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Behavior of matter at superhigh density

V. Canuto* and J. Lodenquai
NORDITA, Copenhagen, Denmark
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The behavior of matter at densities higher than 10^{16} g/cm^3 is investigated. It is found that little information can be gained from the data on neutron stars, whose properties are essentially determined by the behavior of $p = c_s^2\epsilon$ in the region $10^{14} \leq \rho = \epsilon/c^2 \leq 5 \times 10^{15}$ g/cm^3. On the contrary, the study of the properties of the p-p collision in the GeV region seems to provide more stringent conditions. The Landau model is used to analyze data concerning the average multiplicity, the constancy of transverse momenta, and the behavior of the energy per particle. Using a newly derived relation for the hadronic viscosity $\eta \sim T^{1/c_s^2}$, we show that the Landau hydrodynamic model reproduces the same results as the multiperipheral model without the need of any extra hypothesis, if the value $c_s^2 = 1$ is adopted. Once this conclusion is reached, several pion-pion Lagrangians are analyzed and their predictions for c_s^2 investigated. The fundamental assumption of the Landau hydrodynamical model, i.e., the laminar nature of the flow, is also investigated. The result is found that the pion fluid can become turbulent in the later stages of evolution, thus giving rise to eddies or clumps of matter tentatively identified with the clusters found in the p-p scattering.

I. INTRODUCTION

The need for an accurate description of the behavior of matter at very high densities, $\rho > 10^{14}$ g/cm^3, arose from the study of the structure and dynamics of rotating neutron stars (pulsars). Given an equation of state $p = c_s^2\epsilon$, $\epsilon = \rho c^2$ (c_s is the velocity of sound in units of c), where p is the pressure and ϵ the total energy density, the relativistic equations of stellar structure can be integrated to obtain the mass and radius and hence the moment of inertia as functions of the central density.[1] The moment of inertia I is related to the rate of loss of rotational energy and can be determined for at least one pulsar, viz., the Crab pulsar PSR,0534. It was therefore hoped that a comparison of the observational moment of inertia with those obtained theoretically from various equations of state would allow one to narrow down the choice.

In particular, for the Crab pulsar, the lower limit for I is 0.62×10^{44}, 1.72×10^{44}, or 2.69×10^{44} g cm^2, depending on the assumed value for its distance: 1.2, 2, or 2.5 kpc.[2] Present observational data do not allow a narrower margin. All but one of the equations of state published so far give a value for I greater than 10^{44} g cm^2, for a density region of interest in neutron stars. See Fig. 6 of Ref. 3.

In Fig. 1 we present the relation p vs ρ. The curve represents an average of all the nonrelativistic equations of state published so far.[4] They are all based on the best known N-N potential and on the most reliable many-body techniques. All of these satisfy the moment-of-inertia criterion within the mass range $0.13 \leq M/M_\odot \leq 2$ and they are the best one can presently produce. The important feature they all share is their stiffness in the density region of interest for neutron stars, i.e., between 10^{14} and 5×10^{15} g/cm^3. The stiffness is clearly related to their predicted values of the velocity of sound c_s, which are greater than $1/\sqrt{3}$, the value for a noninteracting system.

Since we have good reason to believe in the solid curve in Fig. 1 up to 5×10^{15} g/cm^3, we can expect that as $\rho\to\infty$, the equation of state will approach either of the following two limits:

(a) $c_s^2\to1$, i.e., the pressure p keeps on increasing and approaches ϵ asymptotically;

(b) $c_s^2\to\frac{1}{3}$, i.e., we recover the result for a noninteracting system (asymptotic freedom).

Either of the above cases is consistent with astronomical data, and the choice between (a) and (b) has to be decided by some other means. It appeared to us that one could possibly decide between (a) and (b) by studying the phenomenon of multiparticle production that characterizes the p-p and e^+e^- collisions at very high energies ($>10^2$ GeV).

In the case of a p-p collision, two ultrarelativistic nucleons, Lorentz-contracted into flattened disks, approach each other in the c.m. system. The volume of each disk is ($\hbar = c = 1$)

$$V \sim \frac{4\pi}{3}\frac{1}{m_\pi^3}\left(\frac{2M}{E}\right), \tag{1}$$

where M is the nucleon rest mass, m_π the pion rest mass, and E the c.m. energy of each nucleon. The factor (M/E) is the Lorentz-contraction factor.

The corresponding density is easily seen to be

$$\rho = \frac{\epsilon}{c^2} = \frac{E}{c^2 V} = 1.5\times10^{14} E_L \text{ (g/cm}^3) , \quad (2)$$

where E_L (in GeV) is the kinetic energy of the incident nucleon in the laboratory frame. Typically, for $E_L \sim 10^3$ GeV, $\rho \sim 10^{17}$ g/cm^3. We therefore see that the N-N scattering phenomena can indeed offer the possibility of obtaining information on the high-density portion of the $p=p(\epsilon)$ curve shown in Fig. 1.

In the e^+e^- collision, there is no Lorentz contraction, the electron and positron being treated as point particles. They annihilate into a massive photon which is subsequently converted into hadronic matter. In contrast to the pancake shape of the p-p initial blob, the decay of the massive photon has a spherical geometry. Recently,[5,6] the Landau hydrodynamic model, originally developed for the p-p case, has also been applied to this spherical case and we have therefore decided to use the few data on e^+e^- for the purpose of deciding the possible value of c_s^2.

II. THE LANDAU MODEL[7,8]

The first detailed model describing the evolution of the initial blob of compressed hadronic matter was proposed by Landau in 1953. After a long period of stagnation, the model has been resurrected and is recently the focus of a great deal of attention, particularly in view of the possibility of violation of scaling, a property that is sometimes considered to be an intrinsic feature of the model. This is indeed not the case, as we shall see in the next sections, where it will be shown that the Landau hydrodynamic model has a great degree of flexibility, depending upon the value of the speed of sound and the energy dependence of the hadronic viscosity.

During the expansion of the hadronic matter, the run of the thermodynamic and hydrodynamic variables as temperature and fluid velocity vs space and time must be derived after solving the relativistic Navier-Stokes equation (Greek indices run from 1 to 4)

$$T_{\mu\nu,\nu} = 0 , \quad (3)$$

where the energy-momentum tensor $T_{\mu\nu}$ is written as

$$T_{\mu\nu} = p\,\delta_{\mu\nu} + (p+\epsilon)\,u_\mu u_\nu + \tau_{\mu\nu} = T^{(0)}_{\mu\nu} + \tau_{\mu\nu} . \quad (4)$$

In order for $T_{\mu\nu}$ to be used in Eq. (3), an equation of state $p=p(\epsilon)$ is needed. In principle, such a relation can be derived from the many-body treatment of the Lagrangian describing the system of strongly interacting hadrons. This has actually been done by several authors, who used different Lagrangians and many-body techniques.[4] In spite of the great amount of work done, there is no unanimous consensus on whether p should approach $\frac{1}{3}\epsilon$ or ϵ itself as $\epsilon \to \infty$, even though the more realistic models give rise to $p \to \epsilon$.

In this paper we have therefore decided to adopt the following philosophy in obtaining the high-density equation of state: Assuming the validity of the Landau model, we use its predictions to decide upon the value of c_s^2 in the relation $p = c_s^2\epsilon$ and after having converged hopefully on a unique value of c_s^2, we use it to choose among several proposed pionic Lagrangians.

We shall make use of two quantities that can be compared with observational data, namely, the charged-particle multiplicity N and the constancy of the transverse momentum $\langle p_T \rangle$. In order to carry out this program, one should proceed as follows:

(1) Solve the Navier-Stokes equations with the

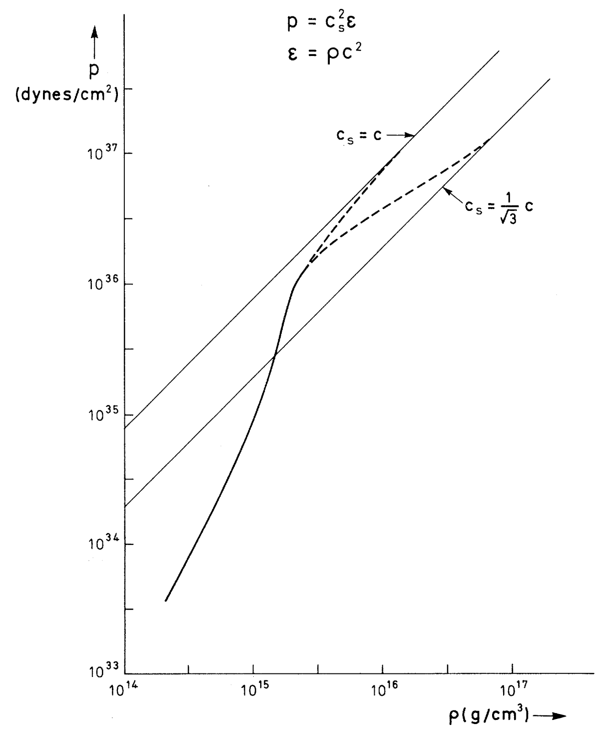

FIG. 1. Curves of pressure p vs density $\rho (=\epsilon/c^2)$ for pure neutron matter. The two lines represent $c_s = c$ and $c_s = (1/\sqrt{3})\,c$, as labeled. The solid curve represents an average of the many-body calculations so far performed, as described in Ref. 4. The dashed lines show how the curve could behave at high densities.

inclusion of the dissipation tensor $\tau_{\mu\nu}$.

(2) Evaluate the hydrodynamic and thermodynamic quantities like v and T vs x and t.

(3) Once that is done the entropy can be evaluated vs x and t. One can then check at which point during the expansion, if ever, the entropy levels off and becomes constant.

Such a program cannot be implemented because we do not know the full solution of Eq. (3) with $\tau_{\mu\nu}\neq 0$. We do not even know the viscosity $\eta=\eta(T)$ to any degree of satisfactory completeness. What we do know from the Landau work is the exact solution of the one-dimensional version of (3) in the case of zero viscosity, $\eta=0$.

In order to make use of such a solution, we shall therefore postulate that there is indeed a point after which the viscosity η becomes negligible, the dissipation is small, and the entropy is conserved. The definition of such a point is one of the crucial parameters of the model and we shall discuss several possibilities.

From the moment the flow becomes inviscid, the entropy is conserved and the multiplicity can be simply computed by saying that $N\sim S$, where S is the total entropy. We have

$$N\sim S=sV\sim\frac{\epsilon}{T}V ,$$

where s is the specific entropy. From the first law of thermodynamics $\epsilon\sim T^{(1+c_s^2)/c_s^2}$ ($\epsilon=E/V$) and we have

$$N\sim E^{1/(1+c_s^2)}\,V^{c_s^2/(1+c_s^2)} , \tag{5}$$

a formula of general validity. The problem will come when we have to specify the volume V.

The evaluation of the average transverse momentum is done using the solution of Eq. (3). The result of a rather long exercise of algebra is[9]

$$\langle p_T\rangle\cong\frac{1}{N}E^{(2c_s^2-1-c_s^4)/2c_s^2}\times\int d\lambda\exp\left(\frac{1-c_s^4}{2c_s^2}(L^{*2}-\lambda^2)^{1/2}\right) ,$$

$$L^*\equiv\ln(E/2M) .$$

To the leading power in E we have

$$\langle p_T\rangle\cong E^{1-c_s^2}N^{-1} , \tag{6}$$

a result again of general validity, since it still depends on N, Eq. (5), which in turn depends upon V, the critical volume starting at which the isentropic flow is ensured.

III. THE ONE-DIMENSIONAL MODEL: *p-p* COLLISION

A. Fully inviscid flow, $\eta=0$

In the original Landau model it was supposed that the viscosity is negligible throughout the entire flow, starting from the very beginning, when the two colliding protons came in contact. Landau based his hypothesis on a rather dubious evaluation of the Reynolds number. In our opinion, it is more satisfactory to consider the fully inviscid model as a first approximation needed for future comparison with a more realistic model.

If we accept that $\eta=0$ always, then the entropy is always constant and we can compute the volume V at any instant, in particular at the initial moment when V is given by Eq. (1). In that case, it follows from Eqs. (5) and (6) that

$$N\simeq E^{(1-c_s^2)/(1+c_s^2)} , \tag{7}$$

$$\langle p_T\rangle\simeq E^{c_s^2(1-c_s^2)/(1+c_s^2)} . \tag{8}$$

The fully inviscid flow, Eqs. (7) and (8), has been recently discussed by Suhonen *et al.*[10] and by Chaichian *et al.*,[9] respectively.

B. Fully viscous flow, $\eta\neq 0$

If we make the more reasonable hypothesis that η is actually different from zero and if we know the functional dependence of η on ϵ (or T) then, even without possessing the full solution of Eq. (3), we can still analyze the problem and decide the critical length or volume after which the flow is actually behaving as inviscid and the Landau solution is applicable. In order to do this, we have to know the law $\eta=\eta(T)$. This touches upon a severely ill-known quantity so far.

As discussed in the Appendix one can have

$$\eta\simeq T^3 \tag{9}$$

or a more general formula (Appendix)

$$\eta\simeq T^{1/c_s^2} . \tag{10}$$

We shall write $\eta\sim T^k$ and specify the exponent k only at the end. Since we do not know the function $T=T(x,t)$ with η included in Eq. (3), the analysis of Eqs. (9) and (10) does not reveal much. We can, however, look at the Reynolds number ($\sim\eta^{-1}$) and decide whether it is ≥ 1. We follow and generalize a method due to Feinberg.[6,8]

The Reynolds number is defined as the ratio of any of the components of $T^{(0)}_{\mu\nu}$, Eq. (4), to any of the components of the viscous stress tensor

$$\tau_{\mu\nu}=-\eta\left(\frac{\partial u_\mu}{\partial x_\nu}+\frac{\partial u_\nu}{\partial x_\mu}+u_\mu u_\beta\frac{\partial u_\nu}{\partial x_\beta}+u_\nu u_\beta\frac{\partial u_\mu}{\partial x_\beta}\right) .$$

Since $\epsilon\sim p$, $u\sim c$ one obtains

$$\mathrm{Re} = \frac{\epsilon L}{\eta c} \ , \tag{11}$$

where L is a typical length of the system. Using $\epsilon = E/V$, $\eta \sim T^k$, and $\epsilon = T^{(1+c_s^2)/c_s^2}$, we obtain

$$\mathrm{Re} = L\left(\frac{E}{V}\right)^{[1+(1-k)c_s^2]/(1+c_s^2)} \ . \tag{12}$$

This expression is still general. Since we are in the one-dimensional case, the volume V, which is Lorentz-contracted, can be written as

$$V \cong L_0^2 L \ . \tag{13}$$

This does not mean that we take $L \sim E^{-1}$, as implied by applying Eq. (1), which we do not use. L_0 is just a constant that represents the uncontracted size of the pionic cloud, $L_0 \sim m_\pi^{-2}$. Equation (12) can then be rewritten for the one-dimensional case as

$$\mathrm{Re} = (L/L_1)^{kc_s^2/(1+c_s^2)} \ , \quad L_1 \equiv E^{-[1+(1-k)c_s^2]/kc_s^2} \ . \tag{14}$$

Since the exponent entering in Re is positive, it follows that Re is large, i.e., η is small ($\mathrm{Re} \sim \eta^{-1}$) only for $L > L_1$, L_1 defining the minimum distance at which we can start applying the concept of constant entropy.

If we now substitute L_1 so defined into (5) and (6) we obtain [after using (13)]

$$N \simeq E^{(k-1)/k} \ , \quad \langle p_T \rangle \simeq E^{(1-kc_s^2)/k} \ . \tag{15}$$

The two laws of viscosity (9) and (10) give respectively

$$k = 3: \quad N \simeq E^{2/3} \ , \tag{16}$$

$$\langle p_T \rangle \simeq E^{(1-3c_s^2)/3} \ ; \tag{17}$$

$$kc_s^2 = 1: \quad N \simeq E^{1-c_s^2} \ , \tag{18}$$

$$\langle p_T \rangle \simeq E^0 \ . \tag{19}$$

In Table I we summarize the results for N and $\langle p_T \rangle$ for the viscid and inviscid one-dimensional case. We list first the distance at which the system becomes inviscid, then the corresponding multiplicity, and finally the average transverse momentum. In the first case $L \sim E^{-1}$, since that is the dependence of $V \sim L_0^2 L$ from Eq. (1).

IV. THREE-DIMENSIONAL CASE: e^+e^- ANNIHILATION

As explained before, in the case of e^+e^- annihilation into hadrons, we cannot use Eq. (13) since there is no contraction. Besides, there is no transverse momentum and we shall therefore use the multiplicity formula and the energy per particle.

A. Fully inviscid flow, $\eta = 0$

If the entropy is constant throughout the entire flow, then we can compute it at the initial moment, i.e., we can take Eq. (5) with $V = V_0 = \mathrm{const}$, so that

$$N \simeq E^{1/(1+c_s^2)} \ . \tag{20}$$

B. Fully viscous flow, $\eta \neq 0$

In the three-dimensional case we shall call $L \equiv R$, following the notation used by Feinberg.[6] Equation (12) is general and it can be used with $V = R^3$, since Eq. (13) is not valid. We then obtain

$$\mathrm{Re} = (R/R_1)^{[(3k-2)c_s^2-2]/(1+c_s^2)} \ , \tag{21}$$

with

$$R_1 \equiv E^{-[1+(1-k)c_s^2]/[(3k-2)c_s^2-2]} \ . \tag{22}$$

1. $\eta = T^3$ (Feinberg[6])

In this case

$$R_1 = E^{-(1-2c_s^2)/(7c_s^2-2)} \ , \tag{23}$$

$$\mathrm{Re} = (R/R_1)^{(7c_s^2-2)/(1+c_s^2)} \ . \tag{24}$$

For $c_s^2 < \frac{2}{7}$

$\mathrm{Re} \gg 1$ for $R < R_1$,

$\mathrm{Re} \ll 1$ for $R > R_1$;

TABLE I. Multiplicity and average transverse momentum for the viscid and inviscid one-dimensional case.

Inviscid flow
(A) $\eta = 0$ (Landau, Ref. 7)
$L \sim E^{-1}$
$N \sim S \sim E^{(1-c_s^2)/(1+c_s^2)}$
$\langle p_T \rangle \sim E^{c_s^2(1-c_s^2)/(1+c_s^2)}$
Viscous flow
(B) $\eta \sim T^3$ (Feinberg, Refs. 6 and 8)
$\mathrm{Re} = \left(\frac{L}{L_1}\right)^{3c_s^2/(1+c_s^2)} \begin{cases} L_1 \sim E^{-(1-2c_s^2)/3c_s^2} \\ N \sim E^{2/3} \\ \langle p_T \rangle \sim E^{(1-3c_s^2)/3} \end{cases}$
(C) $\eta \sim T^{1/c_s^2}$ (present work)
$\mathrm{Re} = \left(\frac{L}{L_1}\right)^{1/(1+c_s^2)} \begin{cases} L_1 \sim E^{-c_s^2} \\ N \sim E^{1-c_s^2} \\ \langle p_T \rangle \sim E^0 \end{cases}$

for $c_s^2 > \frac{2}{7}$

$$\mathrm{Re} \gg 1 \text{ for } R > R_1 ,$$

$$\mathrm{Re} \ll 1 \text{ for } R < R_1 .$$

In the first case, the multiplicity formula, Eq. (5), can be applied only for $R < R_1$ ($\mathrm{Re} \gg 1$, η small) and not for $R > R_1$. It clearly does not do any good to have a formula that can be applied only in the first part of the expansion and not in the final stage where we actually measure it. For $c_s^2 > \frac{2}{7}$, the situation is reversed and we can apply Eq. (5) starting at $R = R_1$. We obtain

$$c_s^2 > \tfrac{2}{7}, \quad N \sim E^{(6c_s^2-2)/(7c_s^2-2)} . \tag{25}$$

Since $c_s^2 < \frac{2}{7}$ is of no interest, we disregard this case.

2. $kc_s^2 = 1$, $\eta \sim T^{1/c_s^2}$ (present work)

In this case

$$\mathrm{Re} = (R/R_1)^{(1-2c_s^2)/(1+c_s^2)} ,$$

$$R_1 \equiv E^{-c_s^2/(1-2c_s^2)} . \tag{26}$$

For $c_s^2 < \frac{1}{2}$, the exponent in Re is positive and $\mathrm{Re} > 1$ for $R > R_1$. A computation analogous to the one that led us to Eq. (25) yields

$$c_s^2 < \tfrac{1}{2}, \quad N \sim E^{(1-3c_s^2)/(1-2c_s^2)} .$$

For $c_s^2 > \frac{1}{2}$, the exponent in Re is negative and $\mathrm{Re} < 1$ (η large) for $R > R_1$, i.e., the entropy increases in the final stage and the multiplicity cannot be computed as before. However, the effective viscosity will most certainly cease when the mean separation among the particles is of the order of their Compton wavelengths. If this is the case, we can then compute the multiplicity following the Pomeranchuk model[8] which gives

$$c_s^2 > \tfrac{1}{2}, \quad N \sim E . \tag{27}$$

The results of the three-dimensional case are summarized in Table II.

TABLE II. Multiplicity for the viscid and inviscid three-dimensional case.

Inviscid fluid
(A) $\eta = 0$ (Landau, Ref. 7)
$V \simeq \mathrm{const}$
$N \simeq E^{1/(1+c_s^2)}$
Viscous fluid
(B) $\eta \sim T^3$ (Feinberg, Refs. 6 and 8)
$R_1 \equiv E^{-(1-2c_s^2)/(7c_s^2-2)}$
$c^2 > \frac{2}{7}$: $N \sim E^{(6c_s^2-2)/(7c_s^2-2)}$
$c^2 < \frac{2}{7}$: S entropy not constant
(C) $\eta \sim T^{1/c_s^2}$ (present work)
$R_1 \equiv E^{-c_s^2/(1-2c_s^2)}$
$c_s^2 < \frac{1}{2}$: $N \simeq E^{(1-3c_s^2)/(1-2c_s^2)}$
$c_s^2 > \frac{1}{2}$: $N \sim E$

V. c_s^2: IS IT $\frac{1}{3}$ OR 1?

We present in Table III the results of Tables I and II for two typical values of c_s^2, one corresponding to a free system and the other to a system of nucleons strongly interacting via vector mesons.

In the p-p collision process, it is a well-established fact that the average transverse momentum $\langle p_T \rangle$ is constant (or perhaps increases logarithmically) with energy over a wide range of energies, say from 10 up to 10^5 GeV. This constancy excludes the third combination

$$c_s^2 = 1, \quad \eta \sim T^3 . \tag{28}$$

The first choice

$$\eta \sim 0, \quad c_s^2 = \tfrac{1}{3}$$

yields $p_T \sim E_L^{1/12}$. Such behavior does not contradict the cosmic-ray data. In fact, the high-energy

TABLE III. The results of Tables I and II for two typical values of c_s^2.[a]

	p-p collision		
c_s^2	η	N	$\langle p_T \rangle$
$\frac{1}{3}$	0	$E^{1/2}$	$E^{1/6}$
$\frac{1}{3}$	T^3	$E^{2/3}$	const
1	0	const	const
1	T^3	$E^{2/3}$	$E^{-2/3}$
1	T	const	const
	e^+e^- annihilation		
c_s^2	η	N	$w = E/N$
$\frac{1}{3}$	0	$E^{3/4}$	$E^{1/4}$
$\frac{1}{3}$	T^3	const	E
1	0	$E^{1/2}$	$E^{1/2}$
1	T	E	const
1	T^3	$E^{4/5}$	$E^{1/5}$

[a] Note: Since the formulas given above are modulus $\ln E$ (Ref. 11), whenever we write const we should understand $\ln E$.

data quoted in Ref. 9 should refer to E_L and not to E. [The author is grateful to Professor E. L. Feinberg for pointing this out. See Fig. 16 of E. L. Feinberg, Phys. Rep. 5C, 237 (1972).] The present experimental data on cosmic rays do not rule out any of the above-mentioned combinations, except the third one. The same conclusion holds true when we analyze the particle multiplicities. A possibility of discriminating among the remaining choices comes from comparing the previous results with the ones of the multiperipheral model, i.e., a $\ln E$ behavior instead of a power law. Such a behavior can be easily achieved within the hydrodynamic model if we limit ourselves to the combinations

$$c_s^2 = 1, \quad \eta = 0, T \ . \tag{29}$$

If we are ready to believe in this chain of arguments, the analysis has led us to a unique candidate for c_s^2. The fact that one obtains the same result with or without viscosity is because the length $L \sim E^{-c_s^2}$ [Table I, case (C)] coincides with the inviscid case for precisely $c_s^2 = 1$, i.e., viscosity becomes negligible from the very beginning.

From the e^+e^- data, we should perhaps only make use of the fact that the energy per particle w is nearly constant. The combination $c_s^2 = \frac{1}{3}$, $\eta = 0$ and $c_s^2 = 1$ and $\eta = T^3$ should be disregarded because of the previous analysis. Concerning the remaining three possibilities we can say the following.

Feinberg[6] favored the combination

$$c_s^2 = \tfrac{1}{3}, \quad \eta \sim T^3$$

which, however, yields too high an energy per particle. For this reason he discussed the formation of γ rays along with hadronic matter, carrying away some of the initial energy, thus weakening the behavior of w vs E. Presumably, if one accepts this way out, the other case $c_s^2 = 1$ and $\eta = 0$ could also be fixed up.

However, without introducing an extra hypothesis, the hydrodynamic model can be made consistent with the data if we choose the combination

$$c_s^2 = 1, \quad \eta \sim T \tag{30}$$

consistent with Eq. (29).

In conclusion, from Eqs. (29) and (30) we propose the combination

$$c_s^2 = 1, \quad \eta \sim T \ .$$

It is clear that if we do not consider c_s^2 as a parameter, but we choose from the very beginning a fixed value, in particular $c_s^2 = \frac{1}{3}$, then the multiplicity will grow like a power law and the result will be in obvious disagreement with the peripheral model. Such a problem has been recently discussed by Thomas,[12] who has shown how one can modify the Landau model in order to bring a power-law agreement with a $\ln E$ behavior.

In the present treatment, we do not need to employ such a procedure since the Landau model itself can give a $\ln E$ behavior, if the velocity of sound and the viscosity are treated in their general form.

Finally, we would like to comment on the fact that recent many-body treatments of a system of strongly interacting baryons (via a vector meson) yield $c_s^2 = 1$ both classically and quantum mechanically.[13]

VI. THE HYDRODYNAMIC TREATMENT OF THE PION-PION LAGRANGIAN

We shall employ here a method originally developed by Milekhin,[14] in which quantum effects are averaged over, on account of the large number of particles present and the fact that we are dealing with large quantum numbers. This is far from saying that only statistical weight factors dominate, as has been done in some recent work.[15]

The emphasis of this paper is on the interaction. We shall keep the number of species to the minimum, just pions, but we shall study the consequences of the interaction. It could be argued that if indeed a system of pions is a good representation of the p-p collision process, it is less so for the neutron-star case. If it is certainly true that a complicated mixture of n, Λ, Σ, etc. is likely to be present in the interiors of neutron stars, it is also true that the studies of such a mixture[16] have indicated that the changes in the equation of state are rather small and most certainly the slope of $p(\epsilon)$ vs ϵ is almost unaltered. Since the behavior of c_s^2 is the main objective of this work, one can consider the system of pions as an excellent representation of the actual, more complicated situation.

The method employed is based on the study of a Lagrangian of the general form

$$\mathcal{L} = \mathcal{L}(\phi^n, \phi_\mu{}^m), \quad \phi_\mu \equiv \partial\phi/\partial x_\mu \tag{31}$$

and on the corresponding energy-momentum tensor

$$\begin{aligned} T_{\mu\nu} &= \mathcal{L}\delta_{\mu\nu} - \frac{\partial \mathcal{L}}{\partial \phi_\nu}\,\phi_\mu \\ &= \mathcal{L}\delta_{\mu\nu} - 2\,\frac{\partial \mathcal{L}}{\partial \phi_\alpha{}^2}\,\phi_\mu\phi_\nu \ . \end{aligned} \tag{32}$$

Since we want to study the average, hydrodynamic behavior of $\mathcal{L}$, we must compare Eq. (32) with Eq. (4), with $\tau_{\mu\nu} = 0$.

If we define

$$u_\mu = \frac{\phi_\mu}{(-\phi_\alpha{}^2)^{1/2}}, \quad u_\mu{}^2 = -1 \tag{33}$$

the identification follows:

$$p=L, \quad \epsilon=2\frac{\partial\mathcal{L}}{\partial\phi_\alpha{}^2}\phi_\beta{}^2-\mathcal{L}\ . \tag{34}$$

We note that p and ϵ are Lorentz-invariant quantities since they are defined to be the proper pressure and energy density, respectively, of the fluid element. Since the kth component of the "momentum density" is

$$p_k=T_{4k}=-\phi_k\frac{\partial\mathcal{L}}{\partial\dot\phi}$$

the the four-vector u_μ defined in Eq. (33) has the property that in a coordinate system where its spatial components vanish (i.e., $\phi_k=0$) the momentum density also vanishes. Furthermore, $u_\mu{}^2=-1$, so that the definition of u_μ coincides with the usual definition of velocity in relativistic hydrodynamics.

A. Uniform system

If the system is sufficiently uniform such that $\phi_k=0$, then, as we have seen above, this condition implies that the momentum density is zero, corresponding to the proper frame. In this case the energy density in Eq. (34) becomes

$$\epsilon=\dot\phi\ \frac{\partial\mathcal{L}}{\partial\dot\phi}-\mathcal{L}\ , \tag{35}$$

and we can therefore solve for $\dot\phi$ in terms of ϕ and ϵ, i.e.,

$$\dot\phi=F(\phi,\epsilon)\ .$$

If we now limit ourselves to fields that are periodic, then the period τ is, by definition,

$$\tau(\epsilon)=\oint\frac{d\phi}{F(\phi,\epsilon)}\ , \tag{36}$$

where the integral is over one cycle. Averaging the pressure over one cycle, we get

$$\begin{aligned}p(\epsilon)&=\frac{1}{\tau(\epsilon)}\oint\mathcal{L}\,dt\\&=\frac{1}{\tau(\epsilon)}\oint\mathcal{L}\frac{d\phi}{\dot\phi}\\&=\frac{1}{\tau(\epsilon)}\oint\frac{\mathcal{L}}{F(\phi,\epsilon)}\,d\phi\ ,\end{aligned} \tag{37}$$

which gives the equation of state $p=p(\epsilon)$.

The use of the time-averaged quantities is valid only if the state of the system at a given point changes little over the period τ. For this to happen, the system must be sufficiently uniform, i.e., we must require

$$L>c_s\tau\ , \tag{38}$$

where L is a characteristic distance over which the properties of the system change considerably. This is equivalent to requiring that the Strouhal number $\mathrm{St}=u\tau/L$ be small. As before, depending on whether the fluid is treated as viscous or otherwise, the length L will vary and we have already defined the corresponding formulas

$(\eta=0)\ L_1\sim E^{-1}$,

$(\eta\neq0)\ L_1$ [Eq. (14)] .

Given a specific Lagrangian, the formula for $\tau(\epsilon)$ will be computed and the corresponding uniformity condition checked (see Table IV).

VII. APPLICATION OF THE PREVIOUS METHOD TO SPECIFIC LAGRANGIANS

The technique previously developed will now be applied to a system of strongly interacting pions. Among the many Lagrangians that have been proposed, we shall study the following four:

$$\text{I:}\quad \mathcal{L}=\mathcal{L}_0-\lambda\phi^{2n}\ , \tag{39}$$

$$\text{II:}\quad \mathcal{L}=\mathcal{L}_0-\lambda\phi^{2n}-\nu\phi_\mu{}^{2m}\ , \tag{40}$$

$$\text{III:}\quad \mathcal{L}=l^{-4}[1-l^4(\phi_\mu{}^2-m_\pi{}^2\phi^2)]^{1/2}\ , \tag{41}$$

$$\text{IV:}\quad \mathcal{L}=-\frac{1}{2}\left[\frac{\phi_\mu{}^2}{(1+F^{-2}\phi^2)^2}+\frac{m_\pi{}^2\phi^2}{1+F^{-2}\phi^2}\right]\ . \tag{42}$$

$\mathcal{L}_0$ is the free pionic Lagrangian density, n and m are integers, while λ, ν, l and F are parameters. Lagrangian III was originally proposed by Born and Infeld[17] for electrodynamics and studied by Heisenberg[18] in connection with high-energy physics. Lagrangian IV, based on chiral symmetry, has been recently proposed by Weinberg.[19]

In Table IV we present, in the first line, the value of c_s for a uniform system, i.e., postulating that condition (38) is satisfied. In the second line we give the value of $\tau(\epsilon)$, the period of the field. In the third line we present the conditions to be satisfied if (38) has to be satisfied.

VIII. THE ϕ^4 INTERACTION: NONUNIFORM SYSTEMS

One of the most popular types of Lagrangians is the one given in Column 1 with $n=2$. With this type of interaction the uniformity condition is not satisfied in the inviscid case, since we must have $n\to\infty$. In the case of a viscous fluid, the condition is

$$kc_s{}^2\gtrsim1\ .$$

For $n=2$, $c_s{}^2=\frac{1}{3}$ and the condition is barely satisfied.

IX. IS THE LANDAU MODEL ACTUALLY APPLICABLE?

We have repeatedly stressed that the lack of a solution of the relativistic Navier-Stokes equations including viscosity forces us to apply the Landau solutions only when η has actually become small.

This demarcation point has been investigated by analyzing the Reynolds number and by working in those regions for which Re is much larger than unity.

As a remedy to an otherwise intractable problem, this method looks at first glance very satisfactory indeed. However, by trying to avoid viscosity in this way, one can run into more serious troubles. In fact, if it is true that a moderately large Re implies small viscosity and nothing else, it is also undeniable that an extremely large Re can signify the onset of turbulence, a much more complex phenomenon.

Should that be the case, the whole treatment of Navier-Stokes equations as describing a laminar flow, regardless of what one does with viscosity, would clearly be a useless exercise in that only a stochastic treatment would be meaningful.

A. Initial stage

The condition for the absence of turbulence, i.e.,

$$\mathrm{Re} = \frac{\epsilon L}{\eta c} < 1 ,$$

can be expressed as

$$L < \epsilon^{[(k-1)c_s^2-1]/(1+c_s^2)} , \tag{43}$$

upon using $\epsilon = T^{(1+c_s^2)/c_s^2}$ and the general power law $\eta \sim T^k$. Since in the initial stage $L \sim E^{-1}$, we must have

$$1 + c_s^2 < 2k c_s^2 .$$

For the two viscosity laws considered earlier we have

$$\eta \sim T^3, \quad \tfrac{1}{5} < c_s^2$$
$$\eta \sim T, \quad c_s^2 > 1 . \tag{44}$$

Since $c_s^2 > 1$ is never satisfied (unless one violates causality) the previous condition is only sufficient and not necessary.

That the initial stage of the expansion should not be turbulent is perhaps to be expected, since L is too small, a result reflected in the mild requirement of Eq. (44).

When the system expands hydrodynamically, its dimension increases and it is quite conceivable that the fluid becomes turbulent, a most natural result if one thinks of a highly compressed gas left to expand. Smoke from a chimney or gas from a jet are but two possible analogies.

TABLE IV. The average, hydrodynamic behavior of four specific Lagrangians.

	$\mathcal{L}_0 - \lambda\phi^{2n}$	$\mathcal{L}_0 - \lambda\phi^{2n} - \nu\phi_\mu^{\ 2m}$	$[1 - l^4(\phi_\mu^{\ 2} - m_\pi^{\ 2}\phi^2)]^{1/2}$	$\frac{m_\pi^{\ 2}\phi^2}{1+F^{-2}\phi^2} + \frac{\phi_\mu^{\ 2}}{(1+F^{-2}\phi^2)^2}$
c_s^2 Uniform system	$\frac{n-1}{n+1}$	$\frac{n-m}{2nm+m-n}$	0	1
Period of field $\tau(\epsilon)$	$\epsilon^{(1-n)/2n}$	$\epsilon^{(m-n)/2mn}$	ϵ^0	$\epsilon^{-1/2}$
Uniformity condition Eq. (38) $\eta = 0$	$\epsilon^{(1/2)n} < 1$	$\epsilon^{[m(n+1)-n]/2mn} < 1$	?	Yes
$\eta \sim T^k$	$\epsilon^{(kc_s^2-1)/(1+c_s^2)} \gtrsim 1$	$\epsilon^{[2mn+(m-n)(1+k)]/2mn} < 1$		$\epsilon^{-(2-k)/2} > \epsilon^{-1/2}$
Constancy of $\langle p_T \rangle$	Yes if $n \gg 1$	Yes if $n - m \simeq nm$	?	Yes
Turbulence in initial stage of expansion	No	No	No	No

B. Final stage

A different situation arises, if we consider the final stage of expansion, defined as the moment when the temperature has dropped to a value $kT \sim m_\pi c^2$. If we use

$$\epsilon = \frac{E}{V} = \frac{E}{\lambda\!\!\!^{-}_\pi{}^2 L}$$

the Reynolds number becomes

$$\mathrm{Re} = \frac{E}{\lambda\!\!\!^{-}_\pi{}^2} \frac{1}{\eta c} . \qquad (45)$$

We have explained in the Appendix how incomplete is our present knowledge about the function $\eta = \eta(T)$. For a pionic Lagrangian with dimensionless coupling constant the relation (A1) holds true and we have

$$\mathrm{Re} = \frac{5\pi^2}{2} \frac{E}{m_\pi c^2} \left(\frac{kT}{m_\pi c^2}\right)^{-3} . \qquad (46)$$

In order to obtain the Reynolds number as a function of the expansion time, a specific model with a definite equation of state must be solved to yield the temperature T as a function of time t, from the initial state to the moment of breakup. The Landau model, for example, provides a relation for $T(t)$, which strongly depends on the assumed equation of state. However, it is definitely true that in any realistic model, kT will approach $m_\pi c^2$ at the breakup point. It is therefore possible to obtain a model-independent estimate of the Reynolds number at the initial and breakup moments and hence to obtain a qualitative behavior of Re as a function of time. At $kT \sim m_\pi c^2$, we obtain

$$\mathrm{Re} = \frac{5\pi^2}{2} \frac{E}{m_\pi c^2} . \qquad (47)$$

Before putting numbers into Eq. (47), we would like to note that such a relation has a general validity and it does not depend upon the particular Lagrangian used, so long as the coupling constant is dimensionless. In fact, independently of the exponent k in the $\eta \sim T^k$ law, when $kT = m_\pi c^2$ (and only then) η can depend solely upon a combination of $\hbar$, c, and m_π and the only way they can enter is in the combination

$$\eta \sim \frac{m_\pi{}^3 c^3}{\hbar} \ (\mathrm{g/cm\ sec}) ,$$

which, substituted into Eq. (45), readily gives Eq. (47).

Since $E^2 = 2Mc^2 E_L$, we have that

$$E_L \sim 10^3 \ \mathrm{GeV}, \ \mathrm{Re} \approx 4000 ,$$

a value virtually sufficient for turbulence to set in (Fig. 2).

Unfortunately, little is known about turbulence and even fewer quantitative statements can be made about the relativistic analog. However, if we limit ourselves to familiar three-dimensional turbulence under laboratory conditions we can make the following observations. The point at which a given type of fluid, in a given geometrical configuration, becomes turbulent depends sensitively on the geometrical boundaries and the way the fluid is injected. These factors can make the Reynolds number increase up to 10 000 or so, before some form of turbulence sets in.[20]

In the case of p-p scattering, no room exists for playing with the way the system is initially put in fluid laminar motion. In fact, since the initial act of putting the fluid in motion is rather violent, one would expect that turbulence should set in at a relatively small Reynolds number.

Even so, the type of motion one has to deal with is probably not fully developed turbulence characterized by a whole spectrum of eddy sizes. For a turbulent medium to reach such a stage, the Reynolds number must be exceedingly large. We can only expect the system to develop large, energy-containing eddies, whose nature and behavior are determined by the geometry of the system.

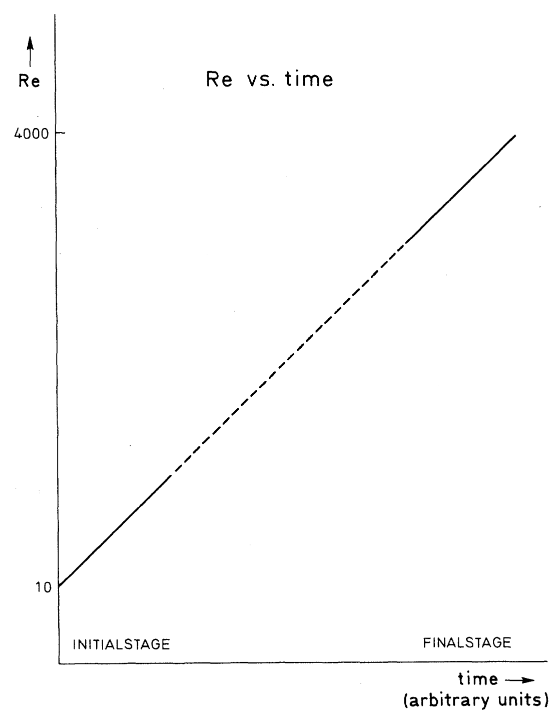

FIG. 2. The expected behavior of the Reynolds number during the expansion of the pionic fluid for $E_L \sim 10^3$ GeV.

X. PREDICTIONS OF THE TURBULENT MODEL FOR *p-p* SCATTERING

A possible experimental consequence of the turbulent nature of the expanding pionic fluid is the formation of clusters of pions, here identified with those eddies that detach themselves from the fluid in the final stage. If we assume that the critical Reynolds number Re_c, above which turbulence sets in, is a universal number, then Eq. (47) implies that the higher the incident energy the earlier the onset of turbulence (Fig. 3). We could therefore expect that the higher the energy, the greater is the number of eddies. However, this does not imply that all the eddies should become clusters of pions. This will be achieved only by those eddies that have acquired enough angular momentum to counterbalance the attractive pion-pion force that would otherwise lead to formation of bound states or resonances like ω and ρ.

Even though it is difficult at the present moment to give a numerical estimate for this limiting value of the angular momentum, the following picture can, however, be put forward. As the system expands with time, the eddies will correspondingly increase in size and only after a certain time t_c, Fig. 3, will they be large enough so that the centrifugal force will be able to counterbalance the pion-pion attractive force. We shall therefore identify these eddies with the observed clusters. Similarly, the resonances should be viewed as those eddies with insufficient angular momentum to prevent the collapse.

If the process described above does indeed take place, then the number and sizes of the pionic clusters will depend only on t_c and not too sensitively on the previous history of the collision, specifically, the collision energy. In this way we produce a more or less constant number of clusters, but do get a larger amount of resonances the higher the collision energy. This result is consistent with the present data.[21]

An analogous situation exists in the problem of formation of galaxies,[22] which are thought to be the remnant eddies of an early turbulent universe. Contrary to the p-p case, the present value of the angular momentum for a few galaxies is known, and one can consequently put limits on when the density of the universe was low enough and the radius of a cluster of $\sim 10^{11} M_\odot$ large enough to contain that given amount of angular momentum.

XI. CONCLUSION

The above analysis indicates that high-energy phenomena characterizing the p-p and e^+e^- systems can be meaningfully used to study the behavior of high-density matter and in particular the equation of state $p = c_s^2 \epsilon$. The analysis is, however, hampered by a lack of a satisfactory knowledge of the viscosity of hadronic matter. A dimensional analysis of the exact relation for the viscosity yields a generalization of Feinberg's T^3 relation of the form T^{1/c_s^2}, provided the coupling constant is dimensionless. By the use of such a relation, evidence is presented that favors the value $c_s^2 \to 1$ over other competing values.

However, the application of the Landau model would be altogether meaningless should the hadronic matter develop any turbulence. It is in fact shown that in spite of our incomplete knowledge of the $\eta(T)$ relation, the system acquires a Reynolds number $\mathrm{Re} \sim 4000$ in the final stage of evolution. Although no definite criteria exist, such a large Re is usually sufficient for laminar flow to break down. This offers the possibility of identifying the observed tendency of the pion gas to cluster with the eddies that characterize any turbulent flow. This will be true only when the angular momentum of each cluster is large enough to prevent the collapse into resonances of the ω and ρ types.

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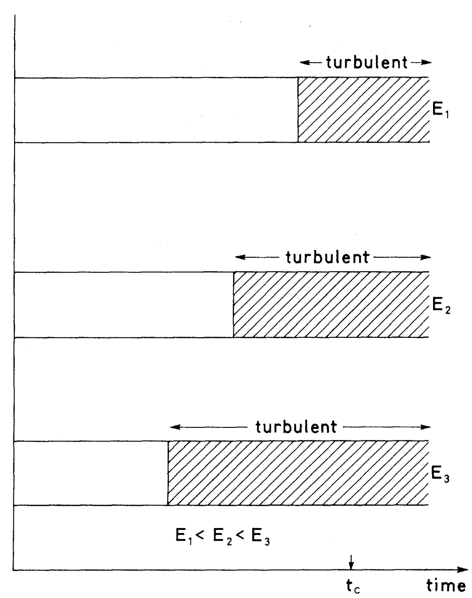

FIG. 3. Pictorial representations of the onset of turbulent flow in the Landau model. Turbulence sets in when the critical Reynolds number is reached. The higher the incident energy, the earlier this point is reached. t_c is the point at which stable eddies can form.

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APPENDIX: THE VISCOSITY OF A HIGH-TEMPERATURE PION GAS

As we have seen from Eq. (11), a value for the viscosity $\eta = \eta(T)$ is needed to evaluate the Reynolds number. It must be said at the outset that presently no satisfactory general relation of such a type is available. Iso *et al.*[23] using a specific Lagrangian and employing a perturbative approach to the Kubo formulas for the transport coefficients, obtained the following result:

$$\eta = \frac{2}{5\pi^2}\hbar^{-2}c^{-3}(kT)^3 . \tag{A1}$$

Feinberg[8] then showed that the T^3 dependence follows more generally from dimensional arguments if the coupling constant entering in the given Lagrangian is dimensionless and, moreover, if the temperature is high enough for the particle masses not to enter. It is clear that the previous arguments are not enough and something better should be done. The role of viscosity has been stressed again recently by Feinberg[6] in his study of the e^+e^- annihilation into hadrons. He has made the point that if (A1) is justified for a value of $c_s^2 = \frac{1}{3}$, it is not clear that it should be so for any other value of c_s^2. Motivated by his remark, we have investigated the problem a little further.

The general formula for η is given by[23] ($\hbar = c = 1$)

$$\eta = \frac{1}{3VT}\int_0^\infty \langle\{J_{11}(0), J_{11}(t)\}\rangle\, dt , \tag{A2}$$

where V is the folume, T the temperature, and J_{ik} is defined as

$$J_{ik}(t) = \int \mathcal{J}_{ik}(r, t)\, d^3r ,$$

$$\mathcal{J}_{ik}(r, t) = -\phi^*(r, t)\,\nabla_i \nabla_k\,\phi(r, t) .$$

Since ($\hbar = c = 1$) the time t has the dimension of E^{-1}, we have that dimensionally (A2) is given by

$$\eta \sim \frac{J^2}{VTE} \sim \frac{\mathcal{J}^2 V^2}{VTE} \sim \frac{\phi^4 p^4 V}{TE} ,$$

where p is the single-particle momentum. In order to compute η, we now need ϕ. The dimensions of ϕ are $(E/L)^{1/2}$ and if the coupling constant entering in the Lagrangian is dimensionless, then the only combination of E, V, $\hbar$, c available is

$$\phi \sim \frac{\hbar c}{(EV)^{1/2}} \sim E^{-1/2}V^{-1/2} ,$$

so that

$$\eta \sim \frac{p^4}{VTE^3} .$$

By definition the pressure is proportional to

$$P \cong \left(\frac{p^2}{E}\right) n ,$$

where $n = N/V$ is the number density and $p^2/E = pv$ is the contribution to P of each particle. We therefore conclude that

$$\eta \sim T^{1/c_s^2} , \tag{A3}$$

upon using $P = c_s^2\epsilon$, $E/V = \epsilon$, and $\epsilon \sim T^{1+1/c_s^2}$. Equation (A3) is the generalization of the Feinberg T^3 law for any Lagrangian with dimensionless coupling constant.

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*Permanent address: Institute for Space Studies, NASA, New York, New York.

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