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Bound-state wave functions and bound-state scattering in relativistic field theory*†

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We describe how a matrix element of an operator may be calculated between bound states, in the framework of conventional relativistic field theory. In the course of doing so, we examine how bound-state creation and annihilation operators may be constructed, the asymptotic condition for bound states, what general types of wave functions are appropriate for describing bound states and why, graphical analysis involving bound states, and questions of renormalizability. The final result is a set of Feynman-type rules for calculating a matrix element. Those for the S matrix are stated explicitly.

I. INTRODUCTION

The distinction between elementary particles and bound states depends purely on the theoretical model with which we describe them. Experimental data, even if extrapolated with the help of some theoretical assumptions about analyticity, can only tell us that a particle has certain quantum numbers, and that it corresponds to a pole in certain S-matrix elements with certain residues. The distinction is dictated by the simplicity of the model which attempts to explain experimental data, and changes as experiments progress into regions of higher energy and probe details over smaller distances. Such has been the case with our conception of molecules, atoms, and nuclei. It has now become practical to ask whether hadrons should be described as bound states, because available experiments suggest that this may clarify some new phenomena in terms of concepts familiar from the past.

Obviously, to be able to answer this question, one must first have a theory of bound states with which to do calculations, and one must understand the theory well enough to make approximations, which are always necessary in practice. In the case of molecules, atoms, and nuclei, one has such a theory, i.e., the nonrelativistic Schrödinger equation, and one understands it. In the case of hadrons, however, no comparable theory has been developed. The new feature that makes the problem more difficult is that at smaller distances relativistic effects become important, especially the existence of antiparticles and the possibility of pair creation. This prevents one from taking immediate advantage of the intuition gained in nonrelativistic problems to devise approximate calculations. A formal framework is needed in which to establish principles and acquire new intuition. Relativistic field theory provides such a framework. In fact it is the only one available at present that is well defined. In this paper we study bound states within this framework.

Since Bethe-Salpeter wave functions[1] were introduced as a way to describe relativistic bound states, many authors[2] have studied their meaning and properties; but a general and systematic study of relativistic bound-state phenomena seems to be lacking. Many natural questions that come to mind have not received clear answers. For example:

1. Is there anything special about Bethe-Salpeter-type wave functions? Are there not other types of wave functions, for example the Tamm-Dancoff-type wave functions,[3] that could serve equally well?

2. If one describes the pion with a Bethe-Salpeter wave function consisting of a quark-antiquark pair $q\bar{q}$, has one unjustifiably neglected contributions from $q\bar{q}q\bar{q}$, $q\bar{q}q\bar{q}q\bar{q}$, etc.?

3. In principle, how does one calculate the matrix element of an operator between bound states, such as the S matrix?

We answer these questions in this paper. Briefly, the answers are as follows:

1. A Bethe-Salpeter-type wave function, defined as any nonvanishing matrix element of a time-ordered product of any number of Heisenberg fields between the bound state and the vacuum state, is the natural description of a bound state, because it occurs in the residues of poles of Green's functions that enter into the reduction formula for bound-state scattering. This, quite apart from other virtues like covariance and renormalizability, makes it appropriate.

2. The $q\bar{q}$ Bethe-Salpeter wave function for the pion is a complete description by itself. One may also consider Bethe-Salpeter-type wave functions containing any number of pairs, e.g., $q\bar{q}q\bar{q}$, $q\bar{q}q\bar{q}q\bar{q}$, etc. These are, however, reducible in the sense that they have poles whose residues contain wave functions with fewer pairs, and can all be expressed in terms of the $q\bar{q}$ wave function, which is irreducible in this sense. There is a reduction formula for pion scattering that involves

any reducible wave function one happens to choose for the pion, but the S-matrix elements calculated therefrom collapse to those calculated with the irreducible choice.

3. There are Feynman-type rules for the calculation of a matrix element of an operator between bound states. For example, for pion-pion scattering the rule for the S matrix is as follows: The pions are to be described by irreducible Bethe-Salpeter wave functions. Start with a quark Green's function with eight external legs, and group the external legs into pion groups by quantum numbers and momenta. In the Feynman graphs for the Green's function, omit all interactions that can be absorbed into the pion groups, and subtract certain well-defined redundant graphs. Finally, multiply by the pion wave functions and integrate. The only nonintuitive part of the procedure is the subtraction of certain redundant graphs. Refer to the main text for a fuller explanation.

What we do in this paper, then, is to describe in principle how one may calculate matrix elements involving bound states, in the framework of conventional field theory. In the course of doing so we examine how bound-state creation and annihilation operators may be constructed, the types and properties of bound-state wave functions, graphical analysis involving bound states, and questions of renormalizability. Not discussed are actual calculations and the physical interpretation of relativistic wave functions. We hope that the formal developments in this paper represent a first step in approaching these problems.

II. BOUND STATES AND GREEN'S FUNCTIONS

Gell-Mann and Low[1] showed that the Bethe-Salpeter wave function of a two-particle bound state is defined by the residue of a pole in the invariant mass of two of the legs of a Green's function with four external lines. We generalize this result to show that a bound state gives rise to a pole in the invariant mass of any group of elementary particles having the same quantum numbers as the bound state, appearing as external legs of any Green's function. The residue of such a pole defines a wave function for the bound state. There are thus many possible wave functions, involving different numbers of constituent elementary particles, all representing the same bound state.

Consider a relativistic quantum field theory involving a set of causal Heisenberg fields $\{\psi_1(x), \psi_2(x), \dots\}$. By the usual asymptotic assumption there are "in" and "out" fields that create asymptotic states made of elementary particles. A bound state in this field theory is defined as a state of discrete mass and definite spin, orthogonal to all the elementary "in" or "out" states. Accordingly, a bound state $|P, s\rangle$ has three characteristics:

$$\mathcal{P}^\mu |P, s\rangle = P^\mu |P, s\rangle \quad (P^2 = m^2), \tag{2.1}$$

$$U(\Lambda)|P, s\rangle = \sum_{s'} |\Lambda P, s'\rangle D^J_{s's}[L^{-1}(\Lambda P)\Lambda L(P)], \tag{2.2}$$

$$\langle 0|\psi_i(x)|P, s\rangle \equiv 0 \text{ , for all } i \tag{2.3}$$

where $\mathcal{P}^\mu$ are the generators of translation for the Heisenberg fields, P^μ is the 4-momentum of the bound state, and Λ is a Lorentz transformation. The label s in $|P, s\rangle$ refers to spin, but may include other internal symmetries in a more general case. We assume throughout this work that for given discrete quantum numbers the bound state is unique, i.e., the mass is not degenerate.

A complete set of states in the Hilbert space of the field theory must specifically include all the bound states. Because the bound states have been chosen to transform covariantly in (2.2), we may normalize them (just as elementary particles) by

$$\langle P', s'|P, s\rangle = 2E_P(2\pi)^3\delta^3(\vec{P}' - \vec{P})\delta_{s's}, \tag{2.4}$$

where $E_P \equiv P^0$. The completeness relation then reads

$$1 = \sum_s \int \frac{d^3P}{(2\pi)^3 2E_P} |P, s\rangle\langle P, s| + \cdots . \tag{2.5}$$

For any bound state $|A\rangle$ in the field theory there is a time-ordered product of elementary fields

$$\mathcal{A}(x) \equiv T[\psi(x_1)\cdots\psi(x_n)], \tag{2.6}$$

for which $\langle 0|\mathcal{A}(x)|A\rangle \neq 0$, where $\langle 0|$ is the vacuum state. Here $\psi(x)$ stands for any Heisenberg field operator $\psi_i(x)$, which may be a boson field, a fermion field, or its adjoint. The distinguishing indices i, which may include spin and internal-symmetry indices, have been suppressed.

We shall call $\langle 0|\mathcal{A}(x)|A\rangle$ a wave function for the bound state $|A\rangle$, in the sense that it gives a description of the state. The relevance of such a description will be demonstrated by showing firstly that it occurs in the residues of poles in certain Green's functions and secondly that we may make use of them to calculate S-matrix elements for reactions involving the bound state. This wave function is clearly nonunique, although for any given bound state there is a unique choice involving the minimum number of fields. We call the minimum choice an *irreducible wave function*, and the others *reducible wave functions*. For example, if a proton is considered to be a bound state in a quark field theory, then its irreducible wave function consists of three quark fields, and the reducible wave functions consist of any number of

pairs of quark-antiquark fields in addition to the three quark fields. The formal developments in this section and the next make no distinction among these choices.

Consider the Green's function

$$\mathcal{G}(x,y)\equiv\langle 0|\,T\mathfrak{A}(x)\mathfrak{B}(y)\,|0\rangle\,, \tag{2.7}$$

where $x\equiv\{x_1,\ldots,x_n\}$, $y\equiv\{y_1,\ldots,y_m\}$, and $\mathfrak{B}(y)$ is a time-ordered product of m elementary Heisenberg fields. In the conventional Feynman-graph representation, $\mathcal{G}(x,y)$ is the sum of all graphs with $n+m$ external legs, which includes the n legs corresponding to the fields in $\mathfrak{A}(x)$. We introduce average and relative coordinates for the two sets x and y by writing

$$\begin{aligned} x_i &= X+\rho_i\,,\quad \sum_{i=1}^{n}\rho_i=0\,,\\ y_i &= Y+\sigma_i\,,\quad \sum_{i=1}^{m}\sigma_i=0\,. \end{aligned} \tag{2.8}$$

This definition is not unique, but merely the simplest. More generally, we may redefine ρ_i such that $\sum a_i\rho_i=0$, with the condition $\sum a_i=1$, so that $X=\sum a_i x_i$. With (2.8), an integration over all coordinates may be written as

$$\begin{aligned} &\int d^4x_1\cdots d^4x_n=\int d^4X\int(d\rho)\,,\\ &\int(d\rho)\equiv\int d^4\rho_1\cdots d^4\rho_n\,\delta\left(n^{-1}\sum_{i=1}^{n}\rho_i\right). \end{aligned} \tag{2.9}$$

Among the time orderings in (2.7) there are those for which all the fields in $\mathfrak{A}(x)$ stand to the left of all those in $\mathfrak{B}(y)$. We isolate their contributions to the Green's function:

$$\mathcal{G}(x,y)=\langle 0|\,\mathfrak{A}(x)\mathfrak{B}(y)\,|0\rangle\,\theta(X_0-Y_0-a)+\mathfrak{R}\,, \tag{2.10}$$

where, for fixed values of $\{\vec{X},\vec{Y},\rho,\sigma\}$, a is some fixed number sufficiently large so that $x_{i0}>y_{j0}$ for all i and j. The remainder term $\mathfrak{R}$ contains a term proportional to $\theta(-X_0+Y_0+a)$, and other terms for which the range of X_0-Y_0 is restricted to finite intervals. Defining the Fourier transform with respect to average coordinates by

$$G(P,Q)\equiv\int d^4X d^4Y e^{i(P\cdot X-Q\cdot Y)}\mathcal{G}(x,y)\,, \tag{2.11}$$

inserting a complete set of intermediate states into the first term of (2.10), and using the representation

$$\theta(t)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}d\omega\,\frac{e^{i\omega t}}{\omega-i\epsilon}, \tag{2.12}$$

we obtain

$$\begin{aligned} G(P,Q)&=(2\pi)^4\delta^4(P-Q)\\ &\times\sum_{s_\alpha}\frac{\langle 0|\,\mathfrak{A}(\rho)\,|\alpha\rangle\langle\alpha|\,\mathfrak{B}(\sigma)\,|0\rangle\,e^{-i(E_\alpha-E)a}}{i(E_\alpha-E+i\epsilon)2E_\alpha}+R\,, \end{aligned} \tag{2.13}$$

where $E\equiv P^0$, $E_\alpha\equiv P^0_\alpha$, and the states $|\alpha\rangle$ all have total 3-momentum $\vec{P}_\alpha=\vec{P}$. The sum over s_α is a sum over quantum numbers of $|\alpha\rangle$. The energy denominators displayed in (2.13) give rise to poles at $E=E_\alpha$. Of the terms in R, the one corresponding to the time ordering $\mathfrak{B}(\sigma)\mathfrak{A}(\rho)$ gives rise to poles at $E=-E_\alpha$, while all others have no poles at all, owing mathematically to the fact that the ranges of X_0-Y_0 are fixed finite intervals. The bound state $|A\rangle$ thus gives rise to a discrete pole at $E=E_A$, which is contained in the first sum in (2.13). Rewriting

$$\begin{aligned} \frac{1}{(E_\alpha-E)2E_\alpha}&=\frac{E_\alpha+E}{(E_\alpha{}^2-E^2)2E_\alpha}\\ &=-\frac{E_\alpha+E}{2E_\alpha}\,\frac{1}{P^2-m_\alpha{}^2}, \end{aligned} \tag{2.14}$$

where $m_\alpha{}^2\equiv E_\alpha{}^2-\vec{P}_\alpha{}^2=E_\alpha{}^2-\vec{P}^2$, we have

$$\begin{aligned} G(P,Q)\xrightarrow[E\to E_A]{}&(2\pi)^4\delta^4(P-Q)\,\frac{i}{P^2-m_A{}^2}\\ &\times\sum_s\langle 0|\,\mathfrak{A}(\rho)\,|P,s\rangle\langle P,s|\,\mathfrak{B}(\sigma)\,|0\rangle\,, \end{aligned} \tag{2.15}$$

where we have written $|A\rangle\equiv|P,s\rangle$. If we had let $E\to -E_A$ we would have picked up the pole corresponding to the antiparticle of the bound state, also located at $P^2=M_A{}^2$, but with residue $\langle 0|\,\mathfrak{B}(\sigma)\,|\bar{A}\rangle\langle\bar{A}|\,\mathfrak{A}(\rho)\,|0\rangle$.

The residue of the bound-state pole in (2.15) is a product of a wave function $\langle 0|\,\mathfrak{A}(\rho)\,|P,s\rangle$ and an adjoint wave function $\langle P,s|\,\mathfrak{B}(\sigma)\,|0\rangle$, which do not necessarily have the same number of constituent fields, and which are in general reducible. By doing the same type of manipulations for a reducible wave function as we have done for $\mathcal{G}(x,y)$, we can show that the Fourier transforms of reducible wave functions have poles in subsets of momenta, whose residues are wave functions containing a smaller number of constituent fields. We may repeat this process until the residues involve irreducible wave functions. We shall show in more detail later that irreducible wave functions satisfy homogeneous Bethe-Salpeter integral equations, whereas reducible wave functions do not—they may be calculated from the former. In deriving a formula for the S matrix involving bound states, however, we may use any wave function, as we show in the next section.

The notation we shall use for wave functions is as follows. By translational invariance

$$\langle 0|\mathfrak{A}(x)|P,s\rangle = e^{-iP\cdot X}\langle 0|\mathfrak{A}(\rho)|P,s\rangle . \tag{2.16}$$

The Fourier transform of the relative wave function is denoted by

$$\chi_{Ps}(p_1,\ldots,p_n) \equiv \int (d\rho)\langle 0|\mathfrak{A}(\rho)|P,s\rangle \times \exp\left(i\sum_{j=1}^{n} p_j\cdot\rho_j\right). \tag{2.17}$$

The corresponding adjoint wave function is

$$\bar{\chi}_{Ps}(p_1,\ldots,p_n) \equiv \int (d\rho)\langle P,s|\bar{\mathfrak{A}}(\rho)|0\rangle \times \exp\left(-i\sum_{j=1}^{n} p_j\cdot\rho_j\right), \tag{2.18}$$

where $\int(d\rho)$ is defined in (2.9), $\bar{\mathfrak{A}}(\rho)$ is obtained from $\mathfrak{A}(\rho)$ by replacing all ψ_i by $\bar{\psi}_i$, the latter being the Hermitian adjoint of the former for boson fields, and Pauli adjoint for spinor fields.

III. REDUCTION FORMULA FOR BOUND-STATE SCATTERING

By a simple generalization of the work of Lehmann, Symanzik, and Zimmermann[4] we shall express any S-matrix element involving bound states as the residue of a set of poles in an off-mass-shell Green's function.[5] First we construct creation and annihilation operators for bound states. Let $|A\rangle = |P,s\rangle$ be a bound state of mass m, and let $\mathfrak{A}(x)\equiv\mathfrak{A}(X,\rho)$ be defined as in (2.6), where average and relative coordinates $\{X,\rho\}$ are defined in (2.8). Operators $\mathfrak{A}^{\text{in}}(X,\rho)$ and $\mathfrak{A}^{\text{out}}(X,\rho)$ that satisfy the Klein-Gordon equation with respect to X with the bound-state mass m are defined by

$$\mathfrak{A}^{\text{in(out)}}(X,\rho) \equiv \mathfrak{A}(X,\rho) - \int d^4X' \Delta^{\text{adv(ret)}}(X-X',m) \times (\Box' + m^2)\mathfrak{A}(X',\rho) . \tag{3.1}$$

We then integrate out the relative coordinates ρ with some weighting function, and smear the average coordinate X over a wave packet as follows. Let

$$U^f_{Ps}(X,\rho) \equiv f_P(X) u_{Ps}(\rho) , \tag{3.2}$$

where $f_P(X)$ is a wave-packet solution of the Klein-Gordon equation with central momentum P that approaches $\exp(-iP\cdot X)$ in the plane-wave limit, and $u_{Ps}(\rho)$ is any function satisfying the following three conditions:

$$u_{Ps}(\rho) \text{ transforms like } \langle 0|\mathfrak{A}(\rho)|P,s\rangle , \tag{3.3}$$

$$\int (d\rho)\bar{u}_{Ps}(\rho)\langle 0|\mathfrak{A}(\rho)|P,s'\rangle = \delta_{ss'} , \tag{3.4}$$

$$\lim_{\rho^0_j\to\pm\infty} u_{Ps}(\rho) = 0 \quad (j=1,\ldots,n) . \tag{3.5}$$

In (3.3), "transform" refers both to Lorentz and internal-symmetry-group transformations on the suppressed indices. The condition (3.4) merely specifies a normalization constant, because any function satisfying (3.3) will have the normalization integral (3.4) proportional to $\delta_{ss'}$ by Schur's lemma. As an example one may take $u_{Ps}(\rho)$ to be the relative wave function $\langle 0|\mathfrak{A}(\rho)|P,s\rangle$ itself, multiplied by a convergence factor $g(\rho_1)\cdots g(\rho_n)$, where

$$g(\rho) = \exp[-\gamma_1(P\cdot\rho)^2 - \gamma_2(\rho\cdot\rho)^2] . \tag{3.6}$$

Now we define a bound-state annihilation operator as

$$a_f^{\text{in(out)}}(P,s) \equiv \int d^3X \int (d\rho)\bar{U}^f_{Ps}(X,\rho) i\overleftrightarrow{\frac{\partial}{\partial X_0}} \times \mathfrak{A}^{\text{in(out)}}(X,\rho) . \tag{3.7}$$

It is easily seen that $a_f^{\text{in(out)}}(P,s)$ is independent of X^0, transforms like a single-particle annihilation operator, and has an energy-momentum content given by

$$[\mathcal{P}^\mu, a_f^{\text{in(out)}}(P,s)] = -P^\mu a_f^{\text{in(out)}}(P,s) \tag{3.8}$$

in the plane-wave limit. Furthermore, (3.1) and (3.4) imply

$$\langle 0|a_f^{\text{in(out)}}(P,s)|P',s'\rangle = 2E_P(2\pi)^3\delta^3(\vec{P}-\vec{P}')\delta_{ss'} . \tag{3.9}$$

Correspondingly, we repeat the smearing on $\mathfrak{A}(X,\rho)$:

$$a_f(P,s,X^0) \equiv \int d^3X \int (d\rho)\bar{U}^f_{Ps}(X,\rho) i\overleftrightarrow{\frac{\partial}{\partial X_0}} \mathfrak{A}(X,\rho) , \tag{3.10}$$

and postulate the asymptotic condition

$$\lim_{X^0\to -\infty(+\infty)} \langle\Phi|a_f(P,s,X^0)|\Psi\rangle = \langle\Phi|a_f^{\text{in(out)}}(P,s)|\Psi\rangle , \tag{3.11}$$

where $|\Psi\rangle$ and $|\Phi\rangle$ are arbitrary normalizable states. The asymptotic condition and the convergence property (3.5) will lead directly to proper commutation relations for $a_f^{\text{in(out)}}(P,s)$ and then to the reduction formula. The key step in these derivations is the typical limiting process

$$\lim_{X^0\to\infty}\left\langle\Phi\left|\int d^3X\int(d\rho)\bar{U}^f_{Ps}(X,\rho)i\overleftrightarrow{\frac{\partial}{\partial X_0}}T[\mathfrak{A}(x)\mathfrak{B}(y)]\right|\Psi\right\rangle = \langle\Phi|a_f^{\text{out}}(P,s)\mathfrak{B}(y)|\Psi\rangle , \tag{3.12}$$

where $\mathcal{S}(y)$ is a time-ordered product of Heisenberg fields.

To derive the commutation relations for annihilation and creation operators, consider states $|A\rangle$ and $|B\rangle$ and time-ordered products $\mathfrak{A}(x)$ and $\overline{\mathfrak{B}}(y)$ for which $\langle 0|\mathfrak{A}(x)|A\rangle \neq 0$ and $\langle B|\overline{\mathfrak{B}}(y)|0\rangle \neq 0$. Let $\mathfrak{A}(x)$ contain r fermion fields and $\overline{\mathfrak{B}}(y)$ contain r' fermion fields. Let $\mathcal{O}$ be an arbitrary operator [e.g., $1, \psi(z), \psi(z)\psi(z'), \ldots$] containing R-fermion fields.

We first show that, because the elementary fields commute or anticommute at spacelike separations, the following interchange of order of integrations is valid:

$$\left\langle \Phi \left| \int dx \int dy \frac{\partial}{\partial X^0}\frac{\partial}{\partial Y^0}\overline{U}_{Ps}(x) i\frac{\overleftarrow{\partial}}{\partial X^0} T[\mathfrak{A}(x)\overline{\mathfrak{B}}(y)\mathcal{O}] i\frac{\overleftarrow{\partial}}{\partial Y^0} U_{P's'}(y) \right| \Psi \right\rangle = \left\langle \Phi \left| \int dy \int dx (\text{same integrand}) \right| \Psi \right\rangle . \tag{3.13}$$

The important point is that the surface terms arising from integrating the total derivatives on both sides of the equation should be equal. Since $\partial/\partial X^0 = \sum \partial/\partial x_j^0$, we need only show

$$\left(\int_{-a}^{a} dx^0 \int_{-b}^{b} dy^0 - \int_{-b}^{b} dy^0 \int_{-a}^{a} dx^0 \right) \frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0} T[\psi(x)\mathfrak{M}\overline{\psi}(y)] = 0 , \tag{3.14}$$

where $\mathfrak{M}$ is an arbitrary operator. We are interested in $a >$ | max. time in $\mathfrak{M}$ | and $b >$ | max. time in $\mathfrak{M}$ | . If, for example, the fields are fermion, the left-hand side is, by direct calculation,

$$(-1)^{M+1}\{\psi(t,\vec{x}), \overline{\psi}(t,\vec{y})\}\mathfrak{M} + (-1)^M \mathfrak{M}\{\psi(-t,\vec{x}), \overline{\psi}(-t,\vec{y})\}, \tag{3.15}$$

where M is the number of fermion fields in $\mathfrak{M}$ and t is the larger of a and b. This is zero if the elementary fields are causal.

Carrying out the integration in (3.13) explicitly, applying the asymptotic condition, and keeping track of the number of fermion field interchanges, we obtain

$$\langle \Phi | [a^{\text{out}}(P) b^{\text{out}\dagger}(P') - (-1)^{rr'} b^{\text{out}\dagger}(P') a^{\text{out}}(P)]\mathcal{O}|\Psi\rangle = (-1)^{(r+r')R}\langle \Phi | \mathcal{O}[a^{\text{in}}(P) b^{\text{in}\dagger}(P') - (-1)^{rr'} b^{\text{in}\dagger}(P') a^{\text{in}}(P)]|\Psi\rangle . \tag{3.16}$$

Taking $\mathcal{O} = 1$ (i.e., $R = 0$) shows that the "in" commutators (anticommutators) equal the "out" commutators (anticommutators). Thus (3.16) may be rewritten

$$\langle \Phi | [a^{\text{in}}(P) b^{\text{in}\dagger}(P') - (-1)^{rr'} b^{\text{in}\dagger}(P') a^{\text{in}}(P)]\mathcal{O}|\Psi\rangle = (-1)^{(r+r')R}\langle \Phi | \mathcal{O}[a^{\text{in}}(P) b^{\text{in}\dagger}(P') - (-1)^{rr'} b^{\text{in}\dagger}(P') a^{\text{in}}(P)|\Psi\rangle . \tag{3.17}$$

Consider the case in which $|A\rangle$ is a fermion and $|B\rangle$ is a boson. Then $(-1)^{rr'} = +1$. Let $|\Psi\rangle = |\Phi\rangle$ and choose $\mathcal{O}^\dagger = [a^{\text{in}}(P), b^{\text{in}\dagger}(P')]$. Then $(-1)^{(r+r')R} = -1$. Hence, (3.17) becomes

$$\langle \Phi | \mathcal{O}^\dagger \mathcal{O} | \Phi \rangle = -\langle \Phi | \mathcal{O}\mathcal{O}^\dagger | \Phi \rangle . \tag{3.18}$$

Since neither the norm of $\mathcal{O}|\Phi\rangle$ nor of $\mathcal{O}^\dagger|\Phi\rangle$ can be negative, both norms must be zero. But $|\Phi\rangle$ is arbitrary; so $\mathcal{O} \equiv \mathcal{O}^\dagger \equiv 0$. Hence, the commutator of a fermion and a boson vanishes.

Consider the case in which $|A\rangle$ and $|B\rangle$ are both fermions or both bosons. Then $(-1)^{(r+r')R} = +1$ and (3.17) becomes

$$\langle \Phi | [a^{\text{in}}(P), b^{\text{in}\dagger}(P')]_\pm \mathcal{O} | \Psi \rangle = \langle \Phi | \mathcal{O}[a^{\text{in}}(P), b^{\text{in}\dagger}(P')]_\pm | \Psi \rangle . \tag{3.19}$$

Since $\mathcal{O}$ is arbitrary, the anticommutator of two fermions is a c number and the commutator of two bosons is a c number. This c number is equal to its vacuum expectation value and therefore vanishes unless $|A\rangle$ and $|B\rangle$ have the same discrete quantum numbers. The only nonvanishing case is

$$[a^{\text{in}}(P), a^{\text{in}\dagger}(P')]_\pm = \langle 0 | [a^{\text{in}}(P), a^{\text{in}\dagger}(P')]_\pm | 0 \rangle = \langle 0 | a^{\text{in}}(P) a^{\text{in}\dagger}(P') | 0 \rangle , \tag{3.20}$$

where $a^{\text{in}}(P)|0\rangle = 0$ because of (3.8) and the assumption that the vacuum is the state of lowest energy. Inserting a complete set of states into (3.20) gives a nonzero contribution only when the intermediate state has the correct quantum numbers and hence, assuming nondegeneracy, is the bound state itself. Using (3.9) gives

$$[a^{\text{in}}(P), a^{\text{in}\dagger}(P')]_\pm = \sum_{s''} \int \frac{d^3p''}{(2\pi)^3 2E''} \langle 0 | a^{\text{in}}(P) | P'', s'' \rangle \langle P'', s'' | a^{\text{in}\dagger}(P') | 0 \rangle = 2E_P (2\pi)^3 \delta^3(\vec{P} - \vec{P}') \delta_{ss'} . \tag{3.21}$$

We can thus construct a creation operator for any single-particle state, either elementary or bound. A complete set of states is built up from either a complete set of single-particle "in" states or from a complete set of single-particle "out" states.

The S-matrix element for the reaction $A+\cdots+C\to D+\cdots+F$ is defined by

$$\langle D\cdots F|\,S\,|A\cdots C\rangle\equiv\langle D\cdots F,\ \text{out}\,|\,A\cdots C,\ \text{in}\rangle. \tag{3.22}$$

The derivation of the reduction formula for bound states proceeds in the usual manner as for elementary particles, because the convergence property (3.5) enables us to make the same formal manipulations. We merely state the result:

$$\langle D\cdots F|\,S-1\,|A\cdots C\rangle=\int dx\cdots dy\,dz\cdots dw\,\bar{U}_D(z)\cdots\bar{U}_F(w)\frac{\vec{K}_D}{i}\cdots\frac{\vec{K}_F}{i}\,\mathcal{G}\,\frac{\overleftarrow{K}_A}{i}\cdots\frac{\overleftarrow{K}_C}{i}U_A(x)\cdots U_C(y), \tag{3.23}$$

$$\mathcal{G}\equiv\langle 0|\,T\mathfrak{D}(z)\cdots\mathfrak{F}(w)\bar{\mathfrak{A}}(x)\cdots\bar{\mathfrak{C}}(y)\,|0\rangle, \tag{3.24}$$

where $\mathfrak{A}(x),\ldots$ are time-ordered products of the type (2.6), and $U_A,\ldots$ are functions of the form (3.2), $K_A,\ldots$ are Klein-Gordon operators in the average coordinates, and $\int dx$ is a $4n$-dimensional integral.

Matrix elements for any operator can be derived similarly. For example, the transition matrix element for a current operator $J_\mu(z)$ between two bound states $|A\rangle$ and $|B\rangle$ is given by

$$\langle B|\,J_\mu(z)\,|A\rangle=\int dx\,dy\bar{U}_B(y)\frac{\vec{K}_B}{i}\times\langle 0|\,T\mathfrak{B}(y)J_\mu(z)\bar{\mathfrak{A}}(x)\,|0\rangle\,\frac{\overleftarrow{K}_A}{i}U_A(x). \tag{3.25}$$

The transformation properties of the U functions specified in (3.3) insure that (3.23) and (3.25) have the correct covariance properties under the Lorentz group and internal-symmetry groups. If the underlying field theory is a gauge theory, then the gauge invariance of $a_f^{\text{in(out)}}(P)$ is guaranteed, at least formally, by (3.5) and the asymptotic condition. For let the gauge-transformed boson or fermion fields be

$$\psi_i'(x)=\psi_i(x)+\delta\psi_i(x), \tag{3.26}$$

where $\delta\psi_i(x)$ contains all dependences on the gauge functions $\Lambda(x)$. The gauge-transformed version of (3.10) is

$$a_f'(P,X^0)=a_f(P,X^0)+\int d^3X\int(d\rho)\bar{U}_P^f(X,\rho)i\frac{\overleftrightarrow{\partial}}{\partial X^0}\delta\mathfrak{A}(x), \tag{3.27}$$

where $\delta\mathfrak{A}(x)$ contains all dependences on $\Lambda(x)$. By (3.5), as $X^0\to-\infty$, each $x_j^0\to-\infty$ for $j=1,\ldots,n$. Because the gauge functions must be square integrable, $\Lambda(x)$ and its derivatives vanish as $x_j^0\to-\infty$. Thus the integral in (3.27) vanishes as $X^0\to-\infty$, so that $a_f(P,-\infty)$ is gauge-invariant. Therefore, $a_f^{\text{in}}(P)$ is gauge-invariant and so is the S matrix.

The reduction formula (3.23), which is a generalization of the familiar one for elementary particle scattering, preserves the usual formal appearance. We recall that in the case of elementary particle scattering, a practical way to use the reduction formula is to expand the Green's function $\mathcal{G}$ in terms of Feynman graphs: If n elementary particles are involved in the reaction, the relevant Green's function is the sum of all Feynman graphs with n external legs (full propagators). Application of the Klein-Gordon operators merely cuts off these legs, which are then replaced by the wave functions of the particles involved in the reaction.

Our generalization extends the treatment to include bound-state scattering. It is of course also valid for elementary scattering, and contains new elements even in the latter case. Regardless of whether the external particles are elementary or bound states, one starts with a Green's function $\mathcal{G}$ whose external legs can be grouped into subsets $A, B, C, \ldots$, such that the quantum numbers of the elementary particles in each subset add up to the desired quantum numbers for the corresponding external particles. Beyond this requirement, however, there is no restriction on the number of elementary particles in each subset. There is, therefore, great arbitrariness in the choice of a Green's function, corresponding to the arbitrariness in the choice of a wave function for an external particle, be it an elementary particle or a bound state.

As a generalization of the usual procedure of summing infinite classes of Feynman graphs to obtain a full propagator for an elementary particle, one must in this case perform graphical summation over the infinite set of graphs that produce the pole of the external particle. The Klein-Gordon operators then extract these poles.

This summation will be carried out explicitly in Sec. VI. From now on we shall focus our attention on bound-state scattering, although the developments will also represent a more general way to treat elementary particle scattering.

If the theory is to be consistent, different choices of $\mathcal{G}$ corresponding to the choices of bound-state wave functions must yield the same S matrix. We shall explicitly demonstrate that this is so in Sec. VI. It will be seen that all choices reduce to the simplest choice, i.e., irreducible wave functions for all the bound states involved.

Another seeming arbitrariness occurs in the smearing functions U. In an explicit calculation, however, they actually drop out, owing to the normalized condition (3.4), as we shall demonstrate in Sec. VI. This is the analog of the fact that when we use the conventional reduction formula for elementary particle scattering the wave-function renormalization constants always drop out of the calculation, although they appear in the reduction formula.

The practical rules for calculating the S matrix are stated in Fig. 12. Before we proceed to its derivation, we first discuss some relevant properties of bound-state wave functions in the next two sections.

IV. IRREDUCIBLE WAVE FUNCTIONS FOR BOUND STATES OF IDENTICAL FERMIONS

A wave function for a bound state has been defined in Sec. II, where a distinction between irreducible and reducible wave functions was also made. A wave function for the bound state $|P,s\rangle$ is of the form $\langle 0|T\psi_i(x_1)\cdots\psi_j(x_n)|P,s\rangle$. Its Fourier transform $\chi_{Ps}(p_1,\ldots,p_n)$ is defined in (2.17). The wave function is called irreducible if χ_{Ps} has no pole in the invariant mass of any proper subset of $\{p_1,\ldots,p_n\}$ that corresponds to a group of elementary particles having the same discrete quantum numbers as the bound state. (We recall that throughout this paper the assumption is made that for given discrete quantum numbers the bound-state mass in nondegenerate.) Wave functions that do have such poles are called reducible.

For example, in the quark model the proton has as irreducible wave function $(\mathcal{P}\mathcal{P}\mathcal{N})$ and reducible wave functions $(\mathcal{P}\mathcal{P}\mathcal{N}\mathcal{P}\bar{\mathcal{P}})$, $(\mathcal{P}\mathcal{P}\mathcal{N}\mathcal{P}\bar{\mathcal{P}}\mathcal{N}\bar{\mathcal{N}})$, etc. A pion has as irreducible wave function a linear combination of $(\mathcal{P}\bar{\mathcal{P}})$, $(\mathcal{N}\bar{\mathcal{N}})$, and $(\lambda\bar{\lambda})$ and reducible wave functions $(\mathcal{P}\bar{\mathcal{P}}\mathcal{P}\bar{\mathcal{P}})$, $(\mathcal{P}\bar{\mathcal{P}}\mathcal{N}\bar{\mathcal{N}}\lambda\bar{\lambda})$, etc. An exotic baryon can have as irreducible wave function $(\mathcal{P}\mathcal{P}\mathcal{P}\bar{\lambda}\mathcal{N})$ and reducible wave function $(\mathcal{P}\mathcal{P}\mathcal{P}\bar{\lambda}\mathcal{N}\mathcal{P}\bar{\mathcal{P}})$, etc.

We shall consider from now on only wave functions composed of fermions and antifermions. In so doing, there is no loss of generality in any theory in which a boson is coupled only to $\bar{\psi}\psi$, where ψ is a fermion operator. Theories with self-coupled boson fields require a separate examination of wave functions involving bosons, and are not discussed in this paper.

An N-fermion wave function is always irreducible because of fermion number conservation. A wave function containing N fermions plus a number of pairs is irreducible only if there exist conserved discrete quantum numbers that forbid the annihilation of the pairs, as in the previous example of the exotic baryon. Similarly, a bound-state wave function consisting of a fermion-antifermion pair is always irreducible. One consisting of n fermion-antifermion pairs with $n>1$ can be irreducible only if quantum numbers forbid their annihilation to $n=1$. We assume that the wave function involving the minimum number of fields allowable by quantum number does not vanish identically. If this is not the case, the simplest wave function would contain supernumerary pairs not called for

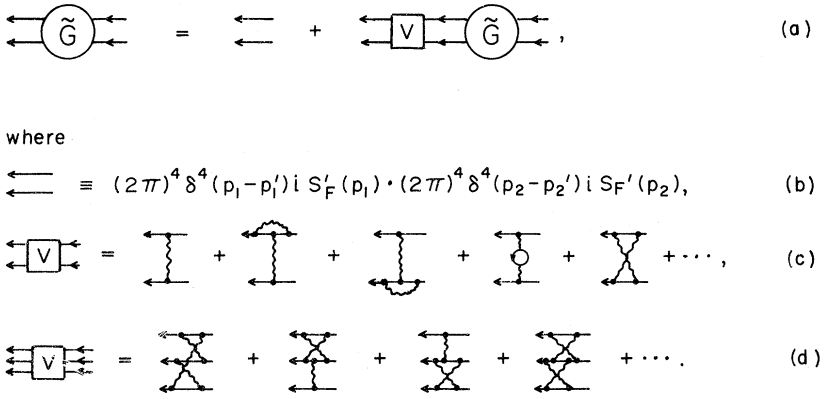


FIG. 1. Nonantisymmetrized Green's function $\tilde{G}$. Solid lines always denote full fermion propagators. Wavy lines always denote bare-boson propagators. All vertices are bare.

by quantum numbers. This is discussed in Sec. VII.

We shall consider in this section and the next the case of identical fermions. Because the irreducible wave functions are only of two types (unless we have the special case noted above), the discussion is simpler, and we can bring out more clearly the principles of graphical summation used in analyzing the reduction formula.

Discrete quantum numbers other than spin will be taken into account in Sec. VII. It will be seen that, although their existence complicates the analysis, the final rules for construction of the S matrix are the same, apart from trivial counting factors.

A. N-fermion wave functions

We consider a theory with only one spinor fermion field $\psi(x)$, interacting through boson exchange. An N-fermion bound-state wave function is always irreducible. To find an integral equation for it, consider the Green's function

$$\mathcal{G} \equiv \langle 0 | T\psi(x_1') \cdots \psi(x_N')\bar{\psi}(x_1) \cdots \bar{\psi}(x_N) | 0 \rangle , \qquad (4.1)$$

and let G be its Fourier transform to momenta $p_1', \ldots, p_N'$; $p_1, \ldots, p_N$:

$$G \equiv \int dx\, dx' \exp\left[i \sum_{j=1}^{N} (x_j' \cdot p_j' - x_j \cdot p_j) \right] \mathcal{G} . \qquad (4.2)$$

We expand G into Feynman graphs in the usual manner. Because of fermion conservation, G is the sum of all Feynman graphs with N continuous fermion lines, which can be individually traced through a Feynman graph. Each fermion line carries two momentum-spin labels: an unprimed p_i where it enters the graph and a primed p_j' where it leaves the graph. Any particular graph is characterized by a one-to-one association between elements of the sets $\{p_1, \ldots, p_N\}$ and $\{p_1', \ldots, p_N'\}$. The identity of the fermions dictates that all possible associations are included in G, with relative signs determined by the signature of the permutation of $\{p_1, \ldots, p_N\}$ that takes the particular association to a standard one. Let us take as standard association $p_i \rightarrowtail p_i'$ $(i = 1, \ldots, N)$. Consider the subset of graphs that have the standard association. The N fermion lines can be distinguished by a *line number* i corresponding to p_i. For convenience we define a *topological graph* to be a Feynman graph with the standard association $p_i' \rightarrowtail p_i$, and with its N fermion lines numbered from 1 to N. Two topological graphs differing only in line numbering are considered to be distinct. A signatured permutation of $\{p_i\}$ then generates a Feynman graph of the same topological connectivity. A topological graph corresponds to a class of $N!$ Feynman graphs, and a Feynman graph belongs to one and only one class. Let

$$\tilde{G} \equiv \text{sum of all topological graphs} . \qquad (4.3)$$

Then

$$G = \sum_P P\delta_P \tilde{G} , \qquad (4.4.)$$

where $P\delta_P$ is a signatured permutation of the labels of the incoming line terminals $\{p_1, \ldots, p_N\}$, and the sum consists of $N!$ terms.

Consider the simplest case $N = 2$, for which $\tilde{G}$ satisfies the well-known integral equation in Fig. 1(a), where the symbols V and $\tilde{G}$ represent functions of four 4-momenta and four spinor indices, and each contains an overall δ^4 of momenta. The product of $V\tilde{G}$ implies a sum over intermediate indices and an integration, $\int d^4k/(2\pi)^4$, over each intermediate momenta. By convention we draw the lines 1 and 2 parallel; then a vertical cut in a graph intersects at least two fermion lines (and possible a number of boson lines). The kernel V is defined as the sum of all connected topological graphs with external legs omitted that cannot be made disconnected by a vertical cut intersecting exactly two internal fermion lines and no boson lines. [See Fig. 1(c)]. This kernel is unrenormalized but may be renormalized by dividing through by four powers of the square root of the fermion field renormalization constant. Similarly, $\tilde{G}$ and wave functions occurring in residues of its poles are unrenormalized but may be renormalized similarly. These renormalizations are quite obvious and we will continue to write unrenormalized equations.

For N arbitrary, irreducible kernels will be defined analogously. Again draw all N-fermion lines parallel to one another. Then a vertical cut intersects at least N-fermion lines. Let L be a subset of the line numbers $\{1, \ldots, N\}$. The irreducible kernel involving L is denoted V_L and is defined as the sum of all connected topological graphs involving the lines L that cannot be made disconnected by cutting exactly n_L (i.e., the number of lines in L) internal fermion lines and no boson lines. Again all external legs are omitted. For example, the three-line kernel is shown in Fig. 1(d). The sum of all topological graphs $\tilde{G}$ is then the sum of all distinct products that can be formed from the set of all possible kernels $\{V_L\}$. It is understood that the identity is included, and a product between V_1 and V_2 includes full propagators that connect the lines that go between V_1 and V_2.

Throughout this paper we define many quantities graphically, such as the V_L. When drawn graphically it is always obvious where full propagators are needed to connect different graphical units;

but they are clumsy to write in algebraic form and tend to obscure. We suppress these propagators, external or internal, in algebraic expressions, since from graphical representations it is always obvious where they should be supplied. Thus, for example, V_1V_2 means V_1IV_2, where I are the suppressed propagators.

The integral equation for $\tilde{G}$ is of the form Fig. 2(a), where the total kernel is denoted by Z. For $N \leq 3$, Z is just the sum of all possible V_L. For $N>3$, however, it is easy to see that this cannot be true, for it would lead to double counting. For example, the kernels V_{12} and V_{34} commute, so that the products $V_{12}V_{34}$ and $V_{34}V_{12}$ correspond to the same set of graphs. In the expansion of $\tilde{G}$, therefore, $V_{12}V_{34}$ should occur only once, whereas in the expansion $(V_{12}+V_{34}+\cdots)(V_{12}+V_{34}+\cdots)$ this product is included twice. Thus Z must contain, in addition to the sum of all V_L, terms that subtract out overcounted terms. The correct rule is given in Theorem 1 of Appendix A which states that

Z = sum of all the V's, minus the product of each pair of commuting V's, plus the product of each triplet of mutually commuting V's, minus the product of each quadruplet of mutually commuting V's, and so on. (4.5)

More explicitly, let $\{L_1, L_2, \ldots, L_g\}$ be g disjoint subsets of the line numbers $\{1, \ldots, N\}$ so that the kernels $\{V_{L_1}, \ldots, V_{L_g}\}$ form a set of g mutually commuting kernels. Then

$$Z = \sum_{g=1} (-1)^{g-1} \sum_{\{L_1,\ldots,L_g\}} V_{L_1} \cdots V_{L_g}, \tag{4.6}$$

where the sum extends over all possible choices of $\{L_1, \ldots, L_g\}$ for a given g. As mentioned before we have suppressed obvious full propagators, which include those for the lines not in the set $\{L_1, \ldots, L_g\}$.

With the integral equation in Fig. 2(a) we can now obtain the Bethe-Salpeter equation for the irreducible bound-state wave function. First rewrite Fig. 2(a) in the form of Fig. 2(b). Then we obtain the integral equation for the fully antisymmetrized Green's function G in the form of Fig. 2(e) by applying the prescription (4.4). Let

$$P \equiv \sum_{i=1}^{N} p_i, \quad P' \equiv \sum_{i=1}^{N} p_i'. \tag{4.7}$$

As we have shown in (2.15), G has a pole at $P^2=M^2$, with residue indicated in (2.15). Thus, letting $P^2 \to M^2$ in Fig. 2(e), we obtain Fig. 3(a). Multiplying the equation in Fig. 3(a) through by P^2-M^2 and setting $P^2=M^2$, we obtain the Bethe-Salpeter equation in Fig. 3(c).

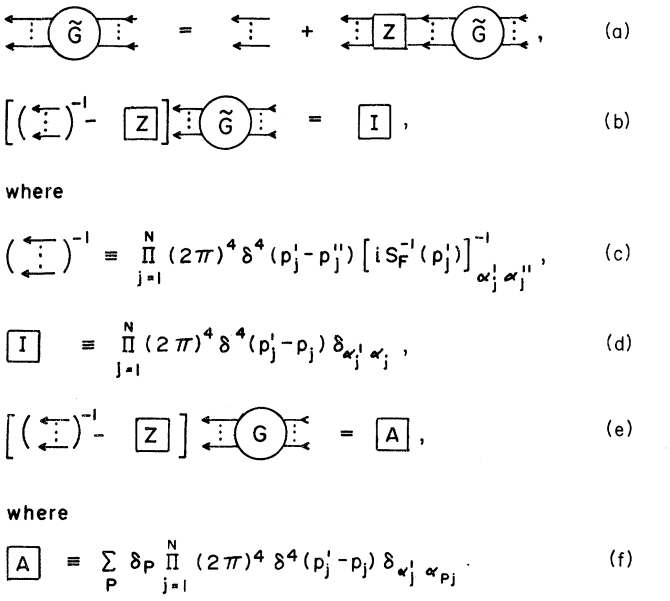


FIG. 2. Integral equation for the nonantisymmetrized Green's function $\tilde{G}$ and for the fully antisymmetrized one G. The kernel Z is defined in (4.5) or (4.6).

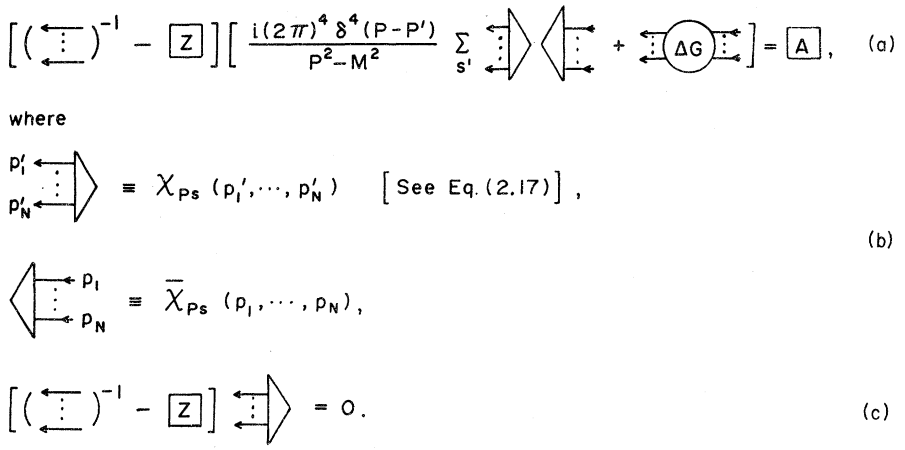


FIG. 3. Derivation of the Bethe-Salpeter equation for an N-fermion irreducible bound-state wave function.

Since Fig. 3(a) is nonlinear in the wave functions, it also fixes their normalization. Multiply the equation in Fig. 3(a) by $P^2 - M^2$, take $\partial/\partial P_\mu$ of both sides, and integrate the resulting equation against $\bar{\chi}_{Ps''}(p_N', \ldots, p_1')$ to get Fig. 4(a). Setting $P^2 = M^2$ and using the Bethe-Salpeter equation give Fig. 4(b), which leads finally to the normalization condition in Fig. 4(c).

Note that Fig. 3(c) and Fig. 4(c) are also satisfied for elementary particles. For example, Fig. 4(c) would read

$$\langle P, s''|\bar{\psi}(0)|0\rangle \frac{\partial}{\partial P_\mu}[iS_F'(p)]^{-1}\langle 0|\psi(0)|P, s'\rangle$$
$$= (2mZ_2)^{1/2}\bar{u}_{Ps''}\frac{\partial}{\partial P_\mu}\left(\frac{\not{P} - m}{iZ_2}\right)u_{Ps'}(2mZ_2)^{1/2}$$
$$= -2i\delta_{s''s'}P^\mu, \qquad (4.8)$$

where the factor $(2m)^{1/2}$ comes from the normalization of states chosen in (2.4).

B. Fermion-antifermion wave functions

The fermion-antifermion wave functions will satisfy an equation and normalization condition similar to those in Fig. 3(c) and Fig. 4(c), but will be further restricted by the orthogonality of the bound state to all elementary bosons. That is, the bound state cannot occur as a pole in the full propagator of any elementary boson.

Let G be the Fourier transform of

$$\langle 0|T\psi(x_1')\psi(x_2')\bar{\psi}(x_1)\bar{\psi}(x_2)|0\rangle$$

to momenta $p_1', p_2'; p_1, p_2$. We seek a pole in the variable $P = p_1 - p_1' = p_2' - p_2$. Its residue will be the transform of

$$\langle 0|T\psi(x_2')\bar{\psi}(x_2)|P, s\rangle\langle P, s|T\psi(x_1')\bar{\psi}(x_1)|0\rangle.$$

Separate out the direct channel disconnected term by writing G in the form of Fig. 5(a). The leftover term G' satisfies the integral equation in Fig. 5(b),

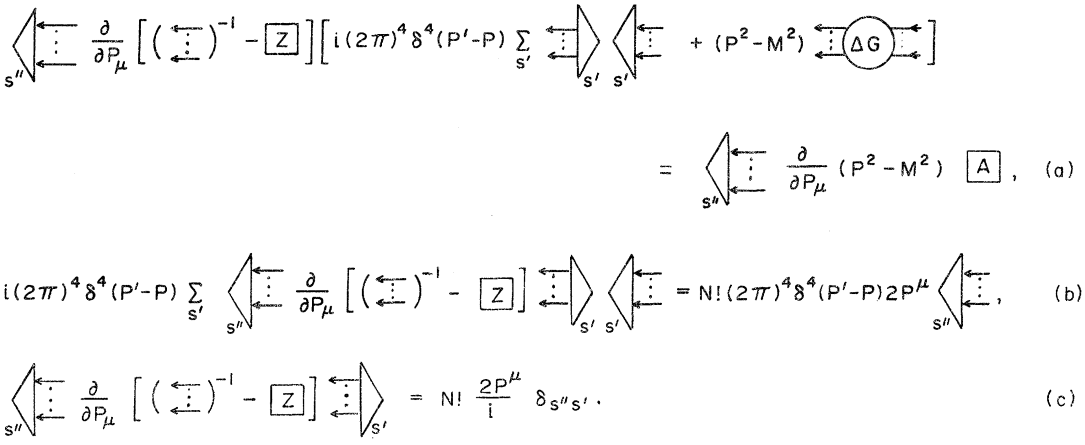


FIG. 4. Derivation of the normalization condition for an N-fermion irreducible bound-state wave function.

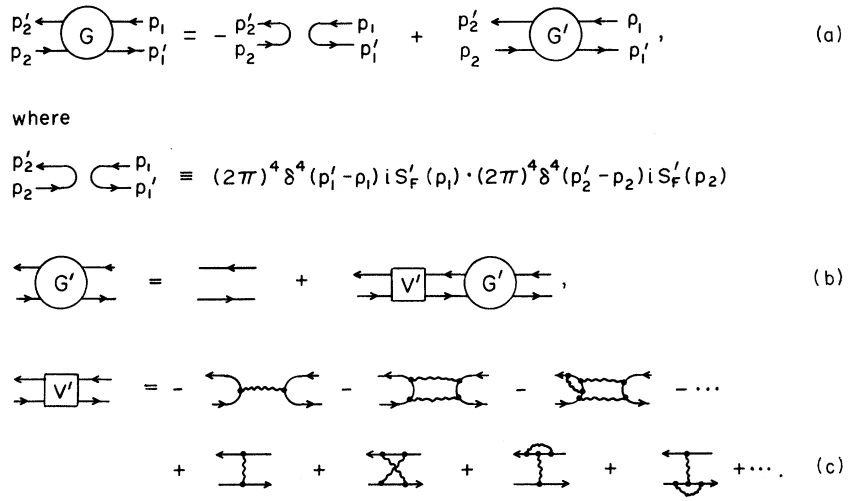


FIG. 5. Integral equation for fermion-antifermion Green's function. It is necessary to isolate the disconnected graph in (a) because its iteration would lead to unwanted vacuum graphs.

with the kernel V' defined in Fig. 5(c) as the sum of all connected Feynman graphs with the indicated external lines that cannot be made disconnected by cutting just one internal fermion line and one internal antifermion line.

Figure 5(b) is of the same form as Fig. 1(a) in the previous section. But here, although G' can be renormalized, the kernel V' cannot, because there are no graphs in V' to renormalize the bare boson propagator in the first graph of Fig. 5(b). Isolate this troublesome graph by defining V'' and G'' as in Fig. 6(a). Both V'' and G'' are renormalizable. Because the full boson propagator satisfies Fig. 6(b), the identity

$$\frac{1}{1-x-x''}=\frac{1}{1-x''}+\frac{1}{1-x''}\left(x+x\frac{1}{1-x-x''}x\right)\frac{1}{1-x''}$$

then shows that Fig. 5(b) is equivalent to Fig. 6(c), which makes the renormalizability of G' manifest.

Because the bound state is orthogonal to all elementary particle states,

$$\langle 0|\, B_i(x)|\, P, s\rangle \equiv 0$$

for all boson fields $B_i(x)$. Thus the full boson propagator in Fig. 6(c) has no pole at the bound-state mass and therefore the pole in G' must come from G''. Furthermore, assuming the pole in G' is of first order requires that the residue of the apparent double pole in Fig. 6(c) vanish, which is stated in Fig. 7(a). We then deduce the Bethe-Salpeter equation and the normalization conditions for the wave function from Fig. 6(a). They are given by Fig. 7(b) and Fig. 7(c).

Figure 7(a) restricts the bound-state wave function to be a particular type of solution to Fig. 7(b) and Fig. 7(c). This constraint involves a single-loop integral in momentum space equivalent to the statement

$$\langle 0|\, \bar{\psi}(x)\Gamma_i\psi(x)|\, P, s\rangle = 0 \tag{4.9}$$

in position space, where Γ_i is the matrix that occurs in the interaction Lagrangian

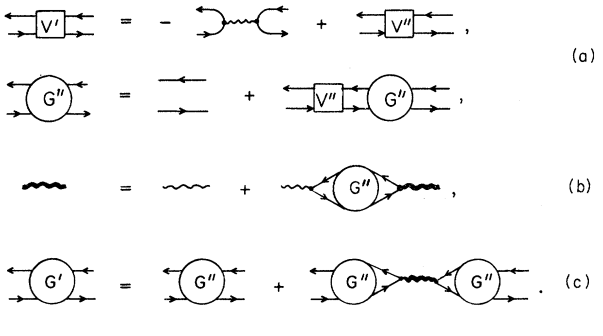


FIG. 6. Redefining the kernel by subtracting out the term with one bare boson propagator because there are no other graphs to renormalize it. Here, a heavy wavy line denotes full boson propagator.

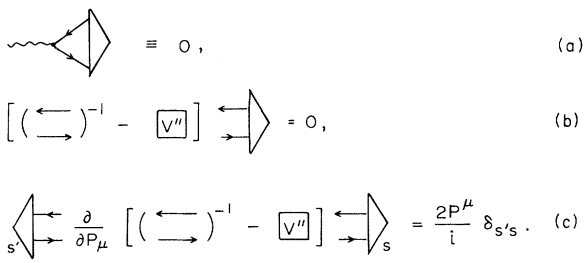


FIG. 7. Bethe-Salpeter equation and normalization condition for fermion-antifermion bound-state wave function. The equation is renormalizable because of (a).

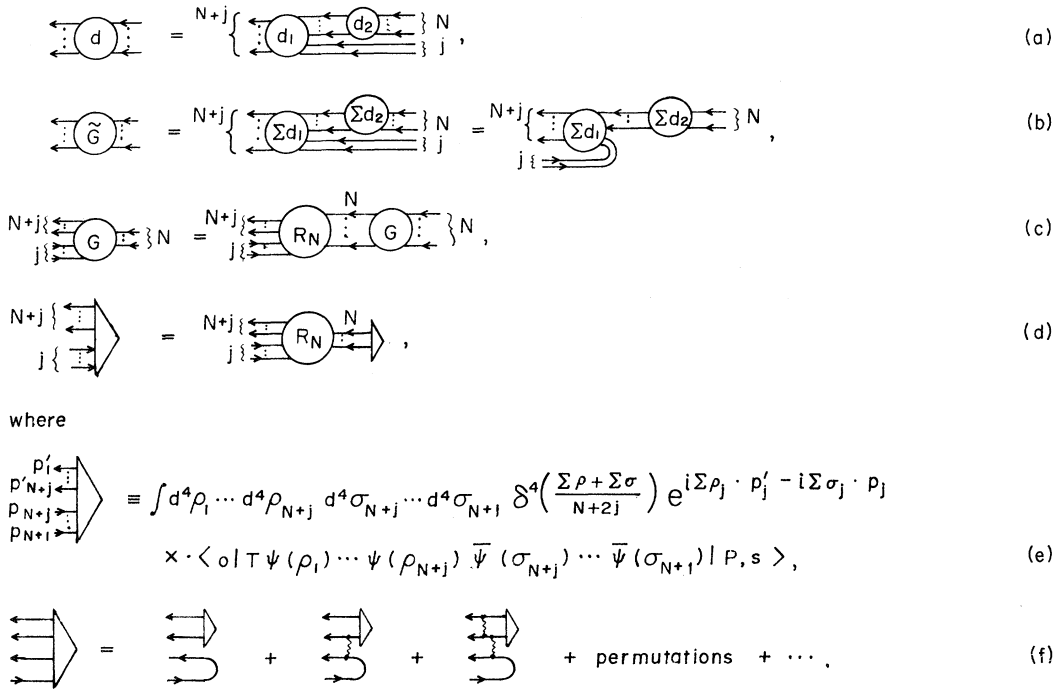


FIG. 8. Derivation of the relation (d) between a reducible wave function of N fermions + pairs and the irreducible one. The quantity R_N is defined in (5.2).

$\sum_i \bar{\psi}(x)\Gamma_i\psi(x)B_i(x)$. This constraint could also be obtained by taking the matrix element of the canonical field equation for $B_i(x)$, or by insisting that there be no pole in the Fourier transform of the vertex function $\langle 0|TB_i(y)\psi(x_1)\bar{\psi}(x_2)|0\rangle$.

V. REDUCIBLE WAVE FUNCTIONS FOR BOUND STATES OF IDENTICAL FERMIONS

A. N fermions + pairs

We have seen that irreducible wave functions satisfy homogeneous integral equations of the Bethe-Salpeter type. We now show that reducible wave functions may be calculated from the irreducible ones. This observation will be essential in calculating S-matrix elements in Sec. VI.

A reducible wave function contains an irreducible "core" plus any number of fermion-antifermion pairs. We consider here the case where the core consists of N fermions, and there are j additional pairs. The appropriate Green's function to start with is of the form (4.1), with N replaced by $N+j$. Its Fourier transform G therefore corresponds to (4.2) with the set of momentum-spin labels replaced by $\{p_1', \ldots, p_{N+j}'; p_1, \ldots, p_{N+j}\}$. Again we follow the convention of drawing all Feynman graphs such that all $N+j$ fermion lines are parallel to one another, and introduce the nonantisymmetrized Green's function $\tilde{G}$, which is the sum of all topological graphs. We look for a bound-state pole in the variable

$$P = \sum_{i=1}^{N} p_i = \sum_{i=1}^{N+j} p_i' - \sum_{i=N+1}^{N+j} p_i . \tag{5.1}$$

Since the fermion lines in $\tilde{G}$ are distinguishable and numbered with line numbers $1, \ldots, N+j$ corresponding to $p_1, \ldots, p_{N+j}$, the pole we seek is produced by interactions among the lines numbered $1, \ldots, N$.

Suppose d is an arbitrary topological graph in $\tilde{G}$. We can write it as a product $d = d_1 d_2$ uniquely, as shown graphically in Fig. 8(a), by demanding that (a) d_2 contain only interactions among lines $1, \ldots, N$; (b) no interaction be included in d_1, if

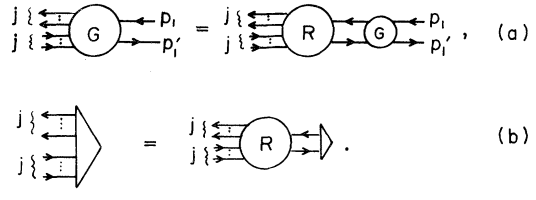


FIG. 9. The relation between a reducible wave function of j pairs ($j>1$) and the irreducible wave function of one pair.

it can be included in d_2.

Because this separation is unique, $\tilde{G}$ is the product of the sum of all d_1 and the sum of all d_2, as illustrated by Fig. 8(b), where in the last step we redraw the graphs such that the j noninteracting fermion lines entering from the right are bent back to become antifermions leaving on the left. The fully antisymmetrized Green's function G is now obtained by use of (4.4), i.e., by summing all $(N+j)!$ signatured permutations of $\{p_1', \ldots, p_{N+j}'\}$, the labels of line terminals with arrows directed out of the graph. The result is shown in Fig. 8(c) which defines R_N:

R_N = sum of all Feynman graphs with external lines as indicated in Fig. 8(c), such that

(a) the first interaction on the right, if it occurs, must not occur among the N right-entering lines, i.e., R_N contains no interaction that can be absorbed into G (the subscript N serves to remind us of this), and

(b) R_N contains no graphs that differ only by a permutation of the momentum-spin labeling of the N right-entering lines.

(5.2)

Because of the restriction (b), R_N is not fully antisymmetric in the $N+j$ lines leaving from the left. Nevertheless, the product $R_N G$ is fully antisymmetric in these $N+j$ lines because G is fully antisymmetric in its N lines. Antisymmetry in the j lines entering from the left is already contained in the definition of R_N. Antisymmetry among all $N+j$ lines entering (left or right) may be confirmed by the following counting: The set of N lines entering from the right may emerge on the left in any of $\binom{N+j}{N}$ possible subsets. For each such subset there are $j!$ possible permutations included in R_N of the j lines entering from the left, and there are $N!$ possible permutations included in G of the N lines entering from the right. Hence, there are a total of $\binom{N+j}{N} N!j! = (N+j)!$ permutations of the $N+j$ entering fermion lines.

By (2.15), G on the left side of Fig. 8(c) has a pole whose residue is the product of an $(N+2j)$-line reducible wave function and an N-line irreducible wave function. The G on the right side of Fig. 8(c) has a pole whose residue is the product of two N-line irreducible wave functions. Therefore, by going to the limit $P^2 \to M^2$, where P is defined in (5.1), we obtain a relation between the reducible wave function and the irreducible one, as stated by Fig. 8(d).

Simple relations like Fig. 8(d) do not exist relating two reducible wave functions, only relating reducible wave functions to the irreducible one. An example is given by Fig. 8(f). In this example there are only $3!/2! = 3$ permutations to be performed because of the restrictions on R_N. The three disconnected pieces in Fig. 8(f) are present because $\langle 0 | T\psi(x_1)\psi(x_2)\psi(y)\bar{\psi}(z) | P, s\rangle$ contains $\langle 0 | T\psi(y)\bar{\psi}(z) | 0\rangle \times \langle 0 | T\psi(x_1)\psi(x_2) | P, s\rangle$ and two other similar terms.

The original definition of a reducible wave function in Sec. II requires that it have a pole in some subset of the momenta representing the very same bound state, and a residue containing a wave function with fewer constituent fields. In Fig. 8(d), R_N is immediately reducible from the left, and contains the same bound-state pole as required plus other subpoles. This observation will be applied in Sec. VI.

B. Fermion-antifermion + pairs

Let G be the Fourier transform of

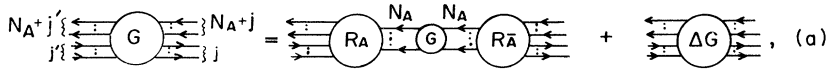
$$\langle 0 | T\psi(x_1') \cdots \psi(x_{j+1}')\bar{\psi}(x_1) \cdots \bar{\psi}(x_{j+1}) | 0\rangle$$

to momenta $p_1', \ldots, p_{j+1}'; p_1, \ldots, p_{j+1}$. To isolate a pole in the variable

$$P = p_1 - p_1' = \sum_{i=2}^{j+1} p_i - \sum_{i=2}^{j+1} p_i', \tag{5.3}$$

group G as in Fig. 9(a). The G on the right side of Fig. 9(a) is the same as in Fig. 5(a). R is the sum of all graphs with the indicated pattern of external propagators, with the restriction that R contain no graph that can be absorbed into G.

As a check that the right side of Fig. 9(a) is fully antisymmetric in the $j+1$ entering fermion lines, we perform the following counting: The line entering from the right may leave on the left



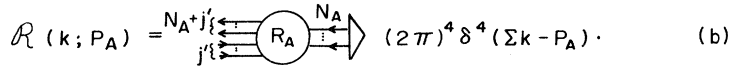


FIG. 10. Extracting a bound-state pole from a Green's function that appears in the reduction formula. R_A has the same meaning as R_N in Fig. 8(d).

in any of j positions, and the line leaving on the right may enter from the left in any of j positions. For each such choice there are $(j-1)!$ possible permutations included in R of the remaining $j-1$ lines that both enter and emerge on the left. Thus, there are $j^2(j-1)!$ possibilities of this type. In addition, the line entering from the right may emerge on the right, in which case R contains $j!$ permutations of the j lines that both enter and emerge on the left. Hence, the right side of Fig. 9(a) contains $j^2(j-1)!+j!=(j+1)!$ permutations of the entering fermion lines, just as it should.

The equation in Fig. 9(a) immediately gives Fig. 9(b), which is the desired relation between a reducible wave function and the irreducible one.

VI. THE S MATRIX

The completely general reduction formula (3.23) can now be applied to the case considered in Secs. IV and V in which there is only one elementary fermion field from which all irreducible wave functions are built. We will see that S-matrix elements do not depend on the choice of the number of constituents or the smearing functions U, and that their calculation always collapses to that involving irreducible wave functions. More importantly, we will derive Feynman-type rules for calculating any S-matrix element involving bound states.

To calculate the scattering $A+\cdots+C\to D+\cdots+F$ requires choosing sets of fields to describe each bound state and corresponding smearing functions U. Each set of fields has an irreducible subset that consists either of N-fermion fields or of one fermion and one antifermion field. The case in which the bound states have irreducible wave functions of the N-fermion type will be treated explicitly. The inclusion of fermion-antifermion types will be a simple extension.

To calculate a scattering amplitude involving a bound state $|A\rangle$, the reduction formula (3.23) requires, as a first step, evaluating

$$\mathfrak{R}(k;P_A)=\lim_{P_A^2\to m_A^2}\int(dp)G(k_1,\ldots,k_m;p_1,\ldots,p_n)i(M_A^2-P_A^2)U_A(p_1,\ldots,p_n)\delta^4(P_A-\textstyle\sum p_i),$$

$$\int(dp)\equiv\int[d^4p_1/(2\pi)^4]\cdots[d^4p_n/(2\pi)^4], \tag{6.1}$$

where all functions of position have been Fourier-transformed, P_A is the total momentum of the bound state $|A\rangle$, and $k\equiv\{k_1,\ldots,k_m\}$ are momenta of external legs of the Green's function G not involved in the bound state $|A\rangle$. The momenta $p_1,\ldots,p_n$ are those of the n elementary particles chosen to constitute a wave function for $|A\rangle$. Let the irreducible wave function for $|A\rangle$ contain N_A fermions so that in general $n=N_A+2j$, where j is the number of fermion-antifermion pairs. It follows from fermion conservation that in $G(k_1,\ldots,k_m;p_1,\ldots,p_n)$, $m=N_A+2j'$ $(j'\geq 0)$.

Now draw all the Feynman graphs of G in the form of the right-hand side of Fig. 10(a), in which the N_A+2j lines constituting the bound-state wave function are on the right, and the rest of the lines on the left. As one reads any graph from right to left, there are no intermediate states with fewer than N_A fermion lines. Hence, G can be uniquely decomposed into two terms as done in Fig. 10(a), where ΔG contains only graphs that have more than N_A fermion lines in any intermediate state; R_A is irreducible in the N_A lines on its right (i.e., contains no interactions that can be slid to the right off the graphs) and contains no graphs that differ from each other only by a permutation of the right legs; $R_{\bar{A}}$ is irreducible in the N_A legs on its left, and contains no graphs that differ from each other only by a permutation of the left legs.

By (2.15) the left side of Fig. 10(a) has a pole whose residue is

$$i(2\pi)^4\delta^4(\textstyle\sum k-\sum p)\sum_{s'}\chi_{s'}(N_A+2j')\bar{\chi}_{s'}(N_A+2j).$$

The first term on the right side of Fig. 10(a) has a pole whose residue is recognizable from Fig. 8(d) as the product of the explicitly reduced forms of the same two wave functions in the previous residue. Therefore, ΔG has no pole, and (6.1) becomes

$$\mathfrak{R}(k;P_A)=(2\pi)^4\delta^4(P_A-\textstyle\sum k)$$

$$\times\sum_{s'}\chi_{s'}(k)\int(dp)\delta^4(P_A-\textstyle\sum p)\bar{\chi}_{s'}(p)U_s(p).$$

By the normalization (3.4) of the U functions, the $\int(dp)$ integral is just a Kronecker δ in the discrete indices s and s'. Hence, $\mathfrak{R}$ is given by Fig. 10(b), where R_A has the same meaning as in Fig. 8(d), and is defined in (5.2). The subscript A merely serves to remind us that R_A contains no interactions that can be slid off the N_A legs on its right. This result shows that neither the number of fields nor the specific choice of the U functions used to define the creation and annihilation opera-

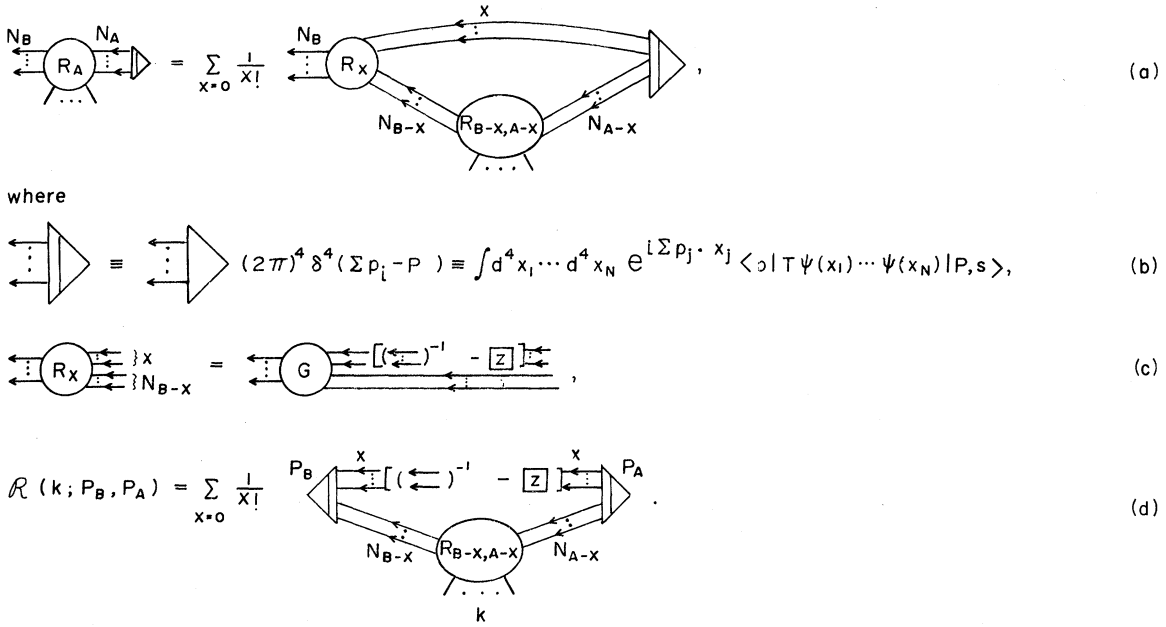


FIG. 11. Extracting a second bound-state pole B, after the first one, A, has been extracted as in Fig. 10.

tors of bound states (or of elementary particles) enters in an S-matrix calculation. Note that, although R_A contains an overall δ^4 of momenta, χ does not, so that their integrated product $R_A\chi$ does not.

If the S-matrix element to be calculated involves only one bound state $|A\rangle$, we are finished, because the remaining elementary particle poles can be immediately extracted by conventional means. If, however, there is another bound state $|B\rangle$, we must extract a pole from the appropriate subset of the k's in $\mathcal{R}(k; P_A)$ of Fig. 10(b). If the chosen set of fields $\mathfrak{G}$ is reducible, again only those graphs with exactly N_B intermediate fermion lines will produce a pole. We may, therefore, adopt the irreducible description of $|B\rangle$ immediately. Suppose the N_B lines are out-going. (The in-going case will be simpler.) Group the graphs of Fig. 10(b) as in Fig. 11(a), where a convenient graphical symbol for a wave function times $(2\pi)^4\delta^4(\sum p_i - P_A)$ is introduced in Fig. 11(b). The principle is that all interactions in R_A involving the N_B lines on the left forming the bound state are grouped into R_x. After this is done, there will in general be x lines free of further interactions, and other lines that have at least one further interaction. The latter are grouped into $R_{B-x,A-x}$, which is defined as the sum of all graphs of the pattern indicated in Fig. 11(a) that (a) contains no graphs in which any of the $N_A - x$ entering

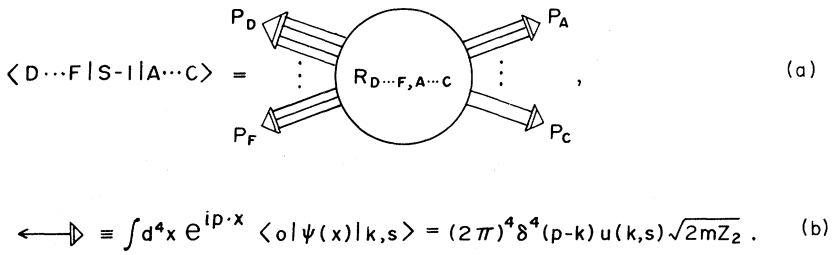


FIG. 12. Graphical rule for the S matrix. All wave functions are irreducible. The graphical symbol for a wave function is defined in Fig. 11(b). The rules for $R_{D\cdots F,A\cdots C}$ are given in the text. The generalization in Sec. VII shows that if any irreducible wave function involves more than one pair, then an extra factor $(N!M!c_0)^{-1}$ must be included for each such wave function, where N = no. of fermions, M = no. of antifermions, c_0 is given in Fig. 22(c).

lines emerge as one of the $N_B - x$ lines without having interacted; (b) contains no graphs reducible in the $N_A - x$ lines on the right or reducible in the $N_B - x$ lines on the left; and (c) contains no graphs that differ from each other only by a permutation of the $N_A - x$ right legs or of the $N_B - x$ left legs.

Finally, R_x is the sum of all graphs of the indicated pattern that contains no graphs reducible in the x lines on the right. Because by construction R_x contains all graphs differing in permutations of the momentun-spin labeling of the N_B lines entering from its right, and because the wave function on the far right of Fig. 11(a) is automatically antisymmetric in its N_A lines, the factor $(x!)^{-1}$ in Fig. 11(a) is needed to prevent overcounting.

For $x=0$ or $x=1$, $R_x=G$ and we may immediately extract the pole for $|B\rangle$. For $x>1$, however, R_x does not contain all the graphs to be a Green's function, because by construction those interactions among the x lines that can be slid off to the right are not there, having been absorbed into the wave function on the far right earlier, in the steps in Fig. 10. The general relation between R_x and G is given by Theorem 2 of Appendix A, and is stated in Fig. 11(c), where Z is the kernel defined in (4.5) or (4.6) for x lines. With Fig. 11(c) inserted into Fig. 11(a) we can easily extract the pole of G at $P_B{}^2=M_B{}^2$. Integrating away the U_B functions as before gives Fig. 11(d), which is the desired result.

If the N_B lines had been in-going, only $x=0$ would have been possible and the N_B lines would have to be reversed. If $|B\rangle$ had an irreducible wave function of the fermion-antifermion type, the appropriate $R_{B,A}$ would have been the same only with $x=0,1$ and a few obvious line reversals.

Extracting poles from the remaining $\{k\}$ requires no new procedures. The general result is Fig. 12(a). In this equation all wave functions are irreducible. Initial and final states differ in the direction of energy flow. Particles and antiparticles differ in the correlation of fermion number flow with energy flow. An overall $(2\pi)^4\delta^4$ of momenta is automatically contained in Fig. 12(a) because of the definition Fig. 11(b). Most importantly, $R_{D\ldots F,A\ldots C}$ is the sum of all Feynman graphs with the indicated pattern of external propagators with the following restrictions:

(a) R contains no graphs that are reducible in the bound groups (i.e., in the set of fermion lines attached to each bound-state wave function);

(b) R contains no graphs that differ from each other only by a permutation of lines within bound groups;

(c) R contains, for each graph in which a subset of x lines of one bound-group terminates entirely within another bound group without having interacted with another line, an additive contribution of compensating graphs, which is obtained by inserting in place of the x noninteracting lines the negative of the interaction kernel Z of (4.5) or (4.6) appropriate to those x lines; and

(d) R contains, for the sum of each graph of (c) and its compensating graphs, a numerical factor $(x!)^{-1}$.

If one of the external particles in Fig. 12(a) is elementary, the formula becomes a more familiar one, because by Fig. 11(b) the wave function reduces to Fig. 12(b). Thus, in this case, the propagator of the particle is not really present, though it appears to be in Fig. 12(a).

The complete S matrix of the theory connects all states made of elementary particles and bound states. Its matrix elements as calculated by Fig. 12(a) satisfy Lorentz covariance, crossing, and unitarity. In Sec. VII other quantum numbers will be introduced and the matrix elements will then be covariant with respect to the appropriate internal-symmetry groups. The analytic properties of these matrix elements are not obvious, owing to the complications introduced by the wave functions. Thus, for example, whereas off-mass-shell elementary particles can be easily defined, off-mass-shell bound states do not have an intermediate natural definition.

We close this section by illustrating the effect of the compensating graphs $-Z$. Consider a term in the graphical expansion of the proton form factor $\langle P'|J(x)|P\rangle$ for a proton $|P\rangle$ whose irreducible wave function is composed of three quarks, as shown in Fig. 13(a). The full kernel for the wave function is $V_{12}+V_{13}+V_{23}+V_{123}$. Therefore, the quantity in the square brackets in Fig. 13(a) may be replaced by $V_{13}+V_{23}+V_{123}$, either to the right or to the left of the interaction with the external field (but not both). Hence, Fig. 13(a) contains graphs like those of Fig. 13(b), and does not contain a graph that is zero order in the quark-gluon coupling constant, as one might think at first glance.

VII. GENERALIZATION TO NONIDENTICAL FERMIONS

In Secs. IV, V, and VI we have dealt with bound states whose irreducible wave functions are built from a single elementary fermion spinor field. We now extend the discussion to the case where there are different fermion species.

It is useful to define two fermion spinor fields ψ_a and ψ_b to be equivalent if there is an interaction $\bar{\psi}_a\Gamma^i_{ab}\psi_b B_i$ which transforms one into the other with the emission of a virtual boson. For example, in the quark model λ is equivalent to $\mathcal{P}$ and $\mathcal{N}$ only if there are strange bosons that couple them. The

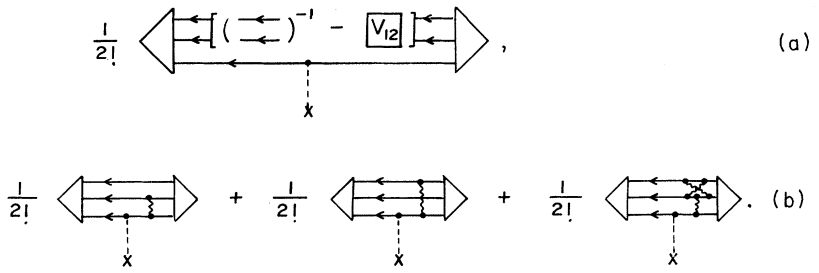


FIG. 13. Illustration of the effect of subtraction of redundant graphs. See text for explanation.

proton field is not equivalent to the electron field in whatever theory. If the discrete quantum numbers distinguishing the equivalent fermions are conserved, as we assume, then it is useful and conventional to regard the set of equivalent fields $\{\psi_a, \psi_b, \ldots\}$ formally as a single multicomponent fermion field, by specifying that at spacelike separations different equivalent fields anticommute (instead of commute), so that wave functions are antisymmetric under the interchange of the coordinates and discrete indices of any two equivalent constituents. Once we adopt this convention, the discussions of N-fermion wave functions and fermion-antifermion wave functions become trivial extensions of previous cases. They merely acquire discrete quantum number indices which are summed over in all integral equations and in the final answer in Fig. 12(a), just as the discrete spinor indices were summed. Thus, all formulas involving these types of wave functions, including the reduction formula, remain the same if the set of spinor indices is enlarged to include other quantum numbers.

The only really new element is that now it is possible to have a bound state whose irreducible wave function contains more than one fermion-antifermion pair, namely, one that contains both a field and the adjoint of an equivalent but not identical field. (If it contained the adjoint of an identical field it would be reducible.) What makes the situation new is that, in addition to graphs of the propagating type shown in Fig. 14(a), there now exist Feynman graphs like those in Fig. 14(b),

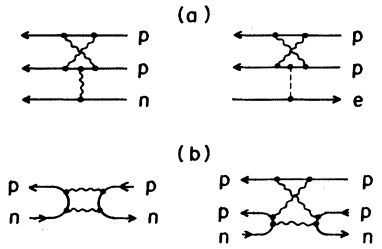


FIG. 14. Typical types of graphs involving nonidentical fermions.

in which annihilations between equivalent, but not identical, fields occur. Our development will show that these annihilation graphs do not contribute to the kernel of the Bethe-Salpeter equation, by reason of renormalizability. This makes the procedure of pole extraction in the reduction formula the same as before, and one will arrive at the same formula as Fig. 12(a) for the S matrix, apart from trivial counting factors.

We are interested in an irreducible wave function consisting of N fermions and M antifermions. Without loss of generality we assume $N \geq M$. To begin, we consider the Green's function G shown in Fig. 15(a), in which N fermion lines and M antifermion lines flow through the graphs, with the

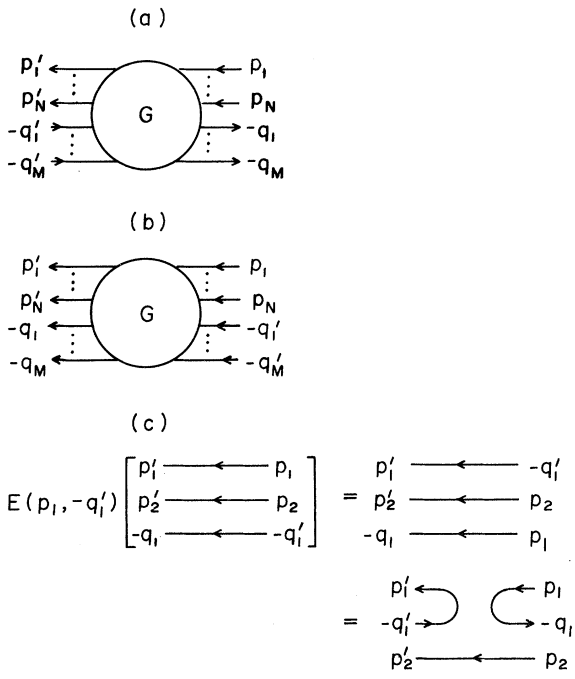


FIG. 15. (a) Convention for momentum assignment for the general Green's function. (b) Redrawing (a) to make all fermion lines flow in the same direction, in order to clarify the process of antisymmetrization. (c) Example of how flow patterns originate.

momenta of the external particles assigned as follows:

$$\begin{aligned} &\text{Initial fermions: } p_1 \cdots p_N , \\ &\text{Initial antifermions: } q_1 \cdots q_M , \\ &\text{Final fermions: } p_1' \cdots p_N' , \\ &\text{Final antifermions: } q_1' \cdots q_M' . \end{aligned} \tag{7.1}$$

We also include in the label p_i or q_j the set of discrete quantum numbers associated with an external particle. The irreducible wave function we are interested in is contained in the residue of a pole in the variable

$$\begin{aligned} P &= \sum_{i=1}^{N} p_i + \sum_{i=1}^{M} q_i \\ &= \sum_{i=1}^{N} p_i' + \sum_{i=1}^{M} q_i' , \end{aligned} \tag{7.2}$$

i.e., in the total momentum that flows from right to left in Fig. 15(a).

If one makes vertical cuts in the graphs of Fig. 15(a), always cutting only fermion or antifermion lines but no boson lines, then owing to the choice of quantum numbers any such cut must intersect at least N internal fermion lines and M internal antifermion lines. That is, by assumption there exists no state containing fewer particles having the given quantum numbers.

If one traces a fermion line as it goes through a graph in G, it either propagates continuously through the graph from right to left (or vice versa), or it may both enter and leave the graph on the same side (left or right). We find it useful to distinguish between propagating lines and back-flowing lines: A *propagating line* is a fermion line that flows from p_i to p_j' or from $-q_i'$ to $-q_j$. A *back-flowing line* is a fermion line that flows from p_i to $-q_j$ or from $-q_i'$ to p_j'.

Back-flowing lines always come in pairs. In any Feynman graph of G, if there are α back-flowing lines on the right, then there are α back-flowing lines on the left. The number of pairs of back-flowing lines α characterizes the *flow pattern* of a graph independent of the number of interaction vertices, the connectivity of boson lines, or the number of closed fermion loops. Because $N \geq M$ by convention, $\alpha = 0, 1, \ldots, M$. Let Γ^α be the sum of all Feynman graphs in G with flow pattern α. Then

$$G = \sum_{\alpha=0}^{M} \Gamma^\alpha . \tag{7.3}$$

To clarify the antisymmetrization of labels on external legs, especially when different flow patterns are possible, redraw the graphs in Fig. 15(a) in the form shown in Fig. 15(b), in which all arrows of fermion lines point in the same direction. Then each fermion line flows through the graph from right to left. Each Feynman graph is characterized by a one-to-one association between elements of the sets $\{p_1, \ldots, p_N; -q_1', \ldots, -q_M'\}$ and $\{p_1', \ldots, p_N'; -q_1, \ldots, -q_M\}$. As before, define *topological graphs* to be the subset of Feynman graphs with the standard association

$$\begin{aligned} &p_i \leftrightarrow p_i' \quad (i = 1, \ldots, N) , \\ &q_j \leftrightarrow q_j' \quad (j = 1, \ldots, M) , \end{aligned} \tag{7.4}$$

and denote the sum of all topological graphs by $\tilde{G}$. Then G is obtainable from $\tilde{G}$ by adding all signatured permutations on the set $\{p_1, \ldots, p_N; -q_1', \ldots, -q_M'\}$, or on the set $\{p_1', \ldots, p_N'; -q_1, \ldots, -q_M\}$. One may also do both and divide by $(N+M)!$.

Because we are eventually interested in the choice of initial and final states as indicated in Fig. 15(a) and not Fig. 15(b), some of the permutations above interchange final and initial labels, and it is necessary to distinguish between those that do and those that do not. Accordingly, we carry out an arbitrary permutation on $\{-q_j'; p_i\}$ in the following manner. Call the M letters that stand to the left of the semicolon the left subset, and the N letters that stand to the right the right subset. First, choose arbitrarily α letters each from the left and right subsets, and move them across the semicolon from one subset to the other, placing them in any order whatsoever. After this is done permute the new set of letters by PQ', where P is a permutation of the new right subset, and Q' is a permutation of the new left subset. In this way a permutation on $\{-q_j'; p_i\}$ is specified by P and Q' and the names $\{i_1, \ldots, i_\alpha\}$ and $\{j_1, \ldots, j_\alpha\}$ of the exchanged letters:

$$\text{Perm} = PQ'[E(i_1, j_1) \cdots E(i_\alpha, j_\alpha)] , \tag{7.5}$$

where $E(i,j)$ interchanges p_i and $-q_j'$. The signature of (7.5) is $\delta_P \delta_{Q'} (-1)^\alpha$, where δ_P, $\delta_{Q'}$ are respectively the signature of P and Q'. As a check, note that the number of operations of the form (7.5) is $N!M!\,\binom{N}{\alpha}\binom{M}{\alpha}$, and the total number of permutations is

$$N!M! \sum_{\alpha=0}^{M} \binom{N}{\alpha} \binom{M}{\alpha} = (N+M)! . \tag{7.6}$$

After the exchanges $E(i,j)$ in (7.5) are carried out, redraw the graphs in the desired division of initial and final states, as indicated in Fig. 15(a). The permutations P and Q' then act respectively on the initial labels $\{p_i\}$ and the final labels $\{-q_i'\}$. The different exchanges give rise to the different flow patterns mentioned earlier. An example is

given in Fig. 15(c).

In general each $E(i,j)$ gives rise to a pair of back-flowing lines, so that the permutation (7.5) gives rise to a graph of flow pattern α. For a given α, there are $\binom{N}{\alpha}\binom{M}{\alpha}$ possible choices of $\{i_1,\ldots,i_\alpha\}$ and $\{j_1,\ldots,j_\alpha\}$, all leading to the same flow pattern. The different choices eventually give the same set of Feynman graphs, after we sum over all permutations PQ' with signature. Therefore, if $\tilde{\Gamma}^\alpha$ is the sum of all topological graphs of flow pattern α, then

$$\Gamma^\alpha=\binom{N}{\alpha}\binom{M}{\alpha}\sum_P P\delta_P\sum_{Q'}Q'\delta_{Q'}\tilde{\Gamma}^\alpha. \tag{7.7}$$

The signature factor $(-1)^\alpha$ is absorbed into $\tilde{\Gamma}^\alpha$. To obtain a more symmetrical representation, we may apply to (7.7) the operation $\sum P'\delta_{P'}\sum Q\delta_Q$, and divide by $N!M!$. Thus we obtain

$$G=\frac{1}{N!M!}\sum_{\alpha=0}^{M}\binom{N}{\alpha}\binom{M}{\alpha}\hat{A}\tilde{\Gamma}^\alpha, \tag{7.8}$$

where, to reiterate, $\tilde{\Gamma}^\alpha$ is the sum of all topological graphs of flow pattern α, and $\hat{A}$ the sum of all signatured permutations of initial and final fermion and antifermion labels within each set separately: $\{p_i\}$, $\{p_i'\}$, $\{-q_i\}$, $\{-q_i'\}$.

Next we regroup the topological graphs in $\tilde{\Gamma}^\alpha$ into irreducible subsets. First, we reemphasize that the choice of quantum numbers is such that any vertical cut in a graph that does not intersect a boson line must intersect at least N fermion lines and M antifermion lines. For example, the graph in Fig. 16(a) does not exist, but the one in Fig. 16(b) does. To define our procedures clearly, we establish special conventions in drawing the topological graphs. For a particular topological graph draw all lines parallel. Disconnected subgraphs, if any, are to be drawn as separate nonoverlapping tiers, as if the graph had been "combed." In each connected subgraph, the back-flowing pairs, if any, are to be paired arbitrarily, with respect to the vertical positioning of each member of the pair. A "combed" graph is shown in Fig. 17(a).

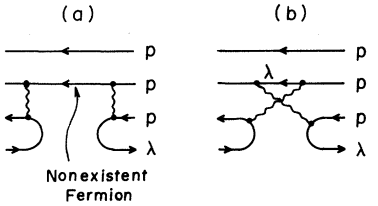


FIG. 16. Graph (a) does not exist because of the choice of quantum numbers. Graph (b) exists.

Now a natural pairing of the back-flowing lines is established as follows. Make all possible distinct cuts in the graph, with each cut intersecting exactly N fermion lines, M antifermion lines, and no boson lines. A graph segment between two successive cuts may be disconnected, even if the whole graph is connected. We specify that the graph shall be so drawn that all segments be already "combed." The graph is then said to be "well-combed." Figure 17(b) shows the well-combed form of Fig. 17(a). If a graph is not in well-combed form, it can always be redrawn in well-combed form by permuting the order of back-flowing lines in the initial and/or final states—like twisting the ends of a rope to disentangle its strands. Ambiguity arises when a permutation of two back-flowing lines in the initial or final state does not destroy the well-combed form. In that case, an arbitrary choice shall be made.

Consider a topological graph drawn in well-combed form. Imagine that all possible cuts that intersect exactly $N+M$ lines are made, and number each line intersected by a cut in an order that is the same for all cuts. That order is specified by the numbering of lines in the initial state, for which the following convention is adopted: Number the ends of a back-flowing line by i and $\bar{\imath}$, respectively, for the fermion and antifermion end, with $i=1,2,\ldots,\alpha$. Number the propagating fermion lines by j, with $j=\alpha+1,\ldots,\alpha+N$. Number the propagating antifermion lines by $\bar{k}$, with $k=\alpha+1,\ldots,\alpha+M$. An example of this convention is shown in Fig. 18. Between two successive cuts lies an irreducible subgraph. An ambiguity in line numbering can occur only when there is an irreducible subgraph involving more than one back-flowing pair. In that case we make an arbitrary choice, and the arbitrariness will be taken into account later in our definition of irreducible kernels.

An *irreducible graph* is defined to be a connected topological graph with a propagating lines (fermion or antifermion) and b back-flowing pairs that can-

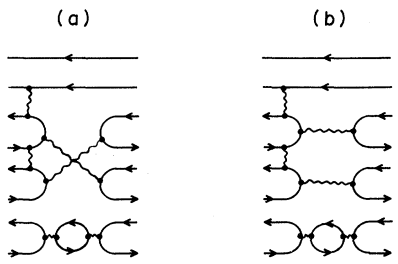


FIG. 17. (a) A graph in "combed" but not "well-combed" form. (b) The "well-combed" form of (a).

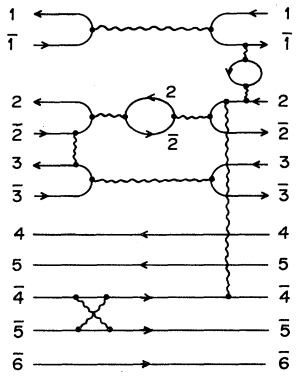


FIG. 18. Example illustrating the convention for line numbering in a topological graph.

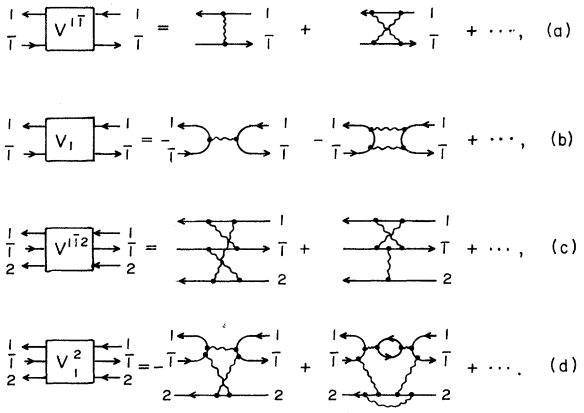


FIG. 19. Examples of irreducible kernels.

not be made disconnected by a vertical cut intersecting $a+2b$ *or fewer* lines (fermion or antifermion), and intersecting no boson lines.

An *irreducible kernel* V_Λ^L is defined as follows:

(a) Choose a set L of line numbers of propagating lines (fermion or antifermion), and choose a set Λ of fermion line numbers of back-flowing pairs:

$$L\subseteq\{1,\ldots,N;\bar{1},\ldots,\bar{M}\}\,,\qquad \Lambda\subseteq\{1,\ldots,N\} \tag{7.9}$$

with the restriction that L and Λ cannot be both empty, and that if $\lambda\in\Lambda$, then $\lambda\notin L$ and $\bar{\lambda}\notin L$:

$$\Lambda\cup L\neq\phi$$

$$\Lambda\cap L=\phi,\quad \bar{\Lambda}\cap L=\phi\,.$$

(b) Draw all irreducible topological graphs involving the lines in L and the pairs in Λ. If there is more than one pair, then a permutation of back-flowing lines in the initial or final state shall not lead to a distinct graph.

(c) Sum all the graphs mentioned above, remove all external legs, and multiply by $(-1)^b$, where b is the number of pairs (i.e., number of elements in Λ).

Some examples of kernels are shown in Fig. 19. Note that the first term of Fig. 19(b) is not renormalizable, just as the first term of Fig. 5(c) is not.

The sum $\tilde{G}^\alpha$ of all topological graphs containing, at most, α pairs is the sum of all distinct products of those kernels containing, at most, α pairs. It, therefore, satisfies the integral equation in Fig. 20, with the kernel Z^α given by

$$Z^\alpha=\sum_{g=1}(-1)^g\sum_{\substack{\{L_1,\ldots,L_g\}\\ \{\Lambda_1,\ldots,\Lambda_g\}}} V_{\Lambda_1}^{L_1}\cdots V_{\Lambda_g}^{L_g}\,, \tag{7.10}$$

where $\{L_1,\ldots,L_g\}$ and $\{\Lambda_1,\ldots,\Lambda_g\}$ are such that $\{V_{\Lambda_1}^{L_1},\ldots,V_{\Lambda_g}^{L_g}\}$ is a set of mutually commuting kernels, and

$$\begin{aligned} &L_i\subseteq\{1,\ldots,N;\bar{1},\ldots,\bar{M}\}\,,\\ &\Lambda_i\subseteq\{1,\ldots,\alpha\}\,,\\ &\Lambda_i\cup L_i\neq\phi\,,\\ &\Lambda_i\cap L_i=\phi\,,\\ &\bar{\Lambda}_i\cap L_i=\phi\,. \end{aligned} \tag{7.11}$$

More explicitly, $\tilde{G}^\alpha$ is the sum of all topological graphs with either no back-flowing lines, or one pair of back-flowing lines in any of $\binom{\alpha}{1}$ positions, or two pairs of back-flowing lines in any of $\binom{\alpha}{2}$ positions, ..., or α pairs of back-flowing lines in one possible position. Hence,

$$\tilde{G}^\alpha \leftrightarrow \sum_{\beta=0}^{\alpha}\binom{\alpha}{\beta}\tilde{\Gamma}^\beta\,, \tag{7.12}$$

where the symbol $\leftrightarrow$ means "topologically equivalent to", that is, both sides of the relation yield the same result upon application of the permutation operator $\hat{A}$ in (7.8). Inverting this leads to

$$\tilde{\Gamma}^\beta \leftrightarrow \sum_{\alpha=0}^{\beta}(-1)^{\beta-\alpha}\binom{\beta}{\alpha}\tilde{G}^\alpha\,. \tag{7.13}$$

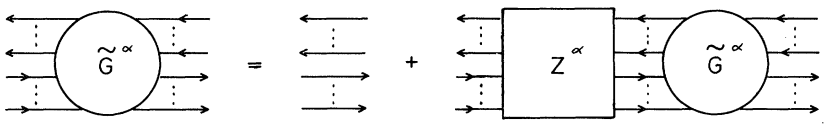


FIG. 20. The integral equation for $\tilde{G}^\alpha$, the sum of all topological graphs containing at most α pairs of back-flowing lines. The equation is not renormalizable if $\alpha>0$, owing to graphs in Z^α that have no renormalizing partners.

Substituting this into (7.8) we obtain

$$G = \hat{A} \sum_{\alpha=0}^{M} c_\alpha \tilde{G}^\alpha , \tag{7.14}$$

$$c_\alpha \equiv \sum_{\beta=\alpha}^{M} (-1)^{\beta-\alpha} [(N-\beta)!(M-\beta)!\beta!\alpha!(\beta-\alpha)!]^{-1} . \tag{7.15}$$

These formulas collapse to those of Sec. IV for the case N arbitrary, $M=0$, and the case $N=M=1$.

A bound-state pole can occur in G only if it occurs in at least one of the $\tilde{G}^\alpha$. Although $\tilde{G}^\alpha$ itself is renormalizable, the integral equation for it in Fig. 20 is not if $\alpha>0$, because its kernel contains bare-boson propagators with no graphs to renormalize them, as in the first graph of Fig. 19(b). More importantly, if $\alpha>0$, then the integral equation for the corresponding bound-state wave function is not renormalizable, except in the special case of $N=M=1$ discussed in Sec. IV B. We recall that in that special case renormalizability is guaranteed by Fig. 7(a), which gets rid of the unrenormalizable graph. In the general case here, in contrast to Sec. IV B, there is no reason for integrals like that of Fig. 21 to vanish automatically. A renormalizable Bethe-Salpeter equation can result only if the bound-state pole in G occurs in $\tilde{G}^0$ alone. By Fig. 20, the wave function then satisfies Fig. 22(a) and Fig. 22(b). These results collapse to those those of Sec. IV A for N arbitrary, $M=0$. For the special case $N=M=1$, one finds that $c_0=0$; thus the pole occurs in $\tilde{G}^1$, and the integral in Fig. 21 does vanish automatically. The discussion then reduces to those in Sec. IV B.

Since the kernel Z^0 of the Bethe-Salpeter equation in Fig. 22(a) involves only graphs with propagating lines (fermion or antifermion), the extraction of the corresponding bound-state pole in the reduction formula proceeds the same way as in Sec. VI. Consequently, the S matrix is still given by Fig. 12(a) except for extra counting factors, as long as spinor indices are generalized to include indices corresponding to other discrete quantum numbers. The extra counting factors arise because of the factor c_0 in Fig. 22(b), which is the weight of $\tilde{G}^0$ in (7.14). The new rule appended to Fig. 12(a) is as follows: For each irreducible wave function in Fig. 12(a) that involves more than one fermion-antifermion pair and which contains N fermions and M antifermions, multiply the matrix element in Fig. 12(a) by $(N!M!c_0)^{-1}$, where c_0 is given by Fig. 22(c). To check that this reduces to the old case, note that $c_0=(N!)^{-1}$ if $M=0$.

In our discussion of identical fermions in Sec. IV, we assumed that irreducible wave functions are constituted either of N fermions or of one pair. This may not be so, if for some reason the minimal wave function vanishes identically. In such a case, the irreducible wave function may involve supernumerary pairs, and the discussion in this section applies without essential change. The final result for the Bethe-Salpeter equation is the same as Fig. 22(a) and Fig. 22(b), and rules for the S matrix are amended by the same factor $(N!M!c_0)^{-1}$ as explained in the last paragraph.

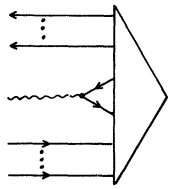


FIG. 21. There is no reason for this to be identically zero. If there were, then Fig. 20 would have been renormalizable for all α.

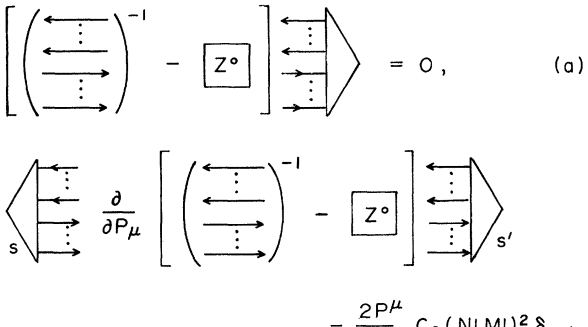
APPENDIX A: TWO THEOREMS

Lemma. Let $V=\{v_1,\ldots,v_r\}$ be a set of r operators. The sum of all distinct products of the v's is given by the expansion of $[1-Z(V)]^{-1}$, where

$$Z(V) \equiv \sum_{i=1}^{r} v_i - \sum_{\langle ij\rangle} v_i v_j + \sum_{\langle ijk\rangle} v_i v_j v_k - \cdots$$

and $\langle ij\rangle$ denotes distinct pairs of commuting operators, $\langle ijk\rangle$ denotes distinct triplets of mutually commuting operators, etc.

Proof. The theorem is obviously true for $r=1$. Assume it is true for $2,\ldots,r$ and consider the set $V'=\{v_1,\ldots,v_r,v'\}$ of $r+1$ operators. We will



$$C_0 \equiv \sum_{\beta=0}^{M} \frac{(-1)^\beta}{(N-\beta)!(M-\beta)!(\beta!)^2} . \tag{c}$$

FIG. 22. Bethe-Salpeter equation and normalization condition for irreducible wave function containing more than one fermion-antifermion pair.

show that $[1 - Z(V')]^{-1}$ is the sum of all distinct products of the $r+1$ v's. First, let C be the set of all operators in V that commute with v'. Then from the definition of Z we may write

$$Z(V') = Z(V) + [1 - Z(C)]v' .$$

Hence,

$$\frac{1}{1 - Z(V')} = \frac{1}{1 - Z(V) - [1 - Z(C)]v'}$$

$$= \{1 + Rv' + Rv'Rv' + \cdots\} \frac{1}{1 - Z(V)} , \tag{A1}$$

where

$$R \equiv \frac{1}{1 - Z(V)} [1 - Z(C)] . \tag{A2}$$

The right side of (A1) is the sum of all distinct products of $\{v_1, \ldots, v_r, v'\}$, as claimed, only if R is the sum of all distinct products of $\{v_1, \ldots, v_r\}$ that cannot be written with a member of C on the right. Call such a product a good product. Every product of $\{v_1, \ldots v_r\}$ is either good or can be written as a good product multiplied from the right by any product of elements in C. Hence, the sum of all distinct products of $\{v_1, \ldots v_r\}$ can be written as the sum of all distinct good products of the v's multiplied from the right by the sum of all distinct products in C. From the inductive hypothesis this means that

$$\frac{1}{1 - Z(V)} = (\text{sum of all distinct good products}) \times \frac{1}{1 - Z(C)} .$$

Thus R in (A2) is the sum of all distinct good products and the lemma is proved.

Theorem 1. If G is the sum of all distinct products of V, then $G = 1 + Z(V)G$.

Theorem 2. If R_C is the sum of all distinct products of V that cannot be written with a member of the set $C \subseteq V$ on the right, then $R_C = G[1 - Z(C)]$.

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