). An interior wave function $\psi_l^{<}(x)$ *may* exist in the region $x<a$; its sole effect on the exterior solution is via the matching BC

$$\begin{aligned}[\psi_l^{>\prime}(x)/\psi_l^{>}(x)]_{x=a}&=[\psi_l^{<\prime}(x)/\psi_l^{<}(x)]_{x=a}\\ &\equiv\lambda_l^{\text{int}}(\kappa^2)\,,\end{aligned}\tag{3}$$

where the prime indicates derivative. One can thus account for the *effect* of the interior by a suitable λ_l^{int}, without postulating either a wave-function description or a short-range potential. This point is of particular importance in the relativistic problem, for which a two-particle description at arbitrarily small separations is essentially meaningless. The essence of the BC approach is to take advantage of the relatively simple exterior representation at the cost of a phenomenological treatment of the interior.

Introducing the function

$$B_l(\kappa) = \lambda_l^{\text{int}}(\kappa^2)\chi_l(a,\kappa) - \chi_l'(a,\kappa)\,, \tag{4}$$

we can write Eq. (3) as

$$\psi_l^{\text{ext}\,\prime}(a) - \lambda_l^{\text{int}}(\kappa^2)\psi_l^{\text{ext}}(a) = B_l(\kappa)t_l(\kappa)\,. \tag{5}$$

The content of Eq. (5) is that both interior and exterior effects can be represented as effective BC's on ψ_l^{ext}, the asymptotic form of the wave function. Since it is our intention to treat two-particle systems phenomenologically, it is sufficient to define

$$\lambda_l^{\text{pot}}(\kappa^2) = \frac{aB_l(\kappa)N_l(\kappa)}{\pi m_r}\,,$$
$$\lambda_l(\kappa^2) = \lambda_l^{\text{int}}(\kappa^2) + \lambda_l^{\text{pot}}(\kappa^2)\,; \tag{6}$$

one can then bring Eq. (5) into the form

$$[\psi_l^{\text{ext}\,\prime}/\psi_l^{\text{ext}}]_{x=a} = \lambda_l(\kappa^2)\,. \tag{7}$$

Given the trivial form of ψ_l^{ext}, one sees clearly that a suitable $\lambda_l(\kappa^2)$ can always be introduced so as to reproduce the exact *experimental* phase shifts via the solution of Eq. (7). The fact that the BC may be applied at a radius for which ψ_l^{ext} has no direct physical interpretation is essentially irrelevant, since our purpose is merely to insure the correct asymptotic behavior.

Taken together, λ_l and the radius (a) provide an alternative representation of the two-particle scattering data. An interesting empirical fact is that for many systems there exists a particular value of the radius for which λ_l appears to approach a constant for large κ.[9] In the context of potential theory one can easily verify that this cannot come about via λ_l^{pot}, which falls to zero; it is a physical property of the interior. The conventional explanation of this phenomenon is that the huge interaction energy in the core is sufficient to swamp even very large κ, but it may also come about as a natural consequence of an underlying composite structure (e.g., quarks), as has recently been pointed out by this author.[10] In any event, it is natural to choose the BC radius singled out in this fashion. The derivative function $\lambda_l(\kappa^2)$ then has a particularly simple structure; it must be a meromorphic function of κ^2 (real for real κ^2).

We shall thus take

$$\lambda_l(\kappa^2) = \lambda_l^\infty + \sum_n \frac{R_{l,n}}{\kappa^2 - \beta_{l,n}{}^2}\,, \tag{8}$$

with the parameters adjusted to fit the two-particle data. In practice, one or two terms in the sum are usually sufficient. It is important to note that Eq. (8) defines a particular analytic continuation of $t_l(\kappa)$ via the explicit formulas which follow from Eq. (7):

$$N_l(\kappa) = (a\lambda_l - l)j_l(a\kappa) + a\kappa j_{l+1}(a\kappa)\,,$$
$$D_l(\kappa) = i\pi\kappa m_r[(a\lambda_l - l)h_l(a\kappa) + a\kappa h_{l+1}(a\kappa)]\,. \tag{9}$$

Therefore, although it is essential to build poles into t_l below threshold ($\kappa^2 < 0$) at the physical energies of bound states, the residues at these poles need not be identical with those generated in potential theory (e.g., via the analytic continuation defined by the Lippmann-Schwinger equation). In addition, of course, there will be no left-hand cuts. Naturally, so long as we are concerned only with the two-particle scattering state these considerations are academic, but they are essential in the three-body treatment to follow. Otherwise, one could easily show that the resulting three-particle amplitude would violate unitarity.

In the above discussion we have made extensive use of the asymptotic representation, $\psi_l^{\text{ext}}(x)$, and it is useful for the subsequent development to restate this in more formal terms. We therefore denote an incoming (plane-wave) state of angular momentum l_0 and c.m. momentum κ by $|\varphi\rangle = |l_0\kappa\rangle$,

$$\langle lx|\varphi\rangle = \delta_{ll_0}(2/\pi)^{1/2}i^l j_l(\kappa x)\,. \tag{10}$$

In this case we can define a free Hamiltonian $H_0 = -\nabla_x{}^2/2m_r$; clearly $(H_0 - E)|\varphi\rangle = 0$ if $E = \kappa^2/2m_r$. Presumably, there is a total Hamiltonian H such that $(H-E)|\psi\rangle = 0$, where $|\psi\rangle$ denotes the physical two-body scattering state. By introducing the (outgoing wave) Green's functions

$$G = (H - E - i\epsilon)^{-1}\,,$$
$$G_0 = (H_0 - E - i\epsilon)^{-1}\,, \tag{11}$$

one can formally define t as an operator by the relation

$$G = G_0 - G_0 t G_0\,, \tag{12}$$

and write the formal solution

$$|\psi\rangle = (1 - G_0 t)|\varphi\rangle\,. \tag{13}$$

It follows that

$$\langle lx|\psi\rangle = (2/\pi)^{1/2} i^l \left[j_l(\kappa x) - \int_0^\infty \frac{dp\, p^2 j_l(px)\langle lp|\,t\,|\varphi\rangle}{p^2/2m_r - E - i\epsilon}\right]. \tag{14}$$

If the potential is generated by particle exchange, and hence bounded by $\exp(-\mu x)$, one can evaluate the integral by the method of residues to obtain Eq. (1) (up to the previously neglected normalization), provided that one identifies $t_l(\kappa)=\langle l\kappa|t|\phi\rangle$. One thus generates the exterior representation by ignoring all singularities of $\langle lp|t|\phi\rangle$ in performing the p integration.

This procedure is readily extended to (nonrelativistic) three-particle states in the following manner. We denote the position of particle α by $\vec{r}_\alpha$ ($\alpha=1,2,3$), and define the reduced masses $\mu_\alpha{}^{-1}=m_\beta{}^{-1}+m_\gamma{}^{-1}$, and $M_\alpha{}^{-1}=m_\alpha{}^{-1}+(m_\beta+m_\gamma)^{-1}$, $\alpha\neq\beta\neq\gamma$. It is useful to introduce the vectors

$$\begin{aligned}\vec{R}&=\sum_\beta m_\beta\vec{r}_\beta\Big/\sum_\beta m_\beta,\\ \vec{x}_\alpha&=\vec{r}_\beta-\vec{r}_\gamma,\\ \vec{y}_\alpha&=\vec{r}_\alpha-(m_\beta\vec{r}_\beta+m_\gamma\vec{r}_\gamma)/(m_\beta+m_\gamma),\end{aligned}\tag{15}$$

and the corresponding momenta

$$\begin{aligned}\vec{P}&=\sum_\beta\vec{k}_\beta,\\ \vec{p}_\alpha&=\mu_\alpha(\vec{k}_\beta/m_\beta-\vec{k}_\gamma/m_\gamma),\\ \vec{q}_\alpha&=M_\alpha[\vec{k}_\alpha/m_\alpha-(\vec{k}_\beta+\vec{k}_\gamma)/(m_\beta+m_\gamma)],\end{aligned}\tag{16}$$

where $\vec{k}_\beta$ is the momentum of particle β, and $(\alpha\beta\gamma)$ are cyclic permutations of (123). It follows that

$$\sum_\beta\vec{r}_\beta\cdot\vec{k}_\beta=\vec{R}\cdot\vec{P}+\vec{x}_\alpha\cdot\vec{p}_\alpha+\vec{y}_\alpha\cdot\vec{q}_\alpha.\tag{17}$$

In addition to the total momentum $\vec{P}$, any two of the six momenta $\vec{p}_\alpha$, $\vec{q}_\alpha$ are linearly independent and serve to completely characterize the three-body state. The free Hamiltonian in the c.m. frame is now

$$H_0=-\nabla_{x_\alpha}{}^2/2\mu_\alpha-\nabla_{y_\alpha}{}^2/2M_\alpha,\tag{18}$$

and an incoming plane wave $|\Phi\rangle$ satisfies $(H_0-W)|\Phi\rangle=0$, where W is the total (kinetic) energy. A three-particle scattering state can be written formally as

$$|\Psi\rangle=(1-G_0T)|\Phi\rangle,\tag{19}$$

where $G_0=(H_0-W-i\epsilon)^{-1}$, and T is the three-body t matrix.

In view of the relation of T to the total Green's function, it is clear that the formal nature of Eq. (19) masks a considerable complexity. In particular, consistency with the two-particle description requires that for *each* $\alpha=(1,2,3)$

$$\langle\vec{x}_\alpha\vec{y}_\alpha|\Psi\rangle\xrightarrow[y_\alpha\to\infty]{}\langle\vec{x}_\alpha|\psi\rangle\langle\vec{y}_\alpha|\phi\rangle,\tag{20}$$

where $|\psi\rangle$ describes the two-body scattering of β and γ, and $|\phi\rangle$ represents α as a noninteracting spectator. In what follows we shall refer to this as the *quasi-two-body limit*. It was in order to simplify this situation that Faddeev introduced the channel decomposition, $T=\sum_\alpha T_\alpha$; one can then write

$$\begin{aligned}|\Psi\rangle&=|\Phi\rangle+\sum_\alpha|\psi^\alpha\rangle,\\ |\psi^\alpha\rangle&=-G_0T_\alpha|\Phi\rangle.\end{aligned}\tag{21}$$

In contrast to $|\Psi\rangle$, the channel state $|\psi^\alpha\rangle$ need only satisfy Eq. (20) for $y_\alpha\to\infty$, inasmuch as

$$\langle\vec{x}_\alpha\vec{y}_\alpha|\psi^\beta\rangle/\langle\vec{x}_\alpha\vec{y}_\alpha|\psi^\alpha\rangle\to 0\tag{22}$$

in that limit.

It is convenient at this point to make an angular momentum decomposition. Assuming an initial state of definite momenta $(\vec{p}_{\alpha_0}\vec{q}_{\alpha_0})$, we may couple $\vec{l}_0(\hat{p}_{\alpha_0})$ and $\vec{\lambda}_0(\hat{q}_{\alpha_0})$ to form the state $|\Phi\rangle=|LMl_0\lambda_0p_{\alpha_0}q_{\alpha_0}\rangle$. Correspondingly, we expand

$$\langle\vec{x}_\alpha\vec{y}_\alpha|\psi^\alpha\rangle=\sum_{LMl\lambda}Y_{LMl\lambda}(\hat{x}_\alpha,\hat{y}_\alpha)\psi^\alpha_{Ll\lambda}(x_\alpha,y_\alpha),\tag{23}$$

where

$$Y_{LMl\lambda}(\hat{x},\hat{y})=\sum_m C(\lambda lL;m,M-m)Y_{\lambda m}(\hat{y})Y_{lM-m}(\hat{x}).\tag{24}$$

For practical applications, the following alternative expression for $Y_{LMl\lambda}$ turns out to be quite useful. Let $\hat{n}$ denote an arbitrary direction in the $\vec{p}_\alpha\vec{q}_\alpha$ plane characterized by the Euler angles $(\alpha\beta\gamma)$, and let θ_z be defined by $\cos\theta_z=\hat{z}\cdot\hat{n}$ for any vector $\vec{z}$. Then[11]

$$Y_{LMl\lambda}(\hat{x},\hat{y})=\sum_{m\mu}C(\lambda lL;m\mu)D^{L*}_{M,m+\mu}(\alpha\beta\gamma)Y_{\lambda m}(\theta_y,0)\times Y_{l\mu}(\theta_x,0).\tag{25}$$

The three-particle generalization of Eq. (14) in this basis follows immediately from Eq. (21). We obtain

$$\psi^\alpha_{Ll\lambda}(x_\alpha,y_\alpha)=-\frac{2}{\pi}i^{l+\lambda}\int_0^\infty dq\,q^2j_\lambda(qy_\alpha)\int_0^\infty\frac{dp\,p^2j_l(px_\alpha)\langle LMl\lambda pq|T_\alpha|\Phi\rangle}{p^2/2\mu_\alpha+q^2/2M_\alpha-W-i\epsilon},\tag{26}$$

where we have dropped indices on the dummy variables, $(p_\alpha q_\alpha)\to(pq)$. Defining the on-shell value of p_α to be

$$\kappa_\alpha=[2\mu_\alpha(W-q_\alpha{}^2/2M_\alpha)]^{1/2}, \tag{27}$$

with the square-root branch cut chosen such that $\mathrm{Im}\kappa_\alpha\geq 0$, the exterior representation again follows from ignoring the p_α singularities of the T_α matrix element. We thus arrive at the expression

$$\psi^{\alpha;\mathrm{ext}}_{Ll\lambda}(x_\alpha,y_\alpha)=-2\mu_\alpha i^{l+\lambda+1}\int_0^\infty dq\,q^2 j_\lambda(qy_\alpha)\kappa_\alpha h_l(\kappa_\alpha x_\alpha)\times T^\alpha_{Ll\lambda}(q), \tag{28}$$

where we have defined

$$T^\alpha_{Ll\lambda}(q_\alpha)=\langle LMl\lambda\kappa_\alpha q_\alpha|T_\alpha|\Phi\rangle. \tag{29}$$

Due to the exponential damping provided by $h_l(\kappa_\alpha x_\alpha)$ when κ_α is (positive) imaginary, it is clear that the only waves which can propagate to large x_α arise from q_α such that κ_α is real; i.e., $q_\alpha{}^2\leq 2M_\alpha W$. For such q_α, $T^\alpha_{Ll\lambda}(q_\alpha)$ is precisely the on-shell (in the nonrelativistic sense) channel amplitude. Comparing Eqs. (1) and (28), we see that the asymptotic form is always specified precisely by appropriate matrix elements of the t operator, which is no surprise.

We now seek a set of BC's which, in analogy to the two-body problem, will enable us to determine the $T^\alpha_{Ll\lambda}$ (and hence all scattering observables). It is at this point that the constraints arising from our assumed knowledge of two-particle properties enter the picture. Recalling the quasi-two-body limit of Eq. (20), we have immediately that, if

$$\Psi^{\mathrm{ext}}(\vec{x}_\alpha,\vec{y}_\alpha)=\sum_\beta\langle\vec{x}_\alpha\vec{y}_\alpha|\psi^{\beta;\mathrm{ext}}\rangle+\langle\vec{x}_\alpha\vec{y}_\alpha|\Phi\rangle,$$

then

$$\lim_{y_\alpha\to\infty}[\partial_{x_\alpha}\Psi^{\mathrm{ext}}_{Ll\lambda}(x_\alpha,y_\alpha)/\Psi^{\mathrm{ext}}_{Ll\lambda}(x_\alpha,y_\alpha)]_{x_\alpha=a_\alpha}=\lambda_{\alpha l}(\kappa_\alpha{}^{02}), \tag{30}$$

where κ^0_α is the value of p_α in the incoming state. Here $\lambda_{\alpha l}(\kappa^2)$ is the same function defined in Eq. (6) with a channel index appended. To make this result more transparent, we note that if α is regarded as the entrance channel ($\alpha=\alpha_0$),

$$T^\alpha_{Ll\lambda}(q_\alpha)=\delta_{ll_0}\delta_{\lambda\lambda_0}\frac{\delta(q_\alpha-q_{\alpha_0})}{q_\alpha{}^2}t_{\alpha l}(\kappa^0_\alpha)+M^{\alpha;L}_{l\lambda l_0\lambda_0}(q_\alpha,q_{\alpha_0}), \tag{31}$$

where M^α arises from multiple scattering terms. In the large-y_α limit the $\delta(q_\alpha-q_{\alpha_0})$ term dominates, producing the behavior noted in Eq. (20).

We thus deduce that any BC of the form

$$\partial_{a_\alpha}\Psi^{\mathrm{ext}}_{Ll\lambda}(a_\alpha,y_\alpha)-\lambda_{\alpha l}\Psi^{\mathrm{ext}}_{Ll\lambda}(a_\alpha,y_\alpha)=F^\alpha_{Ll\lambda}(y_\alpha;W) \tag{32}$$

will automatically build in the two-particle constraints, provided that $F^\alpha_{Ll\lambda}$ tends to zero sufficiently rapidly for large y_α. With somewhat more effort one can show that $F^\alpha_{Ll\lambda}(y_\alpha;W)\propto\exp(-\mu y_\alpha)$, if μ is the mass of the lightest particle exchanged in the pairwise interactions. If one also considers the requirements of three-body unitarity, it turns out that $F^\alpha_{Ll\lambda}$ must have the form

$$F^\alpha_{Ll\lambda}(y_\alpha;W)=\sum_{\beta l'\lambda'}\int_0^\infty dq\,q^2B^{\alpha\beta;L}_{l\lambda l'\lambda'}(y_\alpha,q;W)T^\beta_{Ll'\lambda'}(q), \tag{33}$$

in which $B^{\alpha\beta}$ is an arbitrary real-valued function. Together, Eqs. (32) and (33) provide a natural generalization of Eq. (5). It should be apparent that the result is an implicit integral equation for the $T^\alpha_{Ll\lambda}(q)$; for a discussion of this and some fine points associated with the applicability of Eq. (32) we refer the reader to BCA.

In the context of potential theory, the author has recently shown that for a given set of two-particle phase shifts, a function $B^{\alpha\beta}$ with the stated properties can always be defined so as to reproduce the results of the conventional theory (e.g., the Faddeev equations).[12] A specific choice for $B^{\alpha\beta}$ is thus equivalent to stating the "off-shell" characteristics of the theory. In particular, a model in which $B^{\alpha\beta}\equiv 0$ corresponds to a picture in which the interaction is compressed to the surface of an impenetrable boundary, outside of which the behavior is immediately asymptotic. This is equivalent to the BC model of Feshbach and Lomon applied in the three-particle sector.[13] If one simultaneously goes to the zero range limit ($a_\alpha\to 0$), one obtains the "minimal" three-body equations of Amado.[14] As Amado points out, equations of this degree of complexity are a necessity if one is to achieve an exact solution of the three-body unitarity relations.

In concluding this section, we stress two related aspects of this approach. The first is that even for a crude approximation to $B^{\alpha\beta}$ (say $B^{\alpha\beta}\equiv 0$), we expect to produce reasonable three-body predictions in many cases. The reason for this expectation is primarily the short-range nature of the strong interaction. For systems with $n\geq 3$, the longest-range effect is due to single-particle exchange involving one of the real scattering particles; this is sandwiched between quasi-two-body scattering dominated by on-shell behavior. Thus, the fact that physical wave functions decay exponentially to their asymptotic form rather than switch abruptly at some radius has the character of a perturbation. The existence of this effect in the trinucleon system is exemplified by the equivalent success of simple separable models (or the BC model) in fitting the data, as compared to "realistic" models of the nuclear force.[8]

The second point is that a meaningful analysis of three-particle data should involve the careful separation of features which are a natural consequence of previously known information. Thus, if certain properties of the data require the introduction of explicit parameters into $B^{\alpha\beta}$ in order to achieve a fit, these parameters are truly significant in the sense of summarizing the content of new information in the experiment. This would be the case, for example, if strong three-body forces were present. As was shown in BCA, the BC formalism is especially efficient for such an analysis.

III. RELATIVISTIC GENERALIZATION

In contrast to the Faddeev theory, the BC formalism described in the last section has a remarkably simple extension to relativistic systems. If we denote the 4-vector displacement of particle α by $r_\alpha=(t_\alpha,\vec{r}_\alpha)$, our principal assumptions can be stated as follows:

(1) As a consequence of the short-range nature of the strong interaction, it is meaningful to define an *exterior region* by the requirements $(r_\beta-r_\gamma)^2<-a_\alpha{}^2$, $\alpha=(1,2,3)$. Once in the exterior, particle number is fixed and a wave-function description is valid.

(2) Each particle propagates in the exterior according to the appropriate free Hamiltonian with its physical mass (we deal with "out-states"). For a scalar particle we take $H_\alpha=(m_\alpha{}^2-\nabla_\alpha{}^2)^{1/2}$; for a spin-$\frac{1}{2}$ particle we would use $H_\alpha=i\vec{\alpha}\cdot\vec{\nabla}_\alpha+\beta m_\alpha$. If $|\Psi^{\text{ext}}\rangle$ describes the exterior, we thus require

$$i\frac{\partial}{\partial t_\alpha}|\Psi^{\text{ext}}\rangle=H_\alpha|\Psi^{\text{ext}}\rangle\,. \tag{34}$$

(3) We define the total 4-vector momentum operator $P=\sum_\beta(i\partial/\partial t_\beta,-i\vec{\nabla}_\beta)$, and require that

$$P^2|\Psi^{\text{ext}}\rangle=s|\Psi^{\text{ext}}\rangle\,, \tag{35}$$

where s is the square of the c.m. energy ($\sqrt{s}=\sum_\alpha m_\alpha+W$).

For the purposes of this paper we shall regard these assumptions as physically defensible and be concerned only with their consequences.

One consequence is immediately apparent if we go to a momentum representation in which $k_\alpha=(\epsilon_\alpha,\vec{k}_\alpha)$ is the 4-momentum of particle α. Up to numerical factors, which we introduce for convenience, we then deduce that the relativistic generalization of the three-particle propagator (G_0) is the operator $\mathcal{G}_0\equiv\mathcal{G}_0(s)$, where

$$\langle k_1k_2k_3|\mathcal{G}_0|k_1'k_2'k_3'\rangle=\prod_\beta\delta(k_\beta-k_\beta')\mathcal{G}_0(s|k_1k_2k_3)\,,$$
$$\mathcal{G}_0(s|k_1k_2k_3)=\frac{2\sum_\beta m_\beta\prod_\alpha 2m_\alpha\delta(k_\alpha{}^2-m_\alpha{}^2)\theta(\epsilon_\alpha)}{P^2-s-i\epsilon}\,. \tag{36}$$

It is clear that $\mathcal{G}_0$ has an obvious extension to the n-body case.

Given $\mathcal{G}_0$, a suitable exterior representation can be derived as follows. In analogy to Eq. (19), we take

$$|\Psi\rangle=(1-\mathcal{G}_0\mathcal{T})|\Phi\rangle\,,$$
$$\langle r_1r_2r_3|\Phi\rangle=\frac{1}{(2\pi)^6}\exp\Big(-i\sum_\beta r_\beta\cdot k_\beta^0\Big). \tag{37}$$

The presence of $\mathcal{G}_0$ as a factor puts $\mathcal{T}$ on the mass shell, $\epsilon_\alpha=(m_\alpha{}^2+\vec{k}_\alpha{}^2)^{1/2}$, for which

$$\langle k_1k_2k_3|\mathcal{T}|\Phi\rangle=\delta(\vec{P}-\vec{P}^0)\langle\vec{p}_\alpha\vec{q}_\alpha|T|\Phi\rangle\,. \tag{38}$$

Note that Eqs. (15)–(17) remain valid in this context, although the *values* of the various 3-vectors will be different in diffe ent Lorentz frames. The outgoing exterior wave i: then

$$\langle r_1r_2r_3|\mathcal{G}_0\mathcal{T}|\Phi\rangle=\frac{2\sum_\beta m_\beta}{(2\pi)^6}\int\prod_\alpha\left(\frac{m_\alpha d\vec{k}_\alpha}{\epsilon_\alpha}\right)\frac{\exp(-i\sum_\beta r_\beta\cdot k_\beta)\langle k_1k_2k_3|\mathcal{T}|\Phi\rangle}{(\sum_\beta k_\beta)^2-s-i\epsilon}\,. \tag{39}$$

Just as in the nonrelativistic case (and for the same reason) it is useful to expand $\mathcal{T}=\sum_\alpha\mathcal{T}_\alpha$; it is then convenient to evaluate the $\mathcal{T}_\alpha$ portion in the (instantaneous) c.m. of the $\beta\gamma$ subsystem. In this frame $\vec{P}^0=m_\alpha\vec{q}_\alpha/M_\alpha$, where $\vec{q}_\alpha$ is given in terms of the $\vec{k}_\beta$ by Eq. (16). We then obtain

$$\langle r_1r_2r_3|\mathcal{G}_0\mathcal{T}_\alpha|\Phi\rangle=\frac{2\sum_\beta m_\beta}{(2\pi)^6}\int d\vec{p}_\alpha d\vec{q}_\alpha\prod_\gamma\frac{m_\gamma}{\epsilon_{\alpha\gamma}}\frac{\exp\{-i\sum_\beta\epsilon_{\alpha\beta}t_\beta+i[\vec{y}_\alpha+(m_\alpha/M_\alpha)\vec{R}]\cdot\vec{q}_\alpha+i\vec{x}_\alpha\cdot\vec{p}_\alpha\}_\alpha\langle\vec{p}_\alpha\vec{q}_\alpha|T_\alpha|\Phi\rangle}{f_\alpha(p_\alpha,q_\alpha)-s-i\epsilon}\,, \tag{40}$$

where we have used $\prod_\beta d\vec{k}_\beta=d\vec{P}\,d\vec{p}_\alpha\,d\vec{q}_\alpha$ and defined the quantities

$$\epsilon_{\alpha\beta}=(m_\beta{}^2+p_\alpha{}^2)^{1/2}\,,\quad \beta\neq\alpha$$
$$\epsilon_{\alpha\alpha}=(m_\alpha{}^2+m_\alpha{}^2q_\alpha{}^2/M_\alpha{}^2)^{1/2}\,, \tag{41}$$
$$f_\alpha(p_\alpha,q_\alpha)=\Big(\sum_\beta\epsilon_{\alpha\beta}\Big)^2-\frac{m_\alpha{}^2}{M_\alpha{}^2}q_\alpha{}^2\,,$$

corresponding to the single-particle energies and the invariant momentum squared as evaluated in this frame (note that $\vec{p}_\alpha$, $\vec{q}_\alpha$ are only 3-vectors, so we can continue to use p_α, q_α for their magnitudes with no ambiguity). The notation ${}_\alpha\langle\vec{p}_\alpha\vec{q}_\alpha|T_\alpha|\Phi\rangle$ refers to the operator T defined in Eq. (38) *as evaluated in the* $\beta\gamma$ c.m. frame; in that frame it is clearly a function of the two independent 3-momenta $\vec{p}_\alpha$, $\vec{q}_\alpha$.

We now observe that in the vicinity of the pole $[(f_\alpha(p_\alpha,q_\alpha)=s]$

$$\frac{2\sum_\beta m_\beta\prod_\gamma m_\gamma/\epsilon_{\alpha\gamma}}{f_\alpha(p_\alpha,q_\alpha)-s}\to\frac{\Gamma_\alpha(q_\alpha,s)2\mu_\alpha\omega_\alpha(\kappa_\alpha)}{p_\alpha{}^2-\kappa_\alpha{}^2}\,, \tag{42}$$

where

$$\omega_\alpha{}^{-1}(\kappa_\alpha)=\mu_\alpha(\epsilon_{\alpha\beta}{}^{-1}+\epsilon_{\alpha\gamma}{}^{-1})\,, \tag{43a}$$

$$\Gamma_\alpha(q_\alpha,s)=\frac{\sum_\beta m_\beta}{\sum_\beta\epsilon_{\alpha\beta}}\prod_\beta\frac{m_\beta}{\epsilon_{\alpha\beta}}\,, \tag{43b}$$

and κ_α is the relativistic on-shell value of p_α defined by $f_\alpha(\kappa_\alpha,q_\alpha)=s$. Specifically, κ_α is the solution of

$$(m_\beta{}^2+\kappa_\alpha{}^2)^{1/2}+(m_\gamma{}^2+\kappa_\alpha{}^2)^{1/2}=\Big(s+\frac{m_\alpha{}^2}{M_\alpha{}^2}q_\alpha{}^2\Big)^{1/2}-\Big(m_\alpha{}^2+\frac{m_\alpha{}^2}{M_\alpha{}^2}q_\alpha{}^2\Big)^{1/2}\,. \tag{44}$$

Note that the functions Γ_α, ω_α go independently to unity in the nonrelativistic limit. If $\tau_{\alpha l}(\kappa_\alpha)$ is the proper *relativistic* two-particle c.m. amplitude, $\omega_\alpha(\kappa_\alpha)$ defines the relativistic correction to Eq. (2); i.e.,

$$\tau_{\alpha l}(\kappa_\alpha)=-\omega_\alpha{}^{-1}(\kappa_\alpha)\frac{e^{i\delta_l}\sin\delta_l}{\pi\mu_\alpha\kappa_\alpha}\,. \tag{45}$$

As usual, the exterior representation is obtained by evaluating the p_α integral of Eq. (40) while ignoring the singularities of T_α. If we choose the origin such that $\vec{R}=0$, and make the angular momentum decomposition introduced earlier, we arrive at the expression

$$\psi^{\alpha;\text{ext}}_{Ll\lambda}(x_\alpha,y_\alpha;t_1t_2t_3)=-\frac{2\mu_\alpha}{(2\pi)^3}i^{l+\lambda+1}\int_0^\infty dq\,q^2j_\lambda(qy_\alpha)\exp\Big(-i\sum_\beta\epsilon_{\alpha\beta}t_\beta\Big)\kappa_\alpha h_l(\kappa_\alpha x_\alpha)T^\alpha_{Ll\lambda}(q)\,, \tag{46}$$

where we have defined

$$T^\alpha_{Ll\lambda}(q_\alpha)=\Gamma_\alpha(q_\alpha,s)\omega_\alpha(\kappa_\alpha)\,{}_\alpha\langle LMl\lambda\kappa_\alpha q_\alpha|T_\alpha|\Phi\rangle\,. \tag{47}$$

Comparing to Eqs. (28) and (29), we observe that aside from the presence of the exponential involving the times t_β the only difference in the relativistic version lies in the kinematical relation between q and κ_α and the relationship between $T^\alpha_{Ll\lambda}$ and the physical amplitude [the extra factor of $(2\pi)^{-3}$ is common to $|\Phi\rangle$ and can be ignored]. In the nonrelativistic limit (m_α large compared to the momenta), it is apparent that the time dependence can also be factored out and the resultant expressions are identical.

Having established the exterior representation, our next step is to apply BC's of the type discussed earlier in order to derive an equation for the $T^\alpha_{Ll\lambda}$. It is perhaps obvious that the natural generalization is to apply the BC stated in Eq. (32) *at equal times* $t_\alpha=t$ in the $\beta\gamma$ c.m. One can then establish that the resultant amplitudes do not depend on t, so we can simply set $t=0$. The net result is a set of coupled one-variable integral equations, differing from those stated in BCA primarily by the relativistic kinematics. We defer the specifics to the Appendix.

The input to these equations is again the empirical function $\lambda_l(\kappa^2)$, which is related to the two-particle phase shifts via Eqs. (1) and (7). Since both of these expressions are regarded as applying in the two-body c.m. frame, no difference in interpretation arises except the different kinematical relationship between κ and the two-particle invariant energy. There is one new feature, however, in the structure of the integral equations as a result of Eq. (44). Since q_α is a real spectator momentum, it is clear that the right-hand side of

Eq. (44) is real and positive for $\sqrt{s} > m_\alpha$. In particular, as $q_\alpha \to 0$, the right-hand side approaches $\sqrt{s} - m_\alpha$. Inasmuch as the left-hand side is a sum of two square roots, with the branch cut taken along the negative real axis, it is clear that the equality can be satisfied only if $\kappa_\alpha^2 > -\text{Min}(m_\beta^2, m_\gamma^2)$. We thus distinguish two cases: (a) $m_\beta = m_\gamma$, in which case we can allow any q_α on the interval $(0,\infty)$, and (b) $m_\beta \neq m_\gamma$, in which case q_α is restricted to a *finite* range $(0, Q_\alpha)$, with Q_α defined by Eq. (44), with κ_α^2 set equal to its minimum value. Specifically,

$$Q_\alpha^2 = \frac{M_\alpha^2}{m_\alpha^2} \frac{[s - (m_\alpha + |m_\beta^2 - m_\gamma^2|^{1/2})^2][s - (m_\alpha - |m_\beta^2 - m_\gamma^2|^{1/2})^2]}{4|m_\beta^2 - m_\gamma^2|}. \tag{48}$$

This behavior is significantly different from the nonrelativistic version, in which $\kappa_\alpha^2 \to -\infty$ and $q_\alpha \to +\infty$ regardless of the mass ratios. It is clear, of course, that the differences arise at momenta for which the nonrelativistic approximation is invalid. For example, if we were to consider πd scattering at threshold in the πN channel with a nucleon spectator, we would obtain $q_\alpha < Q_\alpha \simeq (\mu M)^{1/2}$, corresponding to a momentum of $\simeq 2(\mu M)^{1/2}$ for the spectator in the πN c.m. frame (0.7 GeV/c). What is more interesting is that there are also differences with the Faddeev-type relativistic equations discussed earlier. Thus, if we denote the invariant energy squared of particles β and γ by σ_α, our formalism requires that

$$|m_\beta^2 - m_\gamma^2| \leq \sigma_\alpha \leq (\sqrt{s} - m_\alpha)^2, \tag{49}$$

whereas $\sigma_\alpha \to -\infty$ in the equations of Lovelace or Omnes. This is very convenient for us in that the analytic continuation defined by Eq. (8) need not be employed too far from the physical region; the extent to which it may be responsible for the rather different results we shall report is unclear at this time (we shall return to this point in Sec. VI).

The structure of Eq. (44) has also another, rather amusing consequence. We note that if $\sqrt{s} < m_\alpha$, the right-hand side becomes real and negative for all q_α and the equality cannot be satisfied. Thus, if we were to use the equations to define an analytic continuation of the three-body amplitude below the scattering threshold ($W = 0$), the contribution of channel α would vanish for $\sqrt{s} < m_\alpha$. In particular, for $\sqrt{s} < \text{Min}(m_\alpha)$ all of the T_α vanish identically; *there is no exterior representation.* It then follows that if a three-body system is more tightly bound than its lightest constituent, there is no smooth continuation to the domain in which the particles are asymptotically free. This point becomes less academic if we treat the scattering of such a particle with some "elementary" particle as a four-body problem according to the $n = 4$ realization of this formalism. The resultant wave function would contain no outgoing piece corresponding to the three constituents as free particles. Thus, if the lightest nonstrange quark were to have a mass in excess of M, the nucleon could not be decomposed by scattering. The description of such quarks would require the natural complement of this formalism: a purely *interior* representation.

All the information necessary to compute the $T^{\alpha}_{Ll\lambda}$ functions is given explicitly in the Appendix. As we shall demonstrate elsewhere, the use of the Dirac Hamiltonian for H_α does not affect things in a material way; one can again derive an exterior representation which leads to precisely the same equations except for the usual spin-recoupling coefficients familiar in Faddeev theory. The necessary modifications will be indicated in Sec. V when we put isospin into the 3π problem.

IV. PHYSICAL AMPLITUDES AND UNITARITY

In this section we present the necessary formulas for calculating the physical amplitudes from our formalism. We also employ a concise operator notation and some earlier results in order to provide an explicit proof of the three-particle unitarity relations.

If we first consider the description of the scattering process in the three-body c.m., it is clear that any pair $(\vec{p}_\beta^{\,\text{c.m.}}, \vec{q}_\beta^{\,\text{c.m.}})$ of the six vectors defined by Eq. (16) in this frame can be used as independent variables; they are of course restricted by the condition $\sum_\beta \epsilon_\beta = \sqrt{s}$. On the other hand, $\mathcal{T}$ has been written as the sum of the $\mathcal{T}_\alpha$ channel amplitudes, which are most easily expressed in terms of the corresponding pair $(\vec{p}_\alpha^{\,\text{c.m.}}, \vec{q}_\alpha^{\,\text{c.m.}})$. It is thus convenient to define a transformation between these different labels in a given frame of reference. We therefore introduce a Hilbert space of states $|\alpha\vec{p}\vec{q}\rangle$, where the α index tells us that $(\vec{p}, \vec{q})$ are to be interpreted as the numerical values of $(\vec{p}_\alpha, \vec{q}_\alpha)$. These states are taken to satisfy the normalization condition[15]

$$\langle \alpha\vec{p}\vec{q} | \beta\vec{p}'\vec{q}'\rangle = \delta_{\alpha\beta}\delta(\vec{p} - \vec{p}')\delta(\vec{q} - \vec{q}'), \tag{50}$$

with the corresponding completeness relation

$$\int d\vec{p}\, d\vec{q} \sum_\beta |\beta\vec{p}\vec{q}\rangle\langle\beta\vec{p}\vec{q}| = 1. \tag{51}$$

We define an operator I on this space which "interconnects" the various channels by

$$\langle \alpha\vec{p}'\vec{q}'|I|\beta\vec{p}\vec{q}\rangle = -\delta\left(\vec{p}+\frac{\mu_\beta}{m_\gamma}\vec{p}'+\frac{\mu_\beta}{M_\alpha}\vec{q}'\right)\delta\left(\vec{q}-\vec{p}'+\frac{\mu_\alpha}{m_\gamma}\vec{q}'\right) \text{ if } \alpha\beta\gamma \text{ are cyclic},$$
$$= -\delta\left(\vec{p}+\frac{\mu_\beta}{m_\gamma}\vec{p}'-\frac{\mu_\beta}{M_\alpha}\vec{q}'\right)\delta\left(\vec{q}+\vec{p}'+\frac{\mu_\alpha}{m_\gamma}\vec{q}'\right) \text{ if } \beta\alpha\gamma \text{ are cyclic}. \quad (52)$$

The interpretation of I is that if $(\vec{p}',\vec{q}')$ are the values of $(\vec{p}_\alpha,\vec{q}_\alpha)$, then $(\vec{p},\vec{q})$ are the values of $(\vec{p}_\beta,\vec{q}_\beta)$ in that frame. As a consequence of Eq. (52), one can deduce that

$$I=I^T,\quad I^{-1}=\tfrac{1}{2}(1+I),\quad (1-I)=\tfrac{1}{3}(1-I)^2. \quad (53)$$

If we now consider an initial state $|\Phi\rangle = |\alpha_0\vec{p}_0\vec{q}_0\rangle$ in the three-body c.m., and write

$$\langle k_1k_2k_3|\mathcal{T}|\Phi\rangle = \delta(\vec{P})\langle\alpha\vec{p}\vec{q}|T|\alpha_0\vec{p}_0\vec{q}_0\rangle \quad (54)$$

for the c.m. amplitude T, it follows that

$$T=(1-I)\tau, \quad (55)$$

where

$$\langle\beta\vec{p}'\vec{q}'|\tau|\alpha_0\vec{p}_0\vec{q}_0\rangle = \langle\beta\vec{p}'\vec{q}'|T_\beta|\alpha_0\vec{p}_0\vec{q}_0\rangle$$

is the channel amplitude.

The integral equations given in the Appendix provide a means of calculating the (on-shell) operator $\hat{\tau}$. Thus, expanding

$$\langle\beta\vec{p}\vec{q}|\hat{\tau}|\alpha_0\vec{p}_{0\alpha}\vec{q}_{0\alpha}\rangle = \sum_{LM}\sum_{l\lambda l_0\lambda_0} Y_{LMl\lambda}(\hat{p},\hat{q})\,Y_{LMl_0\lambda_0}(\hat{p}_{0\alpha},\hat{q}_{0\alpha})\,\langle\beta LMl\lambda pq|\hat{\tau}|\alpha_0LMl_0\lambda_0p_{0\alpha}q_{0\alpha}\rangle, \quad (56)$$

we have

$$\langle\beta LMl\lambda\kappa_\beta q|\hat{\tau}|\alpha_0LMl_0\lambda_0p_{0\alpha}q_{0\alpha}\rangle \equiv T^\beta_{Ll\lambda}(q). \quad (57)$$

Here $(\vec{p}_{0\alpha},\vec{q}_{0\alpha})$ are the values of $(\vec{p}_{\alpha_0},\vec{q}_{\alpha_0})$ in the $\beta_0\gamma_0$ c.m., provided their values in the three-body c.m. are $(\vec{p}_0,\vec{q}_0)$. The amplitude $T^\beta_{Ll\lambda}(q)$ is the same quantity defined in Eq. (47) of this paper, and in Eq. (19) of BCA.

We must thus provide the connection between the operators τ and $\hat{\tau}$, for which it is necessary to introduce the appropriate Lorentz transformations. Assuming that the vectors describing the on-shell state are $(\vec{p}^{\,\text{c.m.}}_\alpha,\vec{q}^{\,\text{c.m.}}_\alpha)$, it is straightforward to show that the corresponding quantities in the $\beta\gamma$ c.m. are

$$\vec{q}_\alpha = \frac{M_\alpha}{m_\alpha}\frac{\gamma_\alpha\sqrt{s}}{\sqrt{s}-\epsilon_\alpha}\vec{q}^{\,\text{c.m.}}_\alpha,$$
$$\vec{p}_\alpha = \vec{p}^{\,\text{c.m.}}_\alpha + \left[(\gamma_\alpha-1)\frac{\vec{p}^{\,\text{c.m.}}_\alpha\cdot\vec{q}^{\,\text{c.m.}}_\alpha}{q^{\text{c.m.}2}_\alpha}+\frac{\gamma_\alpha\mu_\alpha}{\sqrt{s}-\epsilon_\alpha}\left(\frac{\epsilon_\beta}{m_\beta}-\frac{\epsilon_\gamma}{m_\gamma}\right)\right]\vec{q}^{\,\text{c.m.}}_\alpha, \quad (58)$$

where

$$\beta_\alpha = q^{\text{c.m.}}_\alpha/(\epsilon_\beta+\epsilon_\gamma),\quad \gamma_\alpha=(1-\beta_\alpha^2)^{-1/2}, \quad (59)$$

and the ϵ_β are expressed in terms of $(\vec{p}^{\,\text{c.m.}}_\alpha,\vec{q}^{\,\text{c.m.}}_\alpha)$ via

$$\epsilon_\alpha = (q^{\text{c.m.}2}_\alpha+m_\alpha^2)^{1/2},$$
$$\epsilon_\beta = \left(\left|\vec{p}^{\,\text{c.m.}}_\alpha-\frac{\mu_\alpha}{m_\gamma}\vec{q}^{\,\text{c.m.}}_\alpha\right|^2+m_\beta^2\right)^{1/2}, \quad (60)$$
$$\epsilon_\gamma = \left(\left|\vec{p}^{\,\text{c.m.}}_\alpha+\frac{\mu_\alpha}{m_\beta}\vec{q}^{\,\text{c.m.}}_\alpha\right|^2+m_\gamma^2\right)^{1/2},$$

$\alpha\beta\gamma$ cyclic. To incorporate this information into our operator notation, we define the functions

$$u_\alpha(\vec{p},\vec{q}) = (\gamma_\alpha-1)\frac{\vec{p}\cdot\vec{q}}{q^2}+\frac{\gamma_\alpha\mu_\alpha}{\epsilon_\beta+\epsilon_\gamma}\left(\frac{\epsilon_\beta}{m_\beta}-\frac{\epsilon_\gamma}{m_\gamma}\right),$$
$$v_\alpha(\vec{p},\vec{q}) = \frac{M_\alpha}{m_\alpha}\frac{\gamma_\alpha\sum_\beta\epsilon_\beta}{\epsilon_\beta+\epsilon_\gamma}, \quad (61)$$

and the operator Λ such that

$$\langle\beta\vec{p}\vec{q}|\Lambda|\alpha\vec{p}'\vec{q}'\rangle = \delta_{\alpha\beta}v_\alpha^3\prod_\gamma\frac{\epsilon_{\alpha\gamma}}{\epsilon_\gamma}\delta(\vec{p}'-\vec{p}-u_\alpha\vec{q})\,\delta(\vec{q}'-v_\alpha\vec{q}). \quad (62)$$

Thus Λ takes the c.m. quantities $(\vec{p},\vec{q})$ into the $\beta\gamma$ c.m. quantities $(\vec{p}',\vec{q}')$. Similarly, we can define the inverse transformation Λ^{-1} by

$$\langle\beta\vec{p}'\vec{q}'|\Lambda^{-1}|\alpha\vec{p}\vec{q}\rangle = \delta_{\alpha\beta}\bar{v}_\alpha^3\prod_\gamma\frac{\epsilon_\gamma}{\epsilon_{\alpha\gamma}}\delta(\vec{p}-\vec{p}'-\bar{u}_\alpha q')\delta(\vec{q}-\bar{v}_\alpha\vec{q}')$$
$$= \delta_{\alpha\beta}\,\delta(\vec{p}'-\vec{p}-u_\alpha\vec{q})\delta(\vec{q}'-v_\alpha\vec{q}), \quad (63)$$

where

$$\bar{u}_\alpha(\vec{p}',\vec{q}') = (\gamma_\alpha - 1)\frac{\vec{p}'\cdot\vec{q}'}{q'^2} - \frac{m_\alpha\mu_\alpha}{M_\alpha f_\alpha^{1/2}(p',q')}\left(\frac{\epsilon_{\alpha\beta}}{m_\beta} - \frac{\epsilon_{\alpha\gamma}}{m_\gamma}\right), \tag{64a}$$

$$\bar{v}_\alpha(\vec{p}',\vec{q}') = \frac{m_\alpha}{M_\alpha f_\alpha^{1/2}(p',q')}(\epsilon_{\alpha\beta}+\epsilon_{\alpha\gamma}), \tag{64b}$$

$$\gamma_\alpha = \frac{[f_\alpha(p',q') + (m_\alpha^2/M_\alpha^2)q'^2]^{1/2}}{f_\alpha^{1/2}(p',q')}. \tag{64c}$$

Here the $\epsilon_{\alpha\beta}$ are given in terms of p and q by Eq. (41), and $v_\alpha \bar{v}_\alpha = 1$.

In order to complete the relationship between τ and $\hat{\tau}$, we see from Eq. (47) that in addition to the transformation laws for the labels $(\vec{p},\vec{q})$ given above, we also need to know the Lorentz transformation properties of the T_α operator. This turns out to be trivial since, as we shall see below, τ is a Lorentz invariant. Thus, in order to incorporate the implied relationship into our operator notation, we need only take into account the transformation properties of $\delta(\vec{P}-\vec{P}_0)$ via the factor $\bar{v}_\alpha^3$. We therefore define the diagonal operator χ such that

$$\chi_\alpha(\vec{p},\vec{q}) = \Gamma_\alpha(q,f_\alpha)\omega_\alpha(p)\prod_\beta \epsilon_{\alpha\beta}/\epsilon'_\beta, \tag{65}$$

where the ϵ'_β are the single-particle energies in the three-body c.m. frame. Comparing Eqs. (47) and (57), we deduce that

$$\tau = \Lambda\chi^{-1}\hat{\tau}\Lambda^{-1}, \tag{66}$$

which reduces to an identity in the nonrelativistic limit. Inasmuch as none of the Lorentz transformations take the set $(\vec{p},\vec{q})$ out of the scattering plane, the Euler angles $(\alpha\beta\gamma)$ in the representation of Eq. (25) remain the same. Thus, as in the nonrelativistic case, the c.m. amplitude T (in a state of definite LM) can be expressed in terms of a single integral over the variable $\hat{p}\cdot\hat{q}$ involving the $T^{\beta}_{Ll\lambda}$ amplitudes.

We now consider the statement of three-particle unitarity for our amplitudes. It will be convenient to adopt the notation

$$\Delta t = t(s+i\epsilon) - t(s-i\epsilon) \equiv t_+ - t_- \tag{67}$$

for the discontinuity of an operator across its cut. In addition to the three-body elastic cut with threshold at $W=0$, there will also be cuts corresponding to elastic scattering from two-particle bound states. The discontinuity relations pertinent to the latter are relatively easy to satisfy and we shall not be concerned with them here (a simple proof can be constructed along the lines given in SCI). The development given below will still be relevant in the presence of such thresholds, provided that we interpret the discontinuity as applying to just the three-particle portion of the overlying cuts. We shall first demonstrate that $\mathcal{T}$ satisfies the relation

$$\langle k_1 k_2 k_3 | \Delta\mathcal{T} + \mathcal{T}_+ \Delta\mathcal{G}_0 \mathcal{T}_- | k_1' k_2' k_3'\rangle = 0. \tag{68}$$

We proceed by introducing an operator g^0 which corresponds to $\mathcal{G}_0$ on our Hilbert space. We thus define the diagonal operator g^0 such that

$$g^0_\alpha(\vec{p},\vec{q}) = \frac{2\sum_\beta m_\beta \prod_\gamma m_\gamma/\epsilon_\gamma}{(\sum_\beta \epsilon_\beta)^2 - s - i\epsilon}, \tag{69}$$

where $(\vec{p},\vec{q})$ are taken as the values of $(\vec{p}_\alpha,\vec{q}_\alpha)$ in the three-body c.m. In that frame, Eq. (68) can be expressed in terms of the operator T defined in Eqs. (54) and (55); thus

$$\Delta T = -\tfrac{1}{3}T_+\Delta g^0 T_-. \tag{70}$$

Here the factor of $\frac{1}{3}$ arises because of the completeness relation of Eq. (51) and the fact that

$$\int d\vec{p}\, d\vec{q}(1-I)|\alpha\vec{p}\vec{q}\rangle\langle\alpha\vec{p}\vec{q}|(1-I)$$

is independent of α, as can be verified from Eq. (52). Since the equations given in the Appendix imply that τ (and hence T) has a final factor of $(1-I)$ on the right, Eq. (70) is equivalent to the relation

$$\Delta T = -\int d\vec{p}\, d\vec{q}\, T_+|\alpha\vec{p}\vec{q}\rangle \Delta g^0_\alpha(\vec{p},\vec{q})\langle\alpha\vec{p}\vec{q}|T_-, \tag{71}$$

since I and g^0 commute.

Stated as operator relations, the equations for $\hat{\tau}$ have the same structure as the equations studied in SCI, and hence one must obtain the same result,

$$\Delta\hat{\tau} = -\hat{\tau}_+\Delta G_0 \hat{\tau}_-, \tag{72}$$

where G_0 is defined by

$$\langle\alpha\vec{p}\vec{q}|G_0|\beta\vec{p}'\vec{q}'\rangle = \frac{2\mu_\alpha\delta_{\alpha\beta}\delta(\vec{p}-\vec{p}')\delta(\vec{q}-\vec{q}')}{p^2-\kappa_\alpha^2 - i\epsilon}. \tag{73}$$

Thus, except for the relativistic kinematics reflected in the difference between Eqs. (27) and (44), G_0 is formally identical to the nonrelativistic Green's function of Sec. II. Recalling Eq. (42), the definitions of G_0, g^0, and χ imply the relation

$$\Delta G_0\chi\Lambda^{-1} = \Lambda^{-1}\Delta g^0, \tag{74}$$

since the discontinuity takes G_0 and g^0 to the pole. Invoking Eqs. (66) and (72), we finally obtain

$$\begin{aligned}\Delta\tau &= -\tau_+\Lambda\Delta G_0\chi\Lambda^{-1}\tau_- \\ &= -\tau_+\Delta g^0\tau_- \\ &= -\tfrac{1}{3}\tau_+(1-I)\Delta g^0\tau_-,\end{aligned} \tag{75}$$

where we have employed Eq. (53) and the above noted property of τ. Using Eq. (55) and the commutativity of I and g^0, we have proved Eq. (70), and hence Eq. (68).

We now observe that if one defines a three-body scattering amplitude according to the conventions stated in Goldberger and Watson,[16] one deals with an operator $\mathcal{T}'$ related to the S matrix by the equation

$$S_{fi} = \delta_{fi} - 2\pi i\, \mathcal{T}'_{fi}\,. \tag{76}$$

In analogy to Eq. (38), one defines T' such that

$$\langle k_1 k_2 k_3 | \mathcal{T}' | k_1' k_2' k_3' \rangle = \delta(\vec{P} - \vec{P}')\langle \alpha \vec{p}\vec{q}\,|\, T' \,|\, \alpha \vec{p}'\vec{q}'\rangle\,. \tag{77}$$

Unitarity then requires that T' satisfy the discontinuity relation

$$\Delta T' = -2\pi i \int d\vec{p}\, d\vec{q}\; T'_+ |\alpha\vec{p}\vec{q}\rangle\, \delta(\sqrt{s} - \textstyle\sum_\beta \epsilon_\beta)\langle \alpha \vec{p}\vec{q}\,|\, T'_- \tag{78}$$

in the three-body c.m. On the other hand, Eq. (71) can be reexpressed as

$$\Delta T = -2\pi i \int d\vec{p}\, d\vec{q}\; T_+ |\alpha \vec{p}\vec{q}\rangle\, \overline{\Gamma}_\alpha(\vec{p},\vec{q}\,)\delta \times (\sqrt{s} - \textstyle\sum_\beta \epsilon_\beta)\langle \alpha\vec{p}\vec{q}\,|\, T_-\,, \tag{79}$$

where

$$\overline{\Gamma}_\alpha(\vec{p},\vec{q}\,) = \frac{\sum_\beta m_\beta}{\sqrt{s}} \prod_\gamma \frac{m_\gamma}{\epsilon_\gamma}\,. \tag{80}$$

We therefore infer the relation

$$T' = (\overline{\Gamma})^{1/2}\, T (\overline{\Gamma})^{1/2}\,, \tag{81}$$

connecting T to the conventional amplitude. In view of the transformation properties of $\mathcal{T}'$ (see Ref. 16), Eq. (81) implies that T is a Lorentz invariant.

Together with the integral equations for $\hat{\tau}$ stated in the Appendix, Eqs. (55), (66), and (81) provide the information necessary to calculate the physical amplitudes and hence all scattering observables. As an example, we present numerical results for the three-pion system in the next section.

V. APPLICATIONS TO THE 3π SYSTEM

As a first test of our relativistic formalism, we consider the $J^\pi = 0^-, 1^-$ scattering states ($I = 0, 1, 2$) of the three-pion system. This permits a direct comparison with the previous results noted above.[5,6] For c.m. energies $\sqrt{s} \leq 2$ GeV, one would hope to predict the $\omega(784)$ as a 1^- isoscalar 3π resonance, and perhaps the π itself as a 3π bound state in the 0^- ($I = 1$) channel. In order for such results to be meaningful, no other resonant or bound states should be predicted by the theory. Of course, since it can be shown that our equations constitute a general solution of the three-particle unitarity relations, it is clear that some choice of the A operator defined in the Appendix [analogous to B in Eq. (33)] can always reproduce the physics. Thus, if the qualitative characteristics of the 3π system are to be regarded as a natural consequence of the known two-particle properties (e.g., the ρ), the predicted properties must follow given any reasonable guess as to the off-shell behavior. The numerical results given below demonstrate that this is indeed the case for the channels considered.

As noted in the Appendix, the equations for $\hat{\tau}$ can be expressed as an operator relation $\hat{\tau} = -g\rho X$, where X satisfies

$$X = \Omega + (K_1 + AK_2)X\,, \tag{82}$$

written in terms of states $|\alpha l \lambda q\rangle$ defined (for fixed L) such that

$$\langle \beta l' \lambda' q' | \alpha l \lambda q\rangle = \delta_{\alpha\beta}\delta_{ll'}\delta_{\lambda\lambda'}\frac{\delta(q'-q)}{q^2}. \tag{83}$$

The quantities Ω, K_1, K_2 are totally specified in terms of on-shell two-body information (scattering phase shifts) via the parametrization of $\lambda_l(\kappa^2)$ in Eq. (8), whereas A summarizes both off-shell effects and possible three-body forces.

In order to apply these equations to actual physical systems, one will in general need to add discrete indices describing additional degrees of freedom (spin, isospin). In the present application we are concerned only with isospin, and the necessary modification of Eq. (82) consists of the replacement

$$\langle \beta l' \lambda' q' | K_1 | \alpha l \lambda q\rangle \to C^{I;\,\beta\alpha}_{i'\,;\,i} \langle \beta i' l' \lambda' q' | K_1 | \alpha i l \lambda q\rangle\,, \tag{84}$$

where i' represents the total isospin of the pair $\alpha\gamma$, i similarly labels the $\beta\gamma$ pair, and the expansion of the basis to $|\alpha i l \lambda q\rangle$ merely reminds us that the on-shell two-body parameters contained in K_1 will also depend on i, i'. The matrix $C^{I;\,\beta\alpha}_{i'\,;\,i}$ is just the overlap between a state formed by coupling $(\beta\gamma)_{I=i} + (\alpha)_{I=1}$ to form a state of definite total I, vs coupling $(\gamma\alpha)_{I=i'} + (\beta)_{I=1}$ to total I. Hence C^I is a spin-recoupling coefficient,

$$\begin{aligned} C^{I;\,\beta\alpha}_{i'\,;\,i} &= (-)^N [(2i'+1)(2i+1)]^{1/2} W(i_\alpha i_\gamma I i_\beta\,; i' i)\,,\\ N &= i' + i_\beta - I \ \text{ if } \alpha\beta\gamma \text{ cyclic}\,,\\ &= i + i_\alpha - I \ \text{ if } \beta\alpha\gamma \text{ cyclic}\,, \end{aligned} \tag{85}$$

where i_β is the isospin of particle β, and W is the Racah coefficient as defined by Rose.[11] If our particles had spin which could couple to values s', s in pairs, an additional factor of $C^{S;\,\beta\alpha}_{s'\,;\,s}$ would appear in Eq. (84). For the calculations to be described, we shall need the explicit values

$$C_{00}^{1;\beta\alpha}=\tfrac{1}{3}\,,$$
$$C_{11}^{0;\beta\alpha}=C_{11}^{2;\beta\alpha}=1\,, \tag{86}$$
$$C_{11}^{1;\beta\alpha}=C_{11}^{2;\beta\alpha}=-\tfrac{1}{2}\,.$$

Taking the most interesting case first, we consider the 1^-, $I=0$ (ω) channel. Here, in order to form an isoscalar we need $i=i'=1$, and hence the statistics restrict us to *odd* angular momenta l, l'. Given the strong energy dependence in the p-wave (ρ) channel, plus the angular momentum barriers, one would expect that $l \geq 3$ states could be neglected for energies up to a few GeV. Given $l=1$, we can only have $\lambda=1$ in order to produce a 1^- 3π state. The relevant equation is then particularly simple; Eq. (82) reduces to

$$X(q')=\Omega(q')+\int_0^\infty dq\,q^2\frac{\tilde{N}(q',q;\kappa)}{D_1(\kappa)}X(q)\,, \tag{87}$$

where $D_l(\kappa)$ is defined in Eq. (9), and

$$N(q',q;\kappa)=\sum_\beta N_{1111}^{\alpha\beta;1}(q',q)\,,$$
$$\tilde{N}(q',q;\kappa)=N(q';q;\kappa)+\kappa^{-1}A(q',q;s)\,. \tag{88}$$

Here $N_{l\lambda l'\lambda'}^{\alpha\beta;L}$ is the function defined in the Appendix (the summation is a result of having identical particles), and A is a real-valued function summarizing the off-shell content. In view of the discussion preceding Eq. (33), one would expect A to exhibit exponentially damped behavior in the coordinate representation; this corresponds to poles or branch cuts at complex momenta $q'=i\mu_c$. In addition, if one studies simple off-shell models, one expects an explicit energy dependence of the form $A(q',q;s)\propto(\kappa^2+\mu_c^2)$. The value of μ_c corresponds to the longest-range component of the π-π interaction, so we expect $\mu_c\simeq 2\mu$.

Since it is our intention to study the effect of the off-shell properties on our result, we have taken advantage of the fact that $N^{\alpha\beta;L}$ has an explicit dependence on the momentum $\kappa\equiv\kappa(q,s)$ in writing Eq. (88). It is then convenient to rewrite $\tilde{N}$ in the form

$$\tilde{N}(q',q;\kappa)=N(q',q;\kappa)-\frac{i\mu}{\kappa}N(q',q;i\mu)+\kappa^{-1}\bar{A}(q',q;s)\,; \tag{89}$$

this is justified because $i\mu N(q',q;i\mu)$ has precisely the same analytic structure as A (a cut for $\mathrm{Im}q'\geq\mu$). The point of this decomposition is that as $q\to\infty$, $\kappa\to i\mu$, so the first two terms of Eq. (89) combine to give better convergence properties for the kernel. This simplifies numerical analysis, and there is no real loss in generality. We observe that neglecting A corresponds to neglecting the momentum dependence of form factors (which play a significant role in the convergence properties of Faddeev-type equations), and hence we expect the inclusion of A to improve the convergence of N. Regarding A as a type of vertex correction in this sense, it is reasonable to consider $\bar{A}$ in the form

$$\bar{A}(q',q;s)=i\mu N(q',q;i\mu)g(q')\tilde{g}(q)\,, \tag{90}$$

with $g(0)=\tilde{g}(0)=1$. The on-shell parameters embodied in N thus set the over-all scale, and determine the behavior for small q' and q.

In practice, since $\bar{A}$ is purely phenomenological except at small momenta, it is more efficient to parametrize it directly. For the present purposes we consider a representative model,

$$\bar{A}(q',q;s)=\gamma g(q')g(q)\frac{\mu_0^2+\kappa^2}{\mu_0^2-\mu^2}\,,$$
$$g(q)=\frac{\mu_1^2\mu_2^2q}{(q^2+\mu_1^2)(q^2+\mu_2^2)}\,. \tag{91}$$

To fix the over-all constant γ, we observe that for $\sqrt{s}\to\mu$, $\kappa=i\mu$ *independently* of q, and hence $\bar{A}$ contributes the entire kernel in this limit. The normalization in Eq. (91) has thus been chosen such that

$$\bar{A}(q',q;\mu)\xrightarrow[q',q\to 0]{}\gamma q'q\,, \tag{92}$$

with γ determined by the requirement that $N(q',q;i\mu)$ has the same small-momentum limit.

This prescription is admittedly *ad hoc*, but it preserves the general features of simple off-shell models. The main point has been to provide an estimate for γ, which comes out to $\simeq 0.8$ in this calculation. On general grounds one expects $\mu_0\simeq\mu_1\simeq 2\mu$; the expected analyticity properties of $\hat{\tau}(q)$ (in addition to direct 4π exchange) then imply that $\mu_2\simeq 2\mu_1\simeq 4\mu$. In estimating the off-shell sensitivity, we shall vary the parameters γ, μ_0, μ_1, μ_2 freely about these values.

Schematically, Eq. (87) has the form

$$X=(1-K)^{-1}\Omega\,,\quad K\equiv\tilde{N}/D_1\,. \tag{93}$$

Three-body bound states or resonances thus appear as poles in $\hat{\tau}$, and correspond to discrete values of s for which the inverse $(1-K)^{-1}$ fails to exist, i.e., to values such that there is a nontrivial solution of the homogeneous equation. In order to search for the ω as a 3π resonance, one looks for zeros of the determinant $\mathfrak{D}=|1-K|$. Although the equations actually permit a calculation to be extended onto the second sheet, it is sufficient in practice (and more akin to the experimental situation) to study the behavior of $\mathfrak{D}$ for real values of $\sqrt{s}$. One thus proceeds by approximating the integral as a finite sum through the introduction of appropriate Gaussian points q_i, and weights h_i, converting Eq. (87) into a finite matrix equation.[17] The determinant $\mathfrak{D}$ can then be calculated in an elementary fashion.

In order to describe the p-wave phase shift, we employ the parametrization

$$f_1(\kappa^2) = \alpha + \delta/(\kappa^2 - \beta^2) \equiv a\lambda_1(\kappa^2) + 2\,, \qquad (94)$$

which is of the form given in Eq. (8). There are thus four parameters ($a\alpha\beta\delta$), which we have adjusted to fit the analytical representation of the phase given by Baton *et al.*[18] The numerical values are $a = 0.67$ F, $\alpha = 2.39$, $\beta = 2.20$ F^{-1}, and $\delta = 3.21$ F^{-2}. We observe that for a given choice of a, the behavior of f_1 can be read off from the expression

$$a\kappa \cot(\delta_1 + a\kappa) = (f_1 - a^2\kappa^2)/f_1\,, \qquad (95)$$

if the phase shift (δ_1) is known. In practice, as one would expect, only the values of f_1 in the vicinity of the ρ pole turn out to be significant, and these are well established.[19]

The numerical calculations were performed as described, and a 1^- isoscalar resonance was indeed observed for parameters in the ranges discussed above. With regard to the off-shell parameters ($\gamma\mu_0\mu_1\mu_2$), it was observed that the effects which could be produced by varying all of these independently were quite limited, and the results were always equivalent to calculations in which a single parameter was allowed to vary. The situation is well represented by the choice $\gamma = 0.8$, $\mu_0 = \mu_1 = 2\mu$, with μ_2 taken as the free parameter; the corresponding results are listed in Table I for the entries marked "N."

The inclusion of off-shell corrections can thus shift the position of the resonance to lower energies, while at the same time decreasing its width, but the basic energy variation which leads to the existence of the effect is contained in K_1. The inability of $\bar{A}$ to produce strong effects is a consequence of two factors: (a) The fact that it is *real* limits the role it can play in the interplay of real and imaginary parts leading to the nearby resonance pole; and (b) in contrast to K_1 it has no singularities close to the real axis, and hence has a weaker intrinsic dependence on the energy. As one would expect, the effect is sensitive to the parameters which describe the ρ; this is illustrated in Table I by entries A and B (model A corresponds to a "ρ" of width 60 MeV, model B has the ρ width but a mass of 660 MeV). To illustrate the results in more familiar terms, the first and third entries of the table are plotted in Fig. 1, in which $|1-K|^{-2}$ is used to indicate the rapidly varying factor in an appropriate cross section.

Our result indicates that the ρ *does* essentially imply the existence of the ω, leading naturally to a resonance with the appropriate quantum numbers. The precise position and width of the effect, however, are to some extent dependent on the details of the dynamics. The fact that our calculated widths come out too large as compared to the experimental value ($\simeq$10 MeV) is apparently due to neglect of coupling to the virtual $K\bar{K}$ channel, as we point out below. Just as important, similar calculations for 1^- ($I = 1, 2$) and 0^- ($I = 0, 1, 2$) 3π states with this formalism do *not* predict the existence of resonances; there are no spurious effects of the type reported by Basdevant. The only other "particle" which comes out of the formalism (for c.m. energies less than 2 GeV) is the pion itself, which appears as a 0^- ($I = 1$) 3π bound state for reasonable input (in this channel we couple the $I = 0$ π-π s wave to a $\lambda = 0$ spectator, in all

TABLE I. Variation of the 3π 1^-, $I = 0$ resonance vs the off-shell parameter μ_2 defined in the text. The mass (M) and width (Γ) were determined by extrapolation to the pole. Types N, A, and B refer to the physical parameters of the ρ, a "ρ" of width 60 MeV, and a "ρ" of the proper width but a mass of 660 MeV, respectively.

Type	$\mu_2(\mu)$	M (MeV)	Γ (MeV)
N	4.0	808	120
N	4.3	782	95
N	4.6	752	65
N	4.7	735	45
A	4.0	826	155
B	4.0	757	95

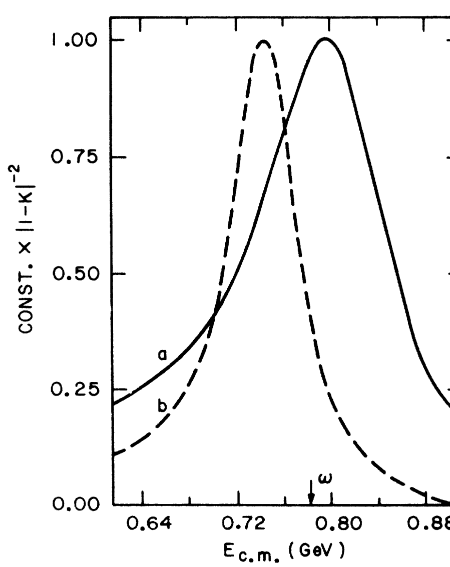

FIG. 1. Variation of $|1-K|^{-2}$ vs c.m. energy in the ω channel; the curves are normalized to unit height. The solid line is $N(4.0)$, the dashed is $N(4.6)$ of Table I.

other channels we take only the p wave).[20] The values of the s-wave parameters required to produce a 0^- bound state at the *exact* pion mass (with $A \equiv 0$) are almost identical with the best fit obtained to the $I = 0$ phase shift of Baton *et al.*[18] Using a $\lambda_0 + 1 = f_0(\kappa^2)$, and parametrizing f_0 according to Eq. (94), we arrive at the values $a = 0.32$ F, $\alpha = 1.28$, $\beta = 3.18$ F^{-1}, and $\delta = 9.04$ F^{-2} (with obvious uncertainties due to the state of our knowledge concerning this phase shift). This calculation has some interesting features and will be reported elsewhere.

It therefore appears that the relativistic BC formalism is capable of providing a viable description of such systems. We hope that subsequent applications to the A_1 and A_2 channels will provide some insight as to the nature of these effects. In concluding this section, we observe that a natural generalization of our technique can be employed to treat several different exit channels simultaneously. Thus, if a reaction can produce different types and numbers of particles in the outgoing state, we regard these as originating within the interaction volume (the interior) and enlarge our exterior description accordingly. The channel wave functions are then coupled via BC's which are a further generalization of Eqs. (32) and (33). For example, above the threshold for $K\bar{K}$ production (1 GeV), our exterior representation should consist of $|\Psi_{3\pi}^{\text{ext}}\rangle$ and $|\Psi_{K\bar{K}}^{\text{ext}}\rangle$, with $\langle\Psi_{3\pi}^{\text{ext}}|\Psi_{K\bar{K}}^{\text{ext}}\rangle = 0$. These will satisfy a coupled set of BC's, which we write schematically as

$$(\partial_1 - \lambda_1)\Psi_{3\pi}^{\text{ext}} = B_{11}T_{3\pi} + B_{12}T_{K\bar{K}}\,,$$
$$(\partial_2 - \lambda_2)\Psi_{K\bar{K}}^{\text{ext}} = B_{21}T_{3\pi} + B_{22}T_{K\bar{K}}\,. \tag{96}$$

Here $\Psi_{K\bar{K}}^{\text{ext}}$ has the form of Eq. (1) and B_{22} is just a number; hence the second equation can be solved explicitly for $T_{K\bar{K}}$ in terms of $T_{3\pi}$, and the result can be substituted into the first equation. The result is an equation involving only $T_{3\pi}$ which is of the type considered above, except that the B operator contains a term having the energy dependence of $T_{K\bar{K}}$, including the ϕ pole at $\sqrt{s} = 1.02$ GeV. The effective size of this term can be fixed by the $\phi \to 3\pi/\phi \to K\bar{K}$ branching ratio; it survives below the $K\bar{K}$ threshold in the sense of an analytic continuation. Treated perturbatively, this contribution has the right behavior (attraction increasing with energy) to significantly lower the ω width calculated above. A detailed examination of this effect is now in progress.

VI. DISCUSSION

There are two essential features of the three-body problem. One of these is unitarity, which expresses the conservation of probability, and links the various physical amplitudes via the discontinuity relations; e.g., Eq. (68). The other is what we have called the quasi-two-body limit; this is merely a consistency condition stating that as one particle is taken to infinity, we must recover ordinary two-body scattering as a special case (for short-range forces). It is trivial to construct models satisfying either of these requirements singly, but taken together they exhibit the fundamental complexity of the problem. This has recently been pointed out (in different but equivalent language) by Amado, who shows that the channel decomposition plus unitarity and the conventional analyticity properties can be used to *derive* the most basic realization of the Faddeev equations.[14] The same derivation produces the zero range limit of the nonrelativistic BC formalism proposed in BCA.

It is then clear that the Faddeev equations (in their most general form) and the BC formalism represent alternative but equivalent solutions of the above problem. The advantages of the BC prescription for computation and data analysis have been pointed out in previous articles.[7,8] In the present paper we have demonstrated that this approach has a straightforward generalization to the relativistic problem. This is in contrast to those theories designed in imitation of the Faddeev approach, for which there is no unambiguous procedure. Furthermore, the most obvious distinction between the corresponding equations suggests that the Faddeev generalizations do not properly take into account the quasi-two-body limit, which we assert must be understood in the *two-particle c.m. frame*. In the exterior (asymptotic) representation, this requires the spectator to have *real* momentum in this frame, and hence leads to the restrictions on the two-particle invariant energy squared given in Eq. (49). In the equations of Freedman, Lovelace, and Namyslowski, for example,[3] σ_α varies to $-\infty$ as a result of arbitrary spectator momentum in the *three-body* c.m. frame; the same property is shared by the other Faddeev-type equations in Ref. 3. A useful by-product of this distinction is the fact that our formalism requires a much more limited analytic continuation to the unphysical region.

Just as in the nonrelativistic problem, our equations in their full generality provide an efficient framework for the analysis of scattering data involving three-particle final states. Since the solutions are automatically unitary, this procedure would avoid the (justified) criticism leveled at current techniques.[21] Obvious applications would include $N\pi \to N\pi\pi$ and the 3π final-state interactions in $N\pi \to N(3\pi)$. Furthermore, via the parametrization of the operator A introduced in this formalism, one is effectively summarizing the full content of *new* information in the scattering experiment, whereas the minimal ($A=0$) model builds in automatically the important features deducible from two-body data. In the case of the ω, the latter constitute almost the entire effect. A nice additional feature, described at the end of Sec. V, is the ability to simultaneously take into account different orthogonal channels, such as 3π vs $K\bar{K}$. This property, which has no counterpart in the Faddeev-type theories, is vital in describing an object such as the ϕ.

The efficacy of the minimal model has been demonstrated in the 3π calculations described in Sec. V. If one considers all isospin 0^- or 1^- states for c.m. energies less than 2 GeV, the only effects predicted by the theory correspond to the only observed particles (the π and ω); the masses are in excellent agreement with experiment. These results are in sharp contrast to the Faddeev-type 3π calculations discussed in the Introduction,[5,6] both in the absence of spurious effects and in the relative insensitivity to the off-shell input. In view of the strong form-factor dependence exhibited by the latter calculations, these discrepancies are not too surprising. The reason is simply that in our formalism the off-shell characteristics are completely distinct from the phase shift, which is held fixed, whereas the two are linked in the separable model. Although our results did not depend strongly on the p-wave phase shift except in the immediate vicinity of the ρ, the variations considered were compatible with the experimental uncertainties, and hence were nowhere as dramatic as those reported by Menessier, Pasquier, and Pasquier.[6] It is thus possible that their sensitivity is not entirely an off-shell effect. On the other hand, several of the spurious levels did not share this sensitivity, and hence it may be that the formal differences discussed above may be quite important in explaining our results. Additional factors, which reflect themselves in different analyticity properties in the unphysical region, may also play a role, although this does not appear very likely. We hope that the next round of calculations planned for the A_1, A_2 channels will produce some insight into this question.

Although there is no room to pursue this topic in detail, it should be clear that our procedure has an obvious generalization to the n-body case. In fact, the 4-body analog corresponds to a two-dimensional integral equation which is comparable in difficulty to solving the Faddeev equations with a local potential (which has been done). With regard to the hadron spectroscopy, it is then clear that a great many interesting problems are now within reach. Furthermore, once one has a scattering theory which is reliable in highly relativistic problems, it is possible to say something useful regarding relativistic corrections, a topic which is poorly understood. For example, a number of contradictory estimates of corrections to the triton binding energy have been suggested.[22] In this case, a comparison of the results of the BC formalism in both its relativistic and nonrelativistic manifestations suggests an *increased* binding of about 0.5 MeV. In addition, the increasing use of pions as probes of the nucleus will require a dependable relativistic treatment of pion-nucleus scattering in order to properly interpret the results. As a first step in this direction, the present formalism has been applied to investigate π-d p-wave scattering at energies up to 350 MeV, and a paper detailing the results has been published.[23] Lastly, a particularly intriguing application has been made to the basic nuclear force problem; a report of this work is already available.[24]

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APPENDIX: THREE-PARTICLE EQUATIONS

Application of the BC's discussed in Sec. III results in a set of coupled one-dimensional integral equations for the functions $T^{\alpha}_{Ll\lambda}(q)$ defined in Eq. (57). For fixed L,M it is useful to define a basis of states $|\alpha l\lambda q\rangle$ with the normalization

$$\langle\beta l'\lambda' q'|\alpha l\lambda q\rangle=\delta_{\alpha\beta}\delta_{ll'}\delta_{\lambda\lambda'}\frac{\delta(q'-q)}{q^2}. \quad (A1)$$

The formal representation of the equations in this basis is identical with that given in BCA for the nonrelativistic problem. In what follows we first state these equations and then sketch a derivation resulting in an expression for the most critical term.

As discussed in Sec. III, the minimum value of κ_α^2 (the on-shell value of p_α^2 in the $\beta\gamma$ c.m.) encountered in our equations is $\kappa_{\alpha m}^2\equiv-\min(m_\beta^2,m_\gamma^2)$. With reference to the BC function $\lambda_{\alpha l}(\kappa_\alpha^2)$ describing the $\beta\gamma$ (two-body) scattering, we define $\lambda^{(0)}_{\alpha l}=\lambda_{\alpha l}(\kappa_{\alpha m}^2)$. In referring to the quantities $N^{(0)}_{\alpha l}$, $D^{(0)}_{\alpha l}$ below, we shall mean the functions $N_{\alpha l}$, $D_{\alpha l}$ defined in Eq. (9) evaluated with $\lambda_{\alpha l}(\kappa_\alpha^2)$ replaced by $\lambda^{(0)}_{\alpha l}$. A consequence of this definition is that $N^{(0)}_{\alpha l}\to N_{\alpha l}$, $D^{(0)}_{\alpha l}\to D_{\alpha l}$ as q_α approaches its maximum allowed value. If we then define an operator X in the above basis such that

$$T^{\alpha}_{Ll\lambda}(q) = -\frac{t_{\alpha l}(\kappa_\alpha)}{N^{(0)}_{\alpha l}(\kappa_\alpha)}\langle \alpha l\lambda q\,|X|\,\alpha_0 l_{0\alpha}\lambda_{0\alpha}q_{0\alpha}\rangle\,, \tag{A2}$$

our equation can be expressed as

$$\begin{aligned} X &= \Omega + KX\,,\\ K &= \theta + K^{(0)}\rho + [(1-\theta)\hat{B} + \theta(\hat{C}-1)](1-R)\rho\,. \end{aligned} \tag{A3}$$

Here $l_{0\alpha}$, $\lambda_{0\alpha}$, $q_{0\alpha}$ are the parameters characterizing the initial plane-wave state in the $\beta_0\gamma_0$ c.m. system ($\beta_0 \neq \gamma_0 \neq \alpha_0$), and the operators entering this equation are defined below.

The operator θ arises from the fact that the BC stated in Eqs. (32) and (33) must be applied in the exterior region, defined by the requirement that for each α, $(r_\beta - r_\gamma)^2 \leq -a_\alpha{}^2$. This is insured by taking the projection operator

$$\mathcal{P}_e(\mathbf{r}_1 r_2 r_3) = \prod_\alpha \theta[-a_\alpha{}^2 - (r_\beta - r_\gamma)^2] \tag{A4}$$

as an explicit factor, where $\theta[d] = 1$ for $d > 0$ and vanishes otherwise. At equal times, $\mathcal{P}_e$ reduces to

$$\mathcal{P}_e(\vec{x}_\alpha, \vec{y}_\alpha) = \prod_\beta \theta[x_\beta - a_\beta]\,, \tag{A5}$$

where x_β, x_γ must be expressed in terms of $\vec{x}_\alpha$, $\vec{y}_\alpha$ via the linear combinations implied by Eq. (15):

$$\begin{aligned} x_\beta &= \left|\frac{\mu_\alpha}{m_\gamma}\vec{x}_\alpha + \vec{y}_\alpha\right|\,,\\ x_\gamma &= \left|\frac{\mu_\alpha}{m_\beta}\vec{x}_\alpha - \vec{y}_\alpha\right|\,, \end{aligned} \tag{A6}$$

$\alpha\beta\gamma$ cyclic. Applying the BC means taking $x_\alpha = a_\alpha$, and thus the domain in which $\mathcal{P}_e \neq 0$ depends solely on y_α and $\hat{x}_\alpha \cdot \hat{y}_\alpha$. In general, there is some maximum value y^0_α such that $\mathcal{P}_e$ vanishes identically for $y_\alpha < y^0_\alpha$ (y^0_α may be zero). The BC only has content for $y_\alpha > y^0_\alpha$, and the operator θ utilized in Eq. (A3) corresponds to $\theta[y^0_\alpha - y_\alpha]$ in the coordinate representation. Explicitly,

$$\begin{aligned} \langle \beta l'\lambda' q'|\,\theta\,|\alpha l\lambda q\rangle &= \delta_{\alpha\beta}\,\delta_{ll'}\,\delta_{\lambda\lambda'}\,\theta_\lambda(q', q; y^0_\alpha)\,,\\ \theta_\lambda(q', q; R) &= \frac{2R^2}{\pi}\left[\frac{q j_{\lambda+1}(Rq) j_\lambda(Rq') - q' j_{\lambda+1}(Rq') j_\lambda(Rq)}{q^2 - q'^2}\right]. \end{aligned} \tag{A7}$$

The operators ρ, R are diagonal in the above basis:

$$\begin{aligned} \rho_{\alpha l}(q) &= t_{\alpha l}(\kappa_\alpha)/t^{(0)}_{\alpha l}(\kappa_\alpha)\,,\\ R_{\alpha l}(q) &= 1 - D^{(0)}_{\alpha l}(\bar{\kappa}_\alpha)/D^{(0)}_{\alpha l}(\kappa_\alpha)\,. \end{aligned} \tag{A8}$$

Here $\bar{\kappa}_\alpha$ refers to $\kappa_\alpha(q)$ evaluated at the kinetic energy $W = \overline{W}$, which is a free parameter ($\overline{W} < 0$). The presence of R is formally necessary to achieve unitarity in the minimal model ($\hat{B} = \hat{C} = 0$), as first pointed out in SCI, but in all the calculations so far performed ($3N$, 3π, πd) there is virtually no sensitivity to $\overline{W}$, and in fact this piece of the kernel is completely negligible.[25] The operators $\hat{B}$, $\hat{C}$ have *arbitrary* real values; the former is analogous to $B^{\alpha\beta}$ in Eq. (33), while the latter arises from an auxiliary BC on the interior segment, $y < y^0_\alpha$, as explained in BCA.

Except for ρ, which corrects the two-particle phases from those implied by $\lambda^{(0)}_{\alpha l}$ to their physical values, all the operators defining K in Eq. (A3) are designed to complement the central operator $K^{(0)}$. That is, the operators θ, R appear to guarantee unitarity and a unique solution, while $\hat{B}$, $\hat{C}$ exhibit the full flexibility allowed by unitarity, and hence summarize the off-shell content. It is clear that $K^{(0)}$, Ω are present in the most trivial realization of the theory, corresponding to a model in which $\lambda_{\alpha l} = \lambda^{(0)}_{\alpha l}$, a constant. This is the "pure" BC model, considered in several previous papers on the nonrelativistic problem.[26] Except for the relativistic kinematics implicit in the relation between q and κ_α, the only differences in the relativistic version are contained in them. In discussing these operators, we shall sketch their derivation in the "pure" BC model.

As explained in Sec. III, the basic BC is to be applied with $\vec{R}$, t_α, t_β, t_γ all set equal to zero in the $\beta\gamma$ c.m. frame. The exterior representation then will depend only on $\vec{x}_\alpha$, $\vec{y}_\alpha$, and the BC can be expressed as

$$(\partial_{a_\alpha} - \lambda^{(0)}_{\alpha l})\langle \alpha LMl\lambda a_\alpha y_\alpha|\mathcal{P}_e|\Psi\rangle = 0\,, \tag{A9}$$

where $\mathcal{P}_e$ has the simple representation given in Eq. (A5). For formal manipulations it is useful to observe that if $\tilde{\psi}_l(p)$ is the Fourier transform of $\psi_l(x)$, then

$$\int_0^\infty dp\, p^2 N^{(0)}_{\alpha l}(p)\tilde{\psi}_l(p) \propto \psi_l'(a_\alpha) - \lambda^{(0)}_{\alpha l}\psi_l(a_\alpha)\,. \tag{A10}$$

This implies that Eq. (A9) is equivalent to the relation

$$\int \prod_\beta dk_\beta e^{i\vec{q}_\alpha\cdot\vec{y}_\alpha} Y^*_{lm}(\hat{p}_\alpha) N^{(0)}_{\alpha l}(p_\alpha)\langle k_1 k_2 k_3|\mathcal{P}_e|\Psi\rangle = 0\,, \tag{A11}$$

where the integration is performed in the $\beta\gamma$ c.m. frame. By decomposing $|\Psi\rangle$ into its three channel pieces, introducing appropriate Lorentz transformations, and taking advantage of the $\delta(\vec{P}-\vec{P}_0)$ factor in $\mathcal{T}$ and the mass-shell δ functions in $\mathcal{G}_0$, this equation can be displayed in a form involving only the independent vectors $\vec{p}_\beta, \vec{q}_\beta$ appropriate to the particular Lorentz frames. This is equivalent to the following formal development, which we state in terms of the $|\alpha\vec{p}\vec{q}\rangle$ Hilbert space defined in Sec. IV.

The $\hat{\tau}$ operator of Eq. (57) is only determined by our formalism for on-shell values $p=\kappa_\alpha(q)$, but it is useful to introduce a function $G_{\alpha l}(p,\kappa_\alpha)$ which is unity on-shell and satisfies

$$\int_0^\infty dp\, p^2 j_l(xp)\frac{G_{\alpha l}(p,\kappa_\alpha)}{p^2-\kappa_\alpha{}^2-i\epsilon}=\frac{\pi i\kappa_\alpha}{2}h_l(x\kappa_\alpha) \quad \text{(A12)}$$

for $x>a_\alpha$; we then regard $\hat{\tau}$ as the operator

$$\langle\alpha LMl\lambda pq|\hat{\tau}=G_{\alpha l}(p,\kappa_\alpha)\langle\alpha LMl\lambda\kappa_\alpha q|\hat{\tau}\ . \quad \text{(A13)}$$

The explicit choice for $G_{\alpha l}$ does not matter since ultimately everything is put on-shell; a particular example might be $j_l(a_\alpha p)/j_l(a_\alpha\kappa_\alpha)$ (another is given in SCI). The initial plane-wave state in the three-body c.m. frame is represented as $|\Phi\rangle=(1-I)|\phi\rangle$, where $|\phi\rangle$ is one of the basis states,

$$|\phi\rangle=|\alpha_0\vec{p}_0\vec{q}_0\rangle\ ,$$
$$\Lambda^{-1}|\phi\rangle=|\alpha_0\vec{p}_{0\alpha}\vec{q}_{0\alpha}\rangle \quad \text{(A14)}$$

[see discussion concerning Eq. (57)]. We may then define the states

$$|\psi\rangle=(1-G_0\hat{\tau})\Lambda^{-1}|\phi\rangle\ ,$$
$$|\Psi\rangle=(1-I)\Lambda|\psi\rangle\ , \quad \text{(A15)}$$

employing G_0 as defined in Eq. (73).

The interpretation of these states is that $|\psi\rangle$ corresponds to the channel wave function in its appropriate c.m. frame, while $|\Psi\rangle$ corresponds to the total wave function in the three-body c.m. frame. They are defined in the space of the two momentum variables $\vec{p},\vec{q}$ and may be Fourier-transformed to obtain the coordinate representation; e.g., for $x>a_\alpha$,

$$\langle\alpha LMl\lambda xy|\psi\rangle =\langle\alpha LMl\lambda xy|\alpha_0\vec{p}_{0\alpha}\vec{q}_{0\alpha}\rangle -2\mu_\alpha i^{l+\lambda+1}\int_0^\infty dq\, q^2 j_\lambda(qy)\kappa_\alpha h_l(\kappa_\alpha x)T^\alpha_{Ll\lambda}(q)\ . \quad \text{(A16)}$$

Comparing to Eq. (46), this is just the exterior representation of the channel function in its c.m. ($\vec{R}=t_\gamma=0$). Similarly, if we evaluate Eq. (37), in the three-body c.m. at equal times $t_\beta=t$, the result in the exterior is proportional to $\exp(-i\sqrt{s}\,t)\langle\alpha\vec{x}\vec{y}|\Psi\rangle$.

It is now useful to introduce the operators g, N such that

$$\langle\alpha\vec{p}\vec{q}|g|\beta\vec{p}'\vec{q}'\rangle=\delta_{\alpha\beta}\delta(\vec{q}'-\vec{q})\frac{\delta(p-p')}{p^2}\times\sum_l\frac{(2l+1)}{4\pi}P_l(\hat{p}\cdot\hat{p}')\frac{G_{\alpha l}(p,\kappa_\alpha)}{D^{(0)}_{\alpha l}(\kappa_\alpha)} \quad \text{(A17)}$$

$$\langle\alpha\vec{p}\vec{q}|N|\beta\vec{p}'\vec{q}'\rangle=\delta_{\alpha\beta}\delta(\vec{q}'-\vec{q})\times\sum_l\frac{(2l+1)}{4\pi}P_l(\hat{p}\cdot\hat{p}')N^{(0)}_{\alpha l}(p')\ .$$

Thus $t\equiv gN$ is a type of off-shell t matrix corresponding to the on-shell value $t^{(0)}_{\alpha l}(\kappa_\alpha)=N^{(0)}_{\alpha l}/D^{(0)}_{\alpha l}$.[27] One can easily verify the relations

$$\Delta t=-t_+\Delta G_0 t_-\ ,$$
$$NG_0 gN=N \quad \text{(A18)}$$

from these definitions; the former is quite useful in the unitarity proof. Employing the $\mathcal{P}_e$ operator defined by

$$\langle\alpha\vec{x}\vec{y}|\mathcal{P}_e|\sigma\vec{x}'\vec{y}'\rangle=\delta_{\alpha\sigma}\delta(\vec{x}-\vec{x}')\delta(\vec{y}-\vec{y}')\mathcal{P}^{(\alpha)}_e(\vec{x},\vec{y}) \quad \text{(A19)}$$

in the coordinate representation, where $\mathcal{P}^{(\alpha)}_e(\vec{x},\vec{y})$ corresponds to $[\mathcal{P}_e(\vec{x}_\alpha,\vec{y}_\alpha)]_{\vec{x}_\alpha=\vec{x},\vec{y}_\alpha=\vec{y}}$, we are ready to restate Eq. (A11) in the form

$$N\mathcal{P}_e\Lambda^{-1}|\Psi\rangle=0\ . \quad \text{(A20)}$$

Since this is to hold independently of $|\phi\rangle$, we deduce the operator relation

$$N\mathcal{P}_e(1-\mathcal{J})(1-G_0\hat{\tau})=0\ ,$$
$$\mathcal{J}=\Lambda^{-1}I\Lambda\ . \quad \text{(A21)}$$

Except for the replacement $I\to\mathcal{J}$, this equation has precisely the same form as the case first studied in SCI.

We now introduce an operator X such that $\tau=-gX$, which is possible as a result of Eq. (A13). The interpretation of X is that

$$\langle\alpha LMl\lambda pq|X=\langle\alpha l\lambda q|X\ , \quad \text{(A22)}$$

where the right-hand side corresponds to the notation of Eq. (A2); there is no dependence on p. At this point we choose to simplify the derivation by assuming that for each l included in the sum stated in Eq. (A17), $\lambda^{(0)}_{\alpha l}=\lambda_\alpha$, independent of l [note that the sum is implicitly truncated to take into account as many partial waves as are necessary to describe the (α) channel function]. This avoids the necessity of going explicitly to the $LMl\lambda$ representation, and is adequate for the simple cases studied to date, but is not essential in our method (formulas for the general case are presented in BCA). This means that there exists an operator $\overline{\mathcal{P}}_e$ such that $N\mathcal{P}_e = \overline{\mathcal{P}}_e N$; $\overline{\mathcal{P}}_e$ is just $\mathcal{P}_e$ with $x_\alpha = a_\alpha$. In accord with the discussion given above, $\overline{\mathcal{P}}_e$ vanishes identically for $y_\alpha < y_\alpha^0$. Defining a generalization of the operator θ above,

$$\langle \alpha \vec{p}\vec{q} \,|\, \theta \,|\, \beta \vec{p}\,' \vec{q}\,' \rangle = \delta_{\alpha\beta}\delta(\vec{p}\,' - \vec{p}) \times \sum_\lambda \frac{(2\lambda+1)}{4\pi} P_\lambda(\hat{q}\cdot\hat{q}')\theta_\lambda(q, q'; y_\alpha^0), \tag{A23}$$

one can show that there exists $\overline{\mathcal{P}}_e^{-1}$ such that $(1-\theta)\overline{\mathcal{P}}_e^{-1}\overline{\mathcal{P}}_e = (1-\theta)$. Thus

$$(1-\theta)\overline{\mathcal{P}}_e^{-1} N\mathcal{P}_e G_0 \hat{\tau} = -(1-\theta) N G_0 g X = -(1-\theta) X, \tag{A24}$$

the last line following from Eq. (A18) and the fact that X is of the form $X = NX'$ (as we see below). Returning to Eq. (A21), we finally obtain

$$(1-\theta)X = (1-\theta)N(\mathcal{J} - 1) + (1-\theta)N\mathcal{J} G_0 g X. \tag{A25}$$

Although written on different bases, Eqs. (A3) and (A25) are directly comparable in view of Eq. (A22); i.e., X has no p dependence. In this model $\rho = 1$ and $\hat{B} = 0$; we thus infer that

$$\langle \alpha l \lambda q \,|\, \Omega \,|\, \alpha_0 l_{0\alpha} \lambda_{0\alpha} q_{0\alpha} \rangle = \langle \alpha L M l \lambda \kappa_\alpha q \,|\, (1-\theta) N(\mathcal{J} - 1) \,|\, \alpha_0 L M l_{0\alpha} \lambda_{0\alpha} \kappa_{0\alpha} q_{0\alpha} \rangle ,$$
$$\langle \alpha l \lambda q \,|\, K^{(0)} \,|\, \beta l' \lambda' q' \rangle = \int_0^\infty dp' p'^2 \langle \alpha L M l \lambda \kappa_\alpha q \,|\, (1-\theta) N \mathcal{J} G_0 g \,|\, \beta L M l' \lambda' p' q' \rangle . \tag{A26}$$

The explicit evaluation of these operators is straightforward, but tedious even in the nonrelativistic case. Naturally, in special cases such as s-wave forces, etc., one can obtain a relatively simple formula.

In conclusion, we note that Eq. (A3) can be expressed in the form

$$\begin{aligned} X &= \Omega + (K_1 + AK_2)X, \\ K_1 &= \theta + K^{(0)}\rho - \theta(1-R)\rho, \\ K_2 &= (1-R)\rho, \\ A &= (1-\theta)\hat{B} + \theta\hat{C}. \end{aligned} \tag{A27}$$

Thus K_1 defines the "minimal" model, and is determined entirely (up to the negligible $\overline{W}$-dependent terms) by the two-particle data. The same is true of K_2, whereas all the off-shell information (including possible three-body forces) is contained in A, which is an arbitrary real-valued operator. We note that in the general case, $\hat{\tau} = -g\rho X$, which is compatible with Eq. (A2). For computational purposes it is convenient to express K in the form

$$\langle \alpha l \lambda q \,|\, K \,|\, \beta l' \lambda' q' \rangle = \frac{N_{l\lambda l'\lambda'}^{\alpha\beta; L}(q', q)}{D_{\beta l'}(\kappa_\beta')} . \tag{A28}$$

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