Comments and Addenda

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Type-N gravitational field with twist – Analytical addendum to Hauser's solution*

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(Received 30 December 1974)

Hauser has presented a type-N gravitational field with twist in terms of a function $f(y)$, which he expressed as the solution of a differential equation. The exact solutions of this equation are presented here as Gauss hypergeometric functions.

Recently,[1] Hauser presented an exact type-N solution of the Einstein field equations with twisting rays. He presents the solution in terms of a function $f(y)$ which satisfies

$$(1+y^2)\frac{d^2f}{dy^2}+\tfrac{3}{16}f=0 \ . \tag{1}$$

Hauser notes that the two solutions of Eq. (1) have the following properties:

(a) The even solution has zeros only at $y=\pm5.5$. The odd solution has a zero only at $y=0$. We will denote these solutions as $f_e(y)$ and $f_o(y)$, with

$$\begin{aligned} f_e(y)&=f_e(-y) \ , \\ f_o(y)&=-f_o(-y) \ . \end{aligned} \tag{2}$$

(b) As $y\to\infty$, two independent solutions behave like

$$\begin{aligned} f_{3/4}&\to C_1 y^{3/4} \ , \\ f_{1/4}&\to C_2 y^{1/4} \ . \end{aligned} \tag{3}$$

We present here the exact solutions of Eq. (1) in terms of the hypergeometric function.[2,3] If we define

$$\begin{aligned} z&\equiv -y^2 \ , \\ \dot f&\equiv \frac{df}{dz} \end{aligned} \tag{4}$$

then Eq. (1) becomes

$$z(1-z)\ddot f+(\tfrac{1}{2}-\tfrac{1}{2}z)\dot f-\tfrac{3}{64}f=0 \ . \tag{5}$$

The two solutions in the neighborhood of zero are given by Erdélyi[2] [Eq. 2.3.1-(1)]:

$$\begin{aligned} u_{1(0)}(z)&={}_2F_1(-\tfrac{1}{8},-\tfrac{3}{8},\tfrac{1}{2},z) \ , \\ iu_{2(0)}(z)&=(-z)^{1/2}{}_2F_1(\tfrac{3}{8},\tfrac{1}{8},\tfrac{3}{2},z) \ . \end{aligned} \tag{6}$$

These solutions are proportional to the even and odd solutions of Eq. (2), with

$$\begin{aligned} f_e(y)&=a_1u_{1(0)}(-y^2) \ , \\ f_o(y)&=a_2u_{2(0)}(-y^2) \ , \end{aligned} \tag{7}$$

where a_1, a_2 are constants. Note that Eq. 15.4.25 of Abramowitz and Stegun[3] implies

$$u_{1(0)}(-y^2)=\frac{\Gamma(\frac{3}{8})\Gamma(\frac{11}{8})}{\sqrt{\pi}\,2^{3/4}}(1+y^2)^{1/4}\{P_{-3/2}^{-1/4}([y^2/(1+y^2)]^{1/2})+P_{-3/2}^{-1/4}(-[y^2/(1+y^2)]^{1/2})\} \tag{8}$$

for $0<y<\infty$.

Expressions for $u_{1(0)}(-y^2)$ and $u_{2(0)}(-y^2)$ for $y=1$ are given by Eq. (6), since these hypergeometric series are absolutely convergent at $|-y^2|=1$.

As $z\to\infty$, these solutions are better understood by analytic continuation [Erdélyi[2], Eq. 2.10-(2)], which gives

$$\begin{aligned} u_{1(0)}(z)&=B_1(-z)^{1/8}{}_2F_1(-\tfrac{1}{8},\tfrac{3}{8},\tfrac{5}{4},1/z) \\ &\quad +B_2(-z)^{3/8}{}_2F_1(-\tfrac{3}{8},\tfrac{1}{8},\tfrac{3}{4},1/z) \ , \\ iu_{2(0)}(z)&=B_3(-z)^{1/8}{}_2F_1(\tfrac{3}{8},-\tfrac{1}{8},\tfrac{5}{4},1/z) \\ &\quad +B_4(-z)^{3/8}{}_2F_1(\tfrac{1}{8},-\tfrac{3}{8},\tfrac{3}{4},1/z) \ , \end{aligned} \tag{9}$$

with

$$B_1 = \frac{\Gamma(\frac{1}{2})\Gamma(-\frac{1}{4})}{\Gamma(-\frac{3}{8})\Gamma(\frac{5}{8})}, \quad B_2 = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}{\Gamma(-\frac{1}{8})\Gamma(\frac{7}{8})},$$
$$B_3 = \frac{\Gamma(\frac{3}{2})\Gamma(-\frac{1}{4})}{\Gamma(\frac{1}{8})\Gamma(\frac{9}{8})}, \quad B_4 = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{8})\Gamma(\frac{11}{8})}. \qquad (10)$$

Thus the solutions of interest as $y \to \infty$ are

$$V_{1(\infty)}(-y^2) = y^{1/4}{}_2F_1(-\tfrac{1}{8}, \tfrac{3}{8}, \tfrac{5}{4}, -1/y^2),$$
$$V_{2(\infty)}(-y^2) = y^{3/4}{}_2F_1(-\tfrac{3}{8}, \tfrac{1}{8}, \tfrac{3}{4}, -1/y^2), \qquad (11)$$

which give the behavior of Eq. (3). Note that Eq. 15.4.11 of Abramowitz and Stegun[3] tells us that

$$V_{1(\infty)}(-y^2) = 2^{1/4}\Gamma(\tfrac{5}{4})(1+y^2)^{1/4}P_{-3/2}^{-1/4}([y^2/(1+y^2)]^{1/2}),$$
$$V_{2(\infty)}(-y^2) = 2^{-1/4}\Gamma(\tfrac{3}{4})(1+y^2)^{1/4}P_{-3/2}^{1/4}([y^2/(1+y^2)]^{1/2}) \qquad (12)$$

for $0 < y < \infty$.

A preliminary computer search[4] for zeros of the even and odd solutions found no zeros in $f_0(y)$ other than at $y = 0$, and found one zero in $f_e(y)$, at $y \approx \pm 5.467$. (This was also found in Ref. 1.)

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*Work supported in part by the U. S. Atomic Energy Commission. Prepared under Contract AT(11-1)-68 for the San Francisco Operations Office, U. S. Atomic Energy Commission.

[1]I. Hauser, Phys. Rev. Lett. 33, 1112 (1974); 33, 1525(E) (1974). We use the notation of this reference.

[2]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. 1.

[3]*Handbook of Mathematical Functions*, edited by M. Abramowitz and I. Stegun, National Bureau of Standards Applied Mathematics Series, No. 55 (U. S. G. P. O., Washington, D. C., 1964).

[4]The computer program for the hypergeometric function was written by Dr. M. Griss.