PHYSICAL REVIEW D VOLUME 11, NUMBER 8 15 APRIL 1975

Spinor-spinor Bethe-Salpeter equations with vector and axial-vector gluons

K. Ladányi
Institute for Theoretical Physics, Roland Eötvös University, Budapest, Hungary
(Received 6 August 1974)

Wick-rotated spinor-spinor Bethe-Salpeter equations are investigated for systems bound to zero total mass by the exchange of vector and axial-vector gluons. At small relative coordinates, the interaction is assumed to be dominated by vector propagators which are chosen in a form suggested by theories of spontaneously broken gauge symmetries. Short-distance expansions are studied for the solutions belonging to representations $(\frac{1}{2}N, \frac{1}{2}N)$ of O(4). Particular attention is paid to the boundary conditions in order to formulate a well-defined eigenvalue problem. All solutions of the indicial equations are explicitly given for arbitrary values of N; the indices involve the coupling parameters characterizing the short-distance behavior of the interaction.

I. INTRODUCTION

During the past few years, considerable efforts have been devoted to understand the properties of the Bethe-Salpeter (BS) equation[1] for systems of spin-$\frac{1}{2}$ fermions; an extensive list of references is available in review articles by Nakanishi[2] and Böhm, Joos, and Krammer.[3] We need not enumerate all the reasons for investigating this problem. In the bound-state region, for example, these equations offer a covariant treatment of quark-parton models along the lines suggested in Refs. 4–8. On the other hand, scattering BS equations are known to be connected to the multiperipheral model of multiple productions at high energy.[9] In particular, spinor-spinor BS equations may lead to a reasonable nonperturbative description of the e^+e^- annihilation into hadrons. As a further aspect, we mention that BS amplitudes involve particular matrix elements of the bilocal generalization of vector and axial-vector currents.[10] Therefore, the short-distance properties of the BS amplitudes are of particular interest; important consequences of a possible noncanonical short-distance behavior have been discussed by Goldberger.[11]

Near the light cone, the behavior of the BS amplitudes may depend sensitively on the leading terms of the interaction kernel. In the present paper we consider Abelian interactions generated by the exchange of neutral vector and axial-vector gluons. According to ladder-type approximations, we shall assume that the light-cone behavior of the interaction is dominated by the propagators of the exchanged particles. The free vector and axial-vector gluon propagators $D^{c(V)}_{\mu\nu}$ and $D^{c(A)}_{\mu\nu}$ will be chosen in a form suggested by theories of spontaneously broken gauge symmetries:

$$D^{c(V,A)}_{\mu\nu}(q;m_{V,A}{}^2)=\frac{1}{m_{V,A}{}^2-q^2-i\epsilon}\left(g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2+i\epsilon}\right)-\beta^{(V,A)}\frac{q_\mu q_\nu}{(q^2+i\epsilon)^2}, \tag{1.1}$$

where $\beta^{(V)}$ and $\beta^{(A)}$ denote the gauge parameters.

Recently, Jackiw and Johnson[12] have proposed dynamical models of spontaneously broken gauge symmetries which may lead to an attractive framework for BS calculations. (For related works, see Ref. 13.) In these theories the masses of fermions and gluons arise spontaneously, without the presence of Higgs scalars.[14] This can happen if there is a massless, bound excitation corresponding to a particular solution of the fermion-antifermion BS equation. Other zero-mass solutions may also yield valuable information since the dominant light-cone properties of the BS amplitudes are expected to be independent of the center-of-mass (c.m.) energy.

In the present paper we restrict ourselves to vanishing c.m. energy and investigate Wick-rotated BS equations of the type

$$(\gamma_\mu\partial_\mu+\kappa)\psi^{(\lambda)}(x)(-\gamma_\nu\overleftarrow{\partial}_\nu+\kappa)=\tilde{V}^{(V)}_{\mu\nu}(x)\gamma_\mu\psi^{(\lambda)}(x)\gamma_\nu+\tilde{V}^{(A)}_{\mu\nu}(x)\gamma_5\gamma_\mu\psi^{(\lambda)}(x)\gamma_5\gamma_\nu, \tag{1.2}$$

with $\lambda=\pm1$. Here the functions $\psi^{(-1)}(x)$ and $\psi^{(1)}(x)$ are the Wick-rotated relative BS amplitudes of fermion-antifermion and fermion-fermion systems, respectively, and x denotes the Euclidean relative coordinates. (Since we are mainly interested in the asymptotic properties of the amplitudes, the calculations will be performed in the coordinate, rather than momentum, space.) The short-distance behavior of the interaction will be fixed as

$$\tilde{V}^{(V,A)}_{\mu\nu}(x)=\frac{\tilde{Z}^{(V,A;0)}}{R^2}\delta_{\mu\nu}+\frac{\tilde{Z}^{(V,A;1)}}{R^2}\hat{x}_\mu\hat{x}_\nu+\cdots \quad \text{for } R\to 0, \tag{1.3}$$

where $\hat{x}_\mu=x_\mu/R$, $R=(x_\mu x_\mu)^{1/2}$. In conventional ladder approximations, the constants $\tilde{Z}^{(V,A;0,1)}$ involve the gauge parameters $\beta^{(V,A)}$ of the propagators (1.1) in a simple form.

The Bethe-Salpeter equation (1.2), (1.3) is invariant under transformations of the group O(4) and offers an aesthetical and attractive possibility for a description of the short-distance dynamics of fermion systems. The purpose of this paper is to investigate the solutions that belong to representations $(\frac{1}{2}N, \frac{1}{2}N)$ of O(4). These solutions dominate the short-distance behavior of the BS amplitudes; on the other hand, they are less known for marginally singular interactions of the type (1.3). Particular attention will be paid to the vector and axial-vector amplitudes $V_\mu^{(V)}(x)$ and $V_\mu^{(A)}(x)$ which are involved in the expansions

$$\psi^{(\lambda)}(x) = i\gamma_5\gamma_\mu V_\mu^{(V)}(x) + \gamma_\mu V_\mu^{(A)}(x) + \cdots .$$

In order to exclude solutions with unreasonable short-distance behavior, we shall impose a stringent subsidiary condition for the space components ($\mu = j = 1, 2, 3$) of the leading vector and axial-vector amplitudes. This condition can be written as

$$V_j^{(V,A)}(x)|_{N=J>0} \sim R^{\rho(J)} Y_{(J-1,J,J-1,M)j}(\Omega) + \cdots \quad \text{for } R \to 0 . \qquad (1.4)$$

Here J is the total angular momentum assignment, and the functions Y denote the components of the four-dimensional three-vector spherical harmonics introduced by Gourdin.[15] [For definition, see Eq. (4.15) of this paper.]

Condition (1.4) is obviously satisfied for regularized interactions ($\tilde{Z}^{(V,A;0,1)} = 0$), and in this case the leading index $\rho(J)$ takes the canonical value $J-1$. On the other hand, the marginally singular interactions (1.3) result in noncanonical values of the indices $\rho(J)$ if no cancellations occur. Apart from this, the short-distance behavior (1.4) may be modified by "spurious" contributions involving terms of the type

$$R^{\rho(J)} Y_{(J+1,J,J\pm1,M)j}(\Omega) .$$

We shall exclude these terms by imposing the subsidiary condition (1.4), which leads to a restriction on the possible choice of the coupling parameters $\tilde{Z}^{(V,A;0)}$ and $\tilde{Z}^{(V,A;1)}$. If this restriction is satisfied, then we obtain a simple factorized form of the indicial equations which can be explicitly solved for arbitrary values of N.

The organization of this paper is as follows. Some general properties of the BS equation are presented in Sec. II. In Sec. III, we derive the radial BS equations by using the standard O(4) expansion of the amplitudes.[4,7,8,16] The boundary conditions, and the short-distance behavior of the solutions, are considered in Sec. IV. A discussion of some consistency problems is left to Sec. V.

II. STRUCTURE OF THE BS EQUATION

The fermion-antifermion BS amplitudes, corresponding to Heisenberg states $\Phi^{(-1)}$, are defined as

$$\tau^{(fa)}(x_1, x_2) = \langle 0 | T\Psi_1(x_1)\bar{\Psi}_2(x_2) | \Phi^{(-1)} \rangle , \qquad (2.1)$$

where the spinor field operators Ψ_1 and Ψ_2 generate spin-$\frac{1}{2}$ fermions of masses κ_1 and κ_2, respectively. We consider also the BS amplitudes

$$\tau^{(ff)}(x_1, x_2) = \langle 0 | T\Psi_1(x_1)\Psi_2(x_2) | \Phi^{(1)} \rangle , \qquad (2.2)$$

where $\Phi^{(1)}$ is a fermion-fermion state. It will be convenient to introduce the notations

$$\begin{aligned} \tau^{(-1)}(x_1, x_2) &= \tau^{(fa)}(x_1, x_2)\gamma_5 , \\ \tau^{(1)}(x_1, x_2) &= \tau^{(ff)}(x_1, x_2)C^{-1}\gamma_5 , \end{aligned} \qquad (2.3)$$

where C is the charge-conjugation matrix. If the fermion self-energy contributions are neglected, the homogeneous BS equation has the form

$$\left(-i\gamma^\mu \frac{\partial}{\partial x_1^\mu} + \kappa_1\right) \tau^{(\lambda)}(x_1, x_2) \left(-i\gamma^\nu \frac{\overleftarrow{\partial}}{\partial x_2^\nu} + \kappa_2\right) = I^{(\lambda)}(x_1 - x_2)\tau^{(\lambda)}(x_1, x_2) , \quad \lambda = \pm 1 . \qquad (2.4)$$

We next consider interactions $I^{(\lambda)}$ suggested by the following terms of the interaction Lagrangian density:

$$L_I' = \sum_{n=1} (G_V' : \bar{\Psi}_n\gamma^\mu\Psi_n\varphi_\mu^{(V)} : + G_A' : \bar{\Psi}_n\gamma_5\gamma^\mu\Psi_n\varphi_\mu^{(A)} :) , \qquad (2.5)$$

where the quantum fields $\varphi_\mu^{(V)}$ and $\varphi_\mu^{(A)}$ generate, respectively, neutral vector and axial-vector mesons whose exchange provides the forces between the fermions. According to ladder-type approximations, the spin structure of the interaction will be approximated in the following form:

$$I^{(\lambda)}(x_1 - x_2)\tau^{(\lambda)}(x_1, x_2) = \lambda G_V'^2 D_{\mu\nu}^{(V)}(x_1 - x_2)\gamma^\mu\tau^{(\lambda)}(x_1, x_2)\gamma^\nu - G_A'^2 D_{\mu\nu}^{(A)}(x_1 - x_2)\gamma_5\gamma^\mu\tau^{(\lambda)}(x_1, x_2)\gamma_5\gamma^\nu . \qquad (2.6)$$

In considering the light-cone behavior of the interaction terms $D_{\mu\nu}^{(V,A)}$, we shall restrict ourselves to a slight modification of the conventional ladder approximation. To obtain a fairly flexible framework for practical calculations, it will be convenient to apply superpositions of the gluon propagators (1.1) as given by[17]

$$D_{\mu\nu}^{(V,A)}(z) = -\frac{1}{(2\pi)^4 i} \int_0^\infty dm^2 \rho^{(V,A)}(m^2) \times \int d^4q\, e^{iqz} D_{\mu\nu}^{c(V,A)}(q; m^2) , \qquad (2.7)$$

where $\int dm^2\rho^{(V,A)}(m^2)$ is finite. We shall regard the form (2.6), (2.7) as a simple parametrization which might be reasonable if the fermion self-energy corrections are absent. We note here that, in general, our interaction terms $D^{(V,A)}_{\mu\nu}(z)$ are not identical with the complete gluon Green's functions which, when calculated in ordinary perturbation theory, involve nonnormalizable Källén-Lehmann spectral functions. A more general treatment of the interaction kernel should be developed along the lines suggested by Domokos and Surányi.[18]

Let us consider BS amplitudes $\tau^{(\lambda)}$ with total four-momentum P_ν. Translational invariance implies

$$\tau^{(\lambda)}(x_1, x_2) = \varphi^{(\lambda)}(z)\exp[-iP_\nu(\mu_1' x_1^\nu + \mu_2' x_2^\nu)], \tag{2.8}$$

where $z = x_1 - x_2$, and the choice of the constants μ_1' and μ_2' is restricted by $\mu_1' + \mu_2' = 1$.

We next restrict ourselves to situations where the Wick rotation can be performed. The Wick-rotated BS amplitude $\psi^{(\lambda)}$ is defined as

$$\varphi^{(\lambda)}(z)|_{\text{rotated}} = \psi^{(\lambda)}(x) = \varphi^{(\lambda)}(-ix_4, x_1, x_2, x_3), \tag{2.9}$$

where $x_j = z^j$ $(j = 1, 2, 3)$, $x_4 = iz^0$, and all the components x_μ are real. After Wick rotation the free scalar propagator takes the form

$$\frac{1}{\pi^2 i}\int d^4q \frac{e^{iqz}}{m^2 - q^2 - i\epsilon}\Bigg|_{\text{rotated}} = V(R; m^2) = (4m/R)K_1(mR), \tag{2.10}$$

where $R = (x_\mu x_\mu)^{1/2}$, and K_1 is the first-order modified Bessel function. One has

$$V(R; m^2) = \frac{4}{R^2} + 2m^2 \ln R + \cdots \quad \text{for } R \to 0. \tag{2.11}$$

We shall use the conventions

$$\tilde\gamma_j = -i\gamma^j \quad (j = 1, 2, 3), \quad \tilde\gamma_4 = \gamma^0, \quad \tilde\gamma_5 = \tilde\gamma_1\tilde\gamma_2\tilde\gamma_3\tilde\gamma_4; \tag{2.12}$$

thus, the commutation relations are $\tilde\gamma_\mu\tilde\gamma_\nu + \tilde\gamma_\nu\tilde\gamma_\mu = 2\delta_{\mu\nu}$.

In the c.m. coordinate frame the BS equation of the amplitude $\varphi^{(\lambda)}(z)$ can be derived by substituting Eq. (2.8) into (2.4) and by setting $P_0 = E$, $P_j = 0$ $(j = 1, 2, 3)$, where E is the total c.m. energy of the system. The Wick-rotated form of this equation is

$$(\tilde\gamma_\mu\partial_\mu - \mu_1'\tilde\gamma_4 E + \kappa_1)\psi^{(\lambda)}(x)(-\tilde\gamma_\nu\overleftarrow{\partial}_\nu - \mu_2'\tilde\gamma_4 E + \kappa_2) = -\lambda G_V{}^2 V^{(V)}_{\mu\nu}(x)\tilde\gamma_\mu\psi^{(\lambda)}(x)\tilde\gamma_\nu + G_A{}^2 V^{(A)}_{\mu\nu}(x)\tilde\gamma_5\tilde\gamma_\mu\psi^{(\lambda)}(x)\tilde\gamma_5\tilde\gamma_\nu, \tag{2.13}$$

with $G_{V,A} = G'_{V,A}/4\pi$ and $\partial_\mu = \partial/\partial x_\mu$. According to Eqs. (2.6), (2.7), and (2.10), we have

$$V^{(V,A)}_{\mu\nu}(x) = V^{(V,A;0)}(R)\delta_{\mu\nu} - \partial_\mu\partial_\nu V^{(V,A;1)}(R), \tag{2.14}$$

with

$$V^{(V,A;0)}(R) = \int_0^\infty dm^2\rho^{(V,A)}(m^2)V(R; m^2), \tag{2.15}$$

$$V^{(V,A;1)}(R) = \int_{+0}^\infty dm^2 \frac{\rho^{(V,A)}(m^2)}{m^2}[V(R; m^2) - V(R; 0)] - \beta^{(V,A)}\left[\int_0^\infty dm^2\rho^{(V,A)}(m^2)\right]\left[\frac{\partial}{\partial m^2}V(R; m^2)|_{m^2\to 0}\right]. \tag{2.16}$$

The interaction terms $V^{(V,A)}_{\mu\nu}(x)$ can also be written as

$$V^{(V,A)}_{\mu\nu}(x) = U^{(V,A;0)}(R)\delta_{\mu\nu} + U^{(V,A;1)}(R)\hat x_\mu \hat x_\nu, \tag{2.17}$$

where $\hat x_\mu = x_\mu/R$, and

$$U^{(V,A;0)}(R) = V^{(V,A;0)}(R) - \frac{1}{R}\frac{dV^{(V,A;1)}(R)}{dR}, \tag{2.18}$$

$$U^{(V,A;1)}(R) = -\frac{d^2V^{(V,A;1)}(R)}{dR^2} + \frac{1}{R}\frac{dV^{(V,A;1)}(R)}{dR}. \tag{2.19}$$

As $R \to 0$, the behavior of the interaction is given by

$$U^{(V,A;l)}(R) = \frac{Z^{(V,A;l)}}{R^2} + \cdots \quad (l = 0, 1), \tag{2.20}$$

where

$$Z^{(V,A;0)} = [1 - \tfrac{1}{2}b^{(V,A)}]Z^{(V,A)}, \quad Z^{(V,A;1)} = b^{(V,A)}Z^{(V,A)}, \tag{2.21}$$

$$Z^{(V,A)} = 4\int_0^\infty dm^2\rho^{(V,A)}(m^2), \quad b^{(V,A)} = 1 - \beta^{(V,A)}. \tag{2.22}$$

In a previous paper[19] the kinematical analysis of BS equations of the type (2.13), (2.17) has been carried out by using basis functions proposed by Gourdin.[15] [Notice that Eq. (2.10) and the definition of the coupling constants $G_{V,A}$ involve normalization factors $(4\pi)^2$ and $1/4\pi$, respectively, which are omitted in Ref. 19.]

The present paper is devoted to a study of the equal-mass BS equation at vanishing c.m. energy, thus

$$E \to 0\,, \quad \kappa_1 = \kappa_2 = \kappa\,, \quad \mu_1' = \mu_2' = \tfrac{1}{2}\,. \tag{2.23}$$

In this case the BS equation (2.13), (2.14) is form invariant under transformations of the group O(4) extended by three-space reflections Π and charge conjugation C. Standard O(4) expansions of the amplitudes lead to a set of radial BS equations; the results are summarized in the next section.

III. RADIAL BS EQUATIONS

The Wick-rotated BS amplitude $\psi^{(\lambda)}$ can be expanded as

$$\psi^{(\lambda)}(x) = i\tilde{\gamma}_5 S(x) + P(x) + i\tilde{\gamma}_5\tilde{\gamma}_\mu V_\mu(x) + \tilde{\gamma}_\mu A_\mu(x) + \tfrac{1}{2}\tilde{\sigma}_{\mu\nu} T'_{\mu\nu}(x)\,, \tag{3.1}$$

$$\tilde{\sigma}_{\mu\nu} = \frac{1}{2i}(\tilde{\gamma}_\mu\tilde{\gamma}_\nu - \tilde{\gamma}_\nu\tilde{\gamma}_\mu)\,, \quad T'_{\mu\nu}(x) = -T'_{\nu\mu}(x)\,, \tag{3.2}$$

where S, P, V_μ, A_μ, and $T'_{\mu\nu}$ are the scalar, pseudoscalar, vector, axial-vector, and tensor amplitudes, respectively. [Notice that our definition (2.3) of the amplitude $\tau^{(\lambda)}$ involves the matrix γ_5.] Substituting expansion (3.1) into the BS equation (2.13), (2.17), (2.23), we obtain a single equation for $P(x)$, and two systems of partial differential equations for the other amplitudes.

The differential equation of the pseudoscalar amplitude $P(x)$ can be written in the form

$$[\Box - I_S(R) - \kappa^2]P(x) = 0\,, \tag{3.3}$$

where

$$I_S(R) = \lambda G_V^2[4U^{(V;0)}(R) + U^{(V;1)}(R)] + G_A^2[4U^{(A;0)}(R) + U^{(A;1)}(R)]\,. \tag{3.4}$$

The amplitudes $S(x)$ and $V_\mu(x)$ satisfy the following system of partial differential equations:

$$[-\Box + I_S(R) - \kappa^2]S(x) + 2\kappa\partial_\mu V_\mu(x) = 0\,, \tag{3.5}$$

$$-2\kappa\partial_\mu S(x) + [-\Box + I_V^{(1)}(R) - I_V^{(0)}(R) + \kappa^2]V_\mu(x) + [2\partial_\mu\partial_\nu - 2I_V^{(1)}(R)\hat{x}_\mu\hat{x}_\nu]V_\nu(x) = 0\,. \tag{3.6}$$

The interaction terms $I_V^{(1)}$ and $I_V^{(0)}$ are given by

$$I_V^{(1)}(R) = \lambda G_V^2 U^{(V;1)}(R) - G_A^2 U^{(A;1)}(R)\,, \tag{3.7}$$

$$I_V^{(0)}(R) = -2\lambda G_V^2 U^{(V;0)}(R) + 2G_A^2 U^{(A;0)}(R)\,. \tag{3.8}$$

Finally, we obtain the system of partial differential equations for the amplitudes $T'_{\mu\nu}(x)$ and $A_\mu(x)$:

$$[-\Box + I_T(R) + \kappa^2]T'_{\mu\nu}(x) - [2\partial_\mu\partial_\rho - 2I_T(R)\hat{x}_\mu\hat{x}_\rho]T'_{\nu\rho}(x) + [2\partial_\nu\partial_\rho - 2I_T(R)\hat{x}_\nu\hat{x}_\rho]T'_{\mu\rho}(x) + 2\kappa i[\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)] = 0\,, \tag{3.9}$$

$$-2\kappa i\partial_\nu T'_{\mu\nu}(x) + [-\Box + I_V^{(1)}(R) - I_V^{(0)}(R) - \kappa^2]A_\mu(x) + [2\partial_\mu\partial_\nu - 2I_V^{(1)}(R)\hat{x}_\mu\hat{x}_\nu]A_\nu(x) = 0\,, \tag{3.10}$$

where

$$I_T(R) = \lambda G_V^2 U^{(V;1)}(R) + G_A^2 U^{(A;1)}(R)\,. \tag{3.11}$$

Let us turn to the separation of the angular variables which are defined by

$$x_1 = r\sin\vartheta\sin\varphi\,, \quad x_2 = r\sin\vartheta\cos\varphi\,,$$

$$x_3 = r\cos\vartheta\,, \quad x_4 = R\cos\Theta\,,$$

with $r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. The calculations will be restricted to BS amplitudes belonging to representations $(\frac{1}{2}N, \frac{1}{2}N)$ of O(4). These amplitudes can be decomposed in four disconnected sectors[4,7,8,16]; we next employ the classification of Refs. 7, 8. Sector I involves scalar and vector amplitudes that may be written as

$$S(x) = S_N(R)Y_{NJM}(\Omega)\,, \tag{3.12}$$

$$V_\mu(x) = V_N^{(1)}(R)Y^{(1)}_{(NJM)\mu}(\Omega) + V_N^{(2)}(R)Y^{(2)}_{(NJM)\mu}(\Omega)\,, \tag{3.13}$$

where J is the total angular momentum of the system. The functions $Y_{NJM}(\Omega)$ and $Y^{(1,2)}_{(NJM)\mu}(\Omega)$ are, respectively, four-dimensional scalar and vector spherical harmonics[4] which are defined in terms of Gegenbauer polynomials[20] $C_n^\nu(\cos\Theta)$ as follows:

$$Y_{NJM}(\Omega) = Y_{NJM}(\Theta, \vartheta, \varphi) = G_N^{(J)}(\Theta)Y_{JM}(\vartheta, \varphi)\,, \tag{3.14}$$

$$G_N^{(J)}(\Theta) = \left[\frac{2^{2J+1}(N+1)(N-J)!}{\pi(N+J+1)!}\right]^{1/2} \times J!\,C_{N-J}^{J+1}(\cos\Theta)\sin^J\Theta\,, \tag{3.15}$$

and

$$Y^{(1)}_{(NJM)\mu}(\Omega) = \hat{x}_\mu Y_{NJM}(\Omega)\,, \quad Y^{(2)}_{(NJM)\mu}(\Omega) = R\partial_\mu Y_{NJM}(\Omega)\,. \tag{3.16}$$

In definition (3.14), the functions Y_{JM} denote the usual three-dimensional scalar spherical harmonics.

The amplitudes of sector IV are the following:

$$T'_{\mu\nu}(x) = T_N^{(1)}(R)Y^{(1)}_{(NJM)\mu\nu}(\Omega)\,, \tag{3.17}$$

$$A_\mu(x) = A_N^{(1)}(R)Y^{(1)}_{(NJM)\mu}(\Omega) + A_N^{(2)}(R)Y^{(2)}_{(NJM)\mu}(\Omega)\,, \tag{3.18}$$

where

$$Y^{(1)}_{(NJM)\mu\nu}(\Omega)=(x_\mu\partial_\nu - x_\nu\partial_\mu)Y_{NJM}(\Omega)\,. \qquad (3.19)$$

We now turn to the radial BS equations of the amplitudes belonging to sectors I and IV. Let us introduce the notations

$$f_1^{(1)\prime}(R)=S_N(R)\,, \qquad (3.20a)$$

$$f_2^{(1)\prime}(R)=V_N^{(1)}(R)\,,$$
$$f_3^{(1)\prime}(R)=[N(N+2)]^{1/2}V_N^{(2)}(R)\,, \qquad (3.20b)$$

and

$$f_1^{(4)\prime}(R)=i[N(N+2)]^{1/2}T_N^{(1)}(R)\,, \qquad (3.21a)$$

$$f_2^{(4)\prime}(R)=A_N^{(1)}(R)\,,$$
$$f_3^{(4)\prime}(R)=[N(N+2)]^{1/2}A_N^{(2)}(R)\,. \qquad (3.21b)$$

According to the method of Böhm, Joos, and Krammer,[3,7,8] we shall apply the radial amplitudes $f_n^{(1)}(R)$ and $f_n^{(4)}(R)$ $(n=1,2,3)$ as defined by the linear transformation

$$f_m^{(\sigma)\prime}(R)=\sum_{n=1}^{3} W_{mn}f_n^{(\sigma)}(R) \quad (m=1,2,3;\ \sigma=1,4)\,, \qquad (3.22)$$

where

$$W=\begin{bmatrix}1 & 0 & 0\\ 0 & \left[\frac{N}{2(N+1)}\right]^{1/2} & -\left[\frac{N+2}{2(N+1)}\right]^{1/2}\\ 0 & \left[\frac{N+2}{2(N+1)}\right]^{1/2} & \left[\frac{N}{2(N+1)}\right]^{1/2}\end{bmatrix}. \qquad (3.23)$$

Starting with Eqs. (3.5), (3.6) and (3.9), (3.10), we arrive at two uncoupled systems of ordinary differential equations (radial BS equations) for the amplitudes $f_n^{(1)}(R)$ and $f_n^{(4)}(R)$. The final result can be written in the form

$$\sum_{n=1}^{3}\hat{B}_{mn}^{(\sigma)}f_n^{(\sigma)}(R)=0 \quad (m=1,2,3;\ \sigma=1,4)\,. \qquad (3.24)$$

The matrix elements $\hat{B}_{mn}^{(1)}$ and $\hat{B}_{mn}^{(4)}$ are explicitly given in Tables I and II, respectively. These formulas involve the following operators:

$$d(-2;N,R)=\frac{d^2}{dR^2}-\frac{2N+1}{R}\frac{d}{dR}+\frac{N(N+2)}{R^2}\,, \qquad (3.25)$$

$$d(-1;N,R)=\frac{d}{dR}-\frac{N}{R}\,, \qquad (3.26)$$

$$d(0;N,R)=\frac{d^2}{dR^2}+\frac{3}{R}\frac{d}{dR}-\frac{N(N+2)}{R^2}\,, \qquad (3.27)$$

$$d(1;N,R)=\frac{d}{dR}+\frac{N+2}{R}\,, \qquad (3.28)$$

$$d(2;N,R)=\frac{d^2}{dR^2}+\frac{2N+3}{R}\frac{d}{dR}+\frac{N(N+2)}{R^2}\,. \qquad (3.29)$$

We end this section with a short summary of sectors II and III. The amplitudes are, respectively,

$$T'_{\mu\nu}(x)=T_N^{(2)}(R)Y^{(2)}_{(NJM)\mu\nu}(\Omega)\,, \qquad (3.30)$$

$$Y^{(2)}_{(NJM)\mu\nu}(\Omega)=\epsilon_{\mu\nu\rho\sigma}Y^{(1)}_{(NJM)\rho\sigma}(\Omega)\,, \qquad (3.31)$$

and

$$P(x)=P_N(R)Y_{NJM}(\Omega)\,. \qquad (3.32)$$

The pseudoscalar amplitude $P_N(R)$ satisfies the well-known equation

$$[d(0;N,R)-I_S(R)-\kappa^2]P_N(R)=0\,. \qquad (3.33)$$

In sector II a similar single radial equation can be derived for the amplitude $T_N^{(2)}(R)$.

IV. SHORT-DISTANCE BEHAVIOR OF THE SOLUTIONS

A. Indicial equations

We focus our attention on sectors I and IV. The BS problem (3.24) is a system of second-order ordinary differential equations involving the interactions (3.4), (3.7), (3.8), and (3.11). These interaction terms are marginally singular at $R\to 0$, if no cancellations occur. According to Eqs. (2.20)–(2.22), we obtain

$$I_{S,T}(R)=\frac{Z_{S,T}(\lambda)}{R^2}+\cdots \quad \text{for } R\to 0\,, \qquad (4.1)$$

$$I_V^{(1,0)}(R)=\frac{Z_V^{(1,0)}(\lambda)}{R^2}+\cdots \quad \text{for } R\to 0\,, \qquad (4.2)$$

TABLE I. List of the matrix elements $\hat{B}_{mn}^{(1)}$.

$\hat{B}_{11}^{(1)}=-d(0;N,R)+I_S(R)-\kappa^2,$

$\hat{B}_{12}^{(1)}=\left(\frac{2N}{N+1}\right)^{1/2}\kappa d(-1;N-1,R),$

$\hat{B}_{13}^{(1)}=-\left[\frac{2(N+2)}{N+1}\right]^{1/2}\kappa d(1;N+1,R),$

$\hat{B}_{21}^{(1)}=-\left(\frac{2N}{N+1}\right)^{1/2}\kappa d(1;N,R),$

$\hat{B}_{22}^{(1)}=-\frac{1}{N+1}[d(0;N-1,R)-I_V^{(1)}(R)]-I_V^{(0)}(R)+\kappa^2,$

$\hat{B}_{23}^{(1)}=-\frac{[N(N+2)]^{1/2}}{N+1}[d(2;N+1,R)-I_V^{(1)}(R)],$

$\hat{B}_{31}^{(1)}=\left[\frac{2(N+2)}{N+1}\right]^{1/2}\kappa d(-1;N,R),$

$\hat{B}_{32}^{(1)}=-\frac{[N(N+2)]^{1/2}}{N+1}[d(-2;N-1,R)-I_V^{(1)}(R)],$

$\hat{B}_{33}^{(1)}=\frac{1}{N+1}[d(0;N+1,R)-I_V^{(1)}(R)]-I_V^{(0)}(R)+\kappa^2.$

TABLE II. List of the matrix elements $\hat{B}^{(4)}_{mn}$.

$$\hat{B}^{(4)}_{11}=d(0;N,R)-I_T(R)+\kappa^2,$$

$$\hat{B}^{(4)}_{12}=-\left[\frac{2(N+2)}{N+1}\right]^{1/2}\kappa d(-1;N-1,R),$$

$$\hat{B}^{(4)}_{13}=-\left(\frac{2N}{N+1}\right)^{1/2}\kappa d(1;N+1,R),$$

$$\hat{B}^{(4)}_{21}=\left[\frac{2(N+2)}{N+1}\right]^{1/2}\kappa d(1;N,R),$$

$$\hat{B}^{(4)}_{22}=-\frac{1}{N+1}[d(0;N-1,R)-I_V^{(1)}(R)]-I_V^{(0)}(R)-\kappa^2,$$

$$\hat{B}^{(4)}_{23}=-\frac{[N(N+2)]^{1/2}}{N+1}[d(2;N+1,R)-I_V^{(1)}(R)],$$

$$\hat{B}^{(4)}_{31}=\left(\frac{2N}{N+1}\right)^{1/2}\kappa d(-1;N,R),$$

$$\hat{B}^{(4)}_{32}=-\frac{[N(N+2)]^{1/2}}{N+1}[d(-2;N-1,R)-I_V^{(1)}(R)],$$

$$\hat{B}^{(4)}_{33}=\frac{1}{N+1}[d(0;N+1,R)-I_V^{(1)}(R)]-I_V^{(0)}(R)-\kappa^2.$$

where

$$Z_S(\lambda)=(4-b^{(V)})\lambda G_{V0}{}^2+(4-b^{(A)})G_{A0}{}^2, \quad (4.3)$$

$$Z_T(\lambda)=b^{(V)}\lambda G_{V0}{}^2+b^{(A)}G_{A0}{}^2, \quad (4.4)$$

$$Z_V^{(1)}(\lambda)=b^{(V)}\lambda G_{V0}{}^2-b^{(A)}G_{A0}{}^2, \quad (4.5)$$

$$Z_V^{(0)}(\lambda)=-(2-b^{(V)})\lambda G_{V0}{}^2+(2-b^{(A)})G_{A0}{}^2, \quad (4.6)$$

with

$$G_{V0}{}^2=G_V{}^2Z^{(V)}, \quad G_{A0}{}^2=G_A{}^2Z^{(A)}. \quad (4.7)$$

Standard theory of differential equations tells us that there exist solutions with powerlike short-distance behavior:

$$f_n^{(\sigma;k)}(R)=a_{n;0}^{(\sigma;k)}R^{\rho(\sigma;k;N)}+\cdots \quad \text{for } R\to 0. \quad (4.8)$$

Here the assignments $k=1,2,\ldots$ are used to indicate independent solutions. For the coefficients $a_{n;0}^{(\sigma;k)}$, the radial BS equations (3.24) lead to the following system of homogeneous linear equations:

$$\sum_{n=1}^{3}\tilde{B}^{(\sigma)}_{mn}(N,\rho(\sigma;k;N))a_{n;0}^{(\sigma;k)}=0 \quad \text{for } m=1,2,3. \quad (4.9)$$

The formulas of the matrix elements $\tilde{B}^{(\sigma)}_{mn}$ are listed in Table III. Since we search for nontrivial solutions of Eqs. (4.9), the indices $\rho(\sigma;k;N)$ must satisfy the indicial equation

$$\text{Determinant}\ |\tilde{B}^{(\sigma)}_{mn}(N,\rho(\sigma;k;N))|=0. \quad (4.10)$$

Let us summarize some results for regularized interactions which are characterized by

$$Z_{S,T}(\lambda)=0, \quad Z_V^{(1,0)}(\lambda)=0. \quad (4.11)$$

In this case the indicial equations (4.10) can be easily solved by using the formulas of Table III. One obtains the following six solutions (canonical indices):

$$\rho(\sigma;1;N)=N+1,$$
$$\rho(\sigma;2;N)=N, \quad (4.12a)$$
$$\rho(\sigma;3;N)=N-1,$$
$$\rho(\sigma;4;N)=-N-1,$$
$$\rho(\sigma;5;N)=-N-2, \quad (4.12b)$$
$$\rho(\sigma;6;N)=-N-3.$$

Consequently, if $N>0$, the radial BS equations (3.24) have three regular solutions and three singular ones as $R\to 0$. Notice that, for $N>0$, the third index $\rho(\sigma;3;N)=N-1$ governs the short-distance behavior of the leading regular solution; the corresponding solutions of Eqs. (4.9) [together with (4.11)] are

$$a_{1;0}^{(\sigma;3)}=0, \quad a_{2;0}^{(\sigma;3)}\neq 0, \quad a_{3;0}^{(\sigma;3)}=0. \quad (4.13)$$

TABLE III. The matrix elements $\tilde{B}^{(\sigma)}_{mn}$.

$$\tilde{B}^{(1)}_{11}(N,\rho)=-\{\rho-[(N+1)^2+Z_S(\lambda)]^{1/2}+1\}\{\rho+[(N+1)^2+Z_S(\lambda)]^{1/2}+1\}, \quad (\text{III}.1)$$

$$\tilde{B}^{(4)}_{11}(N,\rho)=\{\rho-[(N+1)^2+Z_T(\lambda)]^{1/2}+1\}\{\rho+[(N+1)^2+Z_T(\lambda)]^{1/2}+1\}, \quad (\text{III}.2)$$

$$\tilde{B}^{(\sigma)}_{22}(N,\rho)=-\frac{1}{N+1}\{\rho-[N^2+Z_V^{(1)}(\lambda)-(N+1)Z_V^{(0)}(\lambda)]^{1/2}+1\}\{\rho+[N^2+Z_V^{(1)}(\lambda)-(N+1)Z_V^{(0)}(\lambda)]^{1/2}+1\}, \quad (\text{III}.3)$$

$$\tilde{B}^{(\sigma)}_{23}(N,\rho)=-\frac{[N(N+2)]^{1/2}}{N+1}\{\rho+N+2-[1+Z_V^{(1)}(\lambda)]^{1/2}\}\{\rho+N+2+[1+Z_V^{(1)}(\lambda)]^{1/2}\}, \quad (\text{III}.4)$$

$$\tilde{B}^{(\sigma)}_{32}(N,\rho)=-\frac{[N(N+2)]^{1/2}}{N+1}\{\rho-N-[1+Z_V^{(1)}(\lambda)]^{1/2}\}\{\rho-N+[1+Z_V^{(1)}(\lambda)]^{1/2}\}, \quad (\text{III}.5)$$

$$\tilde{B}^{(\sigma)}_{33}(N,\rho)=\frac{1}{N+1}\{\rho-[(N+2)^2+Z_V^{(1)}(\lambda)+(N+1)Z_V^{(0)}(\lambda)]^{1/2}+1\}\{\rho+[(N+2)^2+Z_V^{(1)}(\lambda)+(N+1)Z_V^{(0)}(\lambda)]^{1/2}+1\}, \quad (\text{III}.6)$$

$$\tilde{B}^{(\sigma)}_{a1}(N,\rho)=\tilde{B}^{(\sigma)}_{1a}(N,\rho)=0 \quad \text{for } a=2,3. \quad (\text{III}.7)$$

Now we can return to marginally singular interactions. In this case, the indices $\rho(\sigma; k; N)$ $(k=1, 2, \ldots, 6)$ will be ordered by requiring that they must go smoothly into the values (4.12a), (4.12b) if $Z_{S,T}(\lambda) \to 0$, $Z_V^{(1,0)}(\lambda) \to 0$.

B. The subsidiary condition

We next restrict ourselves to nonvanishing values of N and investigate some general short-distance properties of the vector and axial-vector amplitudes $V_\mu^{(3)}(x)$ and $A_\mu^{(3)}(x)$ that correspond to the leading solutions $f_n^{(1;3)}(R)$ and $f_n^{(4;3)}(R)$, respectively. Let us introduce the notations

$$V_\mu^{(1;3)}(x) = V_\mu^{(3)}(x), \quad V_\mu^{(4;3)}(x) = A_\mu^{(3)}(x).$$

According to Eqs. (3.13), (3.18), and (4.8), the asymptotic solutions can be written as

$$V_\mu^{(\sigma;3)}(x) = c_N^{(\sigma)} R^{\rho(\sigma;3;N)} \times [Y^{(1)}_{(NJM)\mu}(\Omega) + q_N^{(\sigma)} Y^{(2)}_{(NJM)\mu}(\Omega)] + \cdots \quad \text{for } R \to 0, \tag{4.14}$$

where the coefficients $c_N^{(\sigma)}$ and $q_N^{(\sigma)}$ are fixed by the solutions $a_{n;0}^{(\sigma;3)}$ $(n=2,3)$ of Eqs. (4.9). For $\mu = j = 1,2,3$, the form (4.14) involves expressions $\hat{x}_j Y_{NJM}$ and $\partial_j Y_{NJM}$, which can be evaluated in terms of four-dimensional three-vector spherical harmonics $Y_{(NJLM)j}(\Omega)$ defined by[15]

$$Y_{(NJLM)j}(\Omega) = G_N^{(L)}(\Theta) Y_{(JLM)j}(\vartheta, \varphi) \tag{4.15}$$

with Eq. (3.15). Here $Y_{(JLM)j}(\vartheta,\varphi)$ denotes the jth component of the usual three-dimensional vector spherical harmonics.[21] According to Ref. 19, we obtain

$$\begin{aligned} V_j^{(\sigma;3)}(x) = c_N^{(\sigma)} R^{\rho(\sigma;3;N)} \{ & Q(1,1;N,J) Y_{(N+1,J,J+1,M)j}(\Omega)[1 - N q_N^{(\sigma)}] \\ & + Q(-1,1;N,J) Y_{(N-1,J,J+1,M)j}(\Omega)[1 + (N+2) q_N^{(\sigma)}] \\ & + Q(1,-1;N,J) Y_{(N+1,J,J-1,M)j}(\Omega)[1 - N q_N^{(\sigma)}] \\ & + Q(-1,-1;N,J) Y_{(N-1,J,J-1,M)j}(\Omega)[1 + (N+2) q_N^{(\sigma)}] \} + \cdots \quad \text{for } R \to 0. \end{aligned} \tag{4.16}$$

The coefficients Q are given by

$$Q(1,1;N,J) = -\frac{1}{2}\left[\frac{(J+1)(N+J+2)(N+J+3)}{(2J+1)(N+1)(N+2)}\right]^{1/2}, \tag{4.17}$$

$$Q(-1,1;N,J) = \frac{1}{2}\left[\frac{(J+1)(N-J)(N-J-1)}{(2J+1)N(N+1)}\right]^{1/2}, \tag{4.18}$$

$$Q(1,-1;N,J) = -\frac{1}{2}\left[\frac{J(N-J+1)(N-J+2)}{(2J+1)(N+1)(N+2)}\right]^{1/2}, \tag{4.19}$$

$$Q(-1,-1;N,J) = \frac{1}{2}\left[\frac{J(N+J)(N+J+1)}{(2J+1)N(N+1)}\right]^{1/2}. \tag{4.20}$$

For regularized interactions, the short-distance behavior of the amplitudes $V_j^{(\sigma;3)}(x)$ is particularly simple at $N=J>0$. Solutions (4.13) lead to $q_N^{(\sigma)} = 1/N$ and $c_N^{(\sigma)} \neq 0$; thus we have

$$V_j^{(\sigma;3)}(x)|_{N=J>0} = a^{(\sigma;3)} R^{J-1} Y_{(J-1,J,J-1,M)j}(\Omega) + \cdots \quad \text{for } R \to 0. \tag{4.21}$$

In the particular case $N=J=1$, the leading term of (4.21) is independent of the angular variables $\Theta, \vartheta, \varphi$, and, in addition, it is finite at $R=0$.

By considering marginally singular interactions, we observe that, at $N=J>0$, expansion (4.16) may involve terms of the type

$$R^{\rho(\sigma;3;J)} Y_{(J+1,J,J\pm1,M)j}(\Omega). \tag{4.22}$$

As a crucial step in this paper, we shall exclude the terms (4.22) by imposing the requirements

$$q_N^{(\sigma)} = 1/N, \quad c_N^{(\sigma)} \neq 0 \quad \text{for } N>0, \tag{4.23}$$

which yield $a_{2;0}^{(\sigma;3)} \neq 0$ and $a_{3;0}^{(\sigma;3)} = 0$ according to definitions (3.20)–(3.23). These requirements are obviously equivalent to the subsidiary condition

$$V_j^{(\sigma;3)}(x)|_{N=J>0} = a^{(\sigma;3)} R^{\rho(\sigma;3;J)} Y_{(J-1,J,J-1,M)j}(\Omega) + \cdots \quad \text{for } R \to 0. \tag{4.24}$$

We next investigate some consequences of the subsidiary condition (4.24). The requirements $a_{2;0}^{(\sigma;3)} \neq 0$ and $a_{3;0}^{(\sigma;3)} = 0$, together with Eq. (4.9) and Eqs. (III.3)–(III.7) of Table III, imply

$$\begin{aligned} \tilde{B}_{22}^{(\sigma)}(N, \rho(\sigma;3;N)) &= 0, \\ \tilde{B}_{32}^{(\sigma)}(N, \rho(\sigma;3;N)) &= 0. \end{aligned} \tag{4.25}$$

Straightforward calculation shows that Eqs. (4.25) are simultaneously satisfied for any value of N if

$$Z_V^{(1)}(\lambda) = Z_V^{(0)}(\lambda) + \tfrac{1}{4}[Z_V^{(0)}(\lambda)]^2. \tag{4.26}$$

Thus, in this way, the choice of the marginally singular interactions is restricted by the subsidiary condition (4.24).

C. Indices and short-distance expansions

Substituting the restriction (4.26) into the formulas (III.3)–(III.6) of the matrix elements $\tilde{B}_{mn}^{(\sigma)}$ (see Table III), we obtain

$$\tilde{B}_{22}^{(\sigma)}(N,\rho) = -\frac{1}{N+1}\{\rho - [N-1-\tfrac{1}{2}Z_V^{(0)}(\lambda)\} \times\{\rho - [-N-1+\tfrac{1}{2}Z_V^{(0)}(\lambda)\}, \qquad (4.27)$$

$$\tilde{B}_{23}^{(\sigma)}(N,\rho) = -\frac{[N(N+2)]^{1/2}}{N+1}\{\rho - [-N-1+\tfrac{1}{2}Z_V^{(0)}(\lambda)]\} \times\{\rho - [-N-3-\tfrac{1}{2}Z_V^{(0)}(\lambda)]\}, \qquad (4.28)$$

$$\tilde{B}_{32}^{(\sigma)}(N,\rho) = -\frac{[N(N+2)]^{1/2}}{N+1}\{\rho - [N+1+\tfrac{1}{2}Z_V^{(0)}(\lambda)]\} \times\{\rho - [N-1-\tfrac{1}{2}Z_V^{(0)}(\lambda)]\}, \qquad (4.29)$$

$$\tilde{B}_{33}^{(\sigma)}(N,\rho) = \frac{1}{N+1}\{\rho - [N+1+\tfrac{1}{2}Z_V^{(0)}(\lambda)]\} \times\{\rho - [-N-3-\tfrac{1}{2}Z_V^{(0)}(\lambda)]\}. \qquad (4.30)$$

By comparing Eqs. (4.27)–(4.30) with the corresponding formulas of Table III, we observe that the structure of the matrix $\tilde{B}^{(\sigma)}(N,\rho)$ is radically simplified because of the fulfilment of the restriction (4.26). In addition, the matrix elements (III.1) and (III.2) (Table III), (4.27)–(4.30), and (III.7) lead to a factorized form of the indicial equation (4.10) which, in this way, involves the roots $\rho(\sigma;k;N)$ ($k=1,2,\ldots 6$) explicitly. These solutions are the following:

$$\rho(\sigma;1;N) = N+1+\tfrac{1}{2}Z_V^{(0)}(\lambda), \qquad (4.31)$$

$$\rho(1;2;N) = [(N+1)^2+Z_S(\lambda)]^{1/2}-1, \qquad (4.32a)$$

$$\rho(4;2;N) = [(N+1)^2+Z_T(\lambda)]^{1/2}-1, \qquad (4.32b)$$

$$\rho(\sigma;3;N) = N-1-\tfrac{1}{2}Z_V^{(0)}(\lambda), \qquad (4.33)$$

$$\rho(\sigma;4;N) = -N-1+\tfrac{1}{2}Z_V^{(0)}(\lambda), \qquad (4.34)$$

$$\rho(1;5;N) = -[(N+1)^2+Z_S(\lambda)]^{1/2}-1, \qquad (4.35a)$$

$$\rho(4;5;N) = -[(N+1)^2+Z_T(\lambda)]^{1/2}-1, \qquad (4.35b)$$

$$\rho(\sigma;6;N) = -N-3-\tfrac{1}{2}Z_V^{(0)}(\lambda). \qquad (4.36)$$

We proceed to investigate the short-distance expansion of the solutions $f_n^{(\sigma;k)}(R)$. Our aim is to calculate a few dominant terms; therefore, as a first step, it will be convenient to approximate the interactions $I_{S,T}(R)$ and $I_V^{(1,0)}(R)$ by the leading terms that are explicitly contained in Eqs. (4.1) and (4.2). In this case the solutions can be expanded as

$$f_n^{(\sigma;k)}(R) = R^{\rho(\sigma;k;N)}\sum_{h=0} a_{n;h}^{(\sigma;k)}R^h,$$

if the differences $\rho(\sigma;k;N)-\rho(\sigma;k';N)$ of the indices are nonintegers. (Otherwise, expansions of the solutions may involve logarithmic factors.) The expansion coefficients are fixed by standard recursion formulas.[22] The leading terms of the solutions can be written in the form

$$f_n^{(\sigma;k)}(R) = a_{n;\chi(k,n)}^{(\sigma;k)}R^{\rho(\sigma;k;N)+\chi(k,n)}+\cdots \quad \text{for } R\to 0, \qquad (4.37)$$

where

$$\chi(1,1)=1,\quad \chi(1,2)=0,\quad \chi(1,3)=0, \qquad (4.38)$$

$$\chi(2,1)=0,\quad \chi(2,2)=1,\quad \chi(2,3)=1, \qquad (4.39)$$

$$\chi(3,1)=1,\quad \chi(3,2)=0,\quad \chi(3,3)=2, \qquad (4.40)$$

$$\chi(4,1)=1,\quad \chi(4,2)=0,\quad \chi(4,3)=0, \qquad (4.41)$$

$$\chi(5,1)=0,\quad \chi(5,2)=1,\quad \chi(5,3)=1, \qquad (4.42)$$

$$\chi(6,1)=1,\quad \chi(6,2)=2,\quad \chi(6,3)=0. \qquad (4.43)$$

D. Boundary conditions

First, we prepare the discussion of the boundary conditions at $R=0$ by introducing some conventions. We shall call "good" or "bad" indices which take positive (including zero) or negative values, respectively, by setting $Z_{S,T}(\lambda)=0$ and $Z_V^{(1,0)}(\lambda)=0$. In addition, the asymptotic solutions (4.37) involving good or bad indices will be referred to as good or bad solutions, respectively. Thus, if $N>0$, the six independent asymptotic solutions of sector I (or sector IV) consist of three good solutions and three bad ones, which are given by Eq. (4.37) with Eqs. (4.38)–(4.40) and Eq. (4.37) with Eqs. (4.41)–(4.43), respectively. Reference 22 includes a discussion of the solutions at $N=0$.

We now impose the following boundary condition at the origin of the four-dimensional Euclidean space: As $R\to 0$, an acceptable solution must be a linear combination of the good solutions (4.37)–(4.40); thus the bad solutions are to be discarded. Our selection of the good solutions may become meaningless for large values of the coupling parameters. [For example, $\rho(\sigma;3;1)<\rho(\sigma;4;1)$ at $Z_V^{(0)}(\lambda)>2$.] Therefore, in the following, we restrict the choice of the parameters $Z_{S,T}(\lambda)$ and $Z_V^{(1,0)}(\lambda)$ by requiring that, as $R\to 0$, the good solutions must be less singular than the bad ones. Some other restrictions are also necessary in order to guarantee a reasonable set of solutions; related problems will be discussed in Sec. V.

The asymptotic solutions at infinity ($R\to\infty$) can be calculated in a straightforward way.[22] For $N>0$ one obtains three regular (exponentially decreasing) solutions and three irregular (exponentially increasing) ones in both sectors I and IV. According to the usual boundary conditions, the irregular solutions must be absent in order for vanishing bound-state amplitudes at infinity to be guaranteed.

We argue that the previous boundary conditions may lead to a reasonable bound-state problem. Starting with a linear combination of the three good solutions given by Eq. (4.37) with (4.38)–(4.40) near $R=0$, one of the coefficients can be absorbed in the normalization factor of the amplitude $f_n^{(\sigma)}(R)$. Thus, there are two free coefficients as we integrate out to infinity. In addition to these coefficients one needs a third parameter in order to eliminate the three irregular solutions at infinity. In this way, the boundary conditions may be satisfied only at particular values (eigenvalues) of some parameter in the BS equation.

Finally, we summarize the short-distance properties in sector III. The radial equation (3.33) becomes identical to the well-known Goldstein equation[23] in the limit $R \to 0$. The asymptotic solutions have the powerlike behavior

$$P_N^{(k)}(R) = a_0^{(3;k)} R^{\rho(3;k;N)} + \cdots \quad \text{for } R \to 0;\ k=1,2\,. \tag{4.44}$$

The solutions of the indicial equation are

$$\begin{aligned} \rho(3;1;N) &= \rho(1;2;N)\,, \\ \rho(3;2;N) &= \rho(1;5;N) \end{aligned} \tag{4.45}$$

with Eqs. (4.32a) and (4.35a). Here, according to our previous prescriptions, we shall choose $P_N^{(1)}(R)$ as the good solution of the problem, and the bad solution $P_N^{(2)}(R)$ will be discarded.

V. CONSISTENCY PROBLEMS

The previous investigations involve the following dynamical restrictions: (i) the short-distance behavior of the interaction as given by Eqs. (2.17) and (2.20); (ii) the subsidiary condition (4.24); (iii) selection of the good solutions. The field-theoretical basis of the BS equation may lead to other important restrictions; a prominent example is offered by the well-known normalization condition of the amplitudes.[24]

For further insight, we next consider vector and axial-vector interactions subject to the condition

$$b^{(V)} = b^{(A)} = b(\lambda)\,. \tag{5.1}$$

According to the subsidiary condition (4.24), we imposed requirement (4.26) for the parameters $Z_V^{(1)}$ and $Z_V^{(0)}(\lambda)$. If the choice of these parameters is restricted by Eqs. (4.5)–(4.7) and (5.1), then the requirement (4.26) yields a quadratic equation for $b(\lambda)$. The solutions can be written as

$$\begin{aligned} Z_V^{(1)}(\lambda) &= b(\lambda)(\lambda G_{V0}{}^2 - G_{A0}{}^2) \\ &= 2\{(\lambda G_{V0}{}^2 - G_{A0}{}^2) \pm [2(\lambda G_{V0}{}^2 - G_{A0}{}^2)]^{1/2}\}\,. \end{aligned} \tag{5.2}$$

In addition, one obtains

$$Z_V^{(0)}(\lambda) = \pm 2[2(\lambda G_{V0}{}^2 - G_{A0}{}^2)]^{1/2}\,. \tag{5.3}$$

Ladder-type approximations suggest real values for the coupling strengths G_{V0} and G_{A0}. Thus, for fermion-antifermion BS amplitudes ($\lambda=-1$), Eq. (5.2) results in a complex parameter $b(-1)$. Moreover, the parameter $Z_V^{(0)}(-1)$ becomes purely imaginary in this case. For fermion-fermion amplitudes, on the other hand, both $b(1)$ and $Z_V^{(0)}(1)$ are real if $G_{V0}{}^2 > G_{A0}{}^2$. In the particular situation $G_{V0}{}^2 = G_{A0}{}^2$, $\lambda=1$, the parameters $Z_V^{(1)}(1)$ and $Z_V^{(0)}(1)$ vanish by cancellation and, in this way, the requirement (4.26) is satisfied for arbitrary values of $b(1)$.

We next return to fermion-antifermion systems. The BS amplitudes have some remarkable properties if the short-distance behavior of the interaction is prescribed by Eqs. (4.1), (4.2) and (5.1)–(5.3). For example, let us consider the radial amplitudes $f_2^{(\sigma;3)}(R)$, which are the dominant components of the leading solutions of sectors I and IV for nonvanishing values of N. Equations (4.37) and (4.40), together with (4.33) and (5.3), imply an oscillating short-distance behavior:

$$f_2^{(\sigma;3)}(R) = a_{2;0}^{(\sigma;3)} R^{N-1} e^{-2iG_0 \ln R} + \cdots \quad \text{for } R \to 0,\ \lambda=-1 \tag{5.4}$$

with

$$G_0{}^2 = \tfrac{1}{2}(G_{V0}{}^2 + G_{A0}{}^2)\,. \tag{5.5}$$

Notice that $|f_2^{(\sigma;3)}(0)|$ is finite at $N=1$.

In the case $N=0$, the scalar and pseudoscalar amplitudes play an important role at short distances. We now suggest the following conditions:

$$0 < |\mathrm{Tr}\,\tau^{(\mathrm{fa})}(x_1, x_1)| < \infty \quad \text{for } N=0\,, \tag{5.6}$$

$$0 < |\mathrm{Tr}\,\tau^{(\mathrm{fa})}(x_1, x_1)\gamma_5| < \infty \quad \text{for } N=0\,. \tag{5.7}$$

The explicit form of the index (4.32a) implies that conditions (5.6) and (5.7) can be satisfied only if $Z_S(-1)=0$. This requirement, together with Eqs. (4.3), (5.1), and (5.2), yields

$$G_{V0}{}^2 = G_{A0}{}^2 = G_0{}^2\,, \tag{5.8}$$

$$b(-1) = 2(1 + iG_0{}^{-1})\,. \tag{5.9}$$

Inspection of the indices $\rho(\sigma;k;N)$ shows that, for marginally singular interactions satisfying restrictions (5.1), (5.8), and (5.9), the good solutions are less singular than the bad ones. In this way, our selection of the good solutions leads to well-defined eigenvalue problems along the lines discussed in Sec. IV. Let us notice that real BS eigenvalues of physical quantities are not excluded in spite of the complex value of $b(-1)$. For example, one may obtain the real ground-state c.m.

energy, but, in general, this implies a further restriction on the space-time dependence of the integration.

Finally, we include some remarks on the fermion-fermion BS equation involving interactions restricted by Eqs. (5.1) and (5.8). If $b(1)\neq 4$, for example, then only the indices $\rho(\sigma;2;N)$ and $\rho(\sigma;5;N)$ have noncanonical values in both sectors I and IV. In addition, we observe that all the indices of sectors I and III are canonical by setting $b(1)=4$.

Motivated by the results of this paper, we suggest further study of the spinor-spinor BS equation that involves vector and axial-vector interactions satisfying the restrictions suggested in Secs. IV and V. We mention here that it is difficult to find other interactions of the type (2.17)-(2.20) without producing solutions with unpleasant short-distance properties.

VI. COMMENTS

We have discussed that spinor-spinor BS equations with marginally singular interactions may lead to standard eigenvalue problems; however, boundary conditions and other consistency requirements impose severe restrictions on the possible choice of the interactions. The framework of the present paper should be extended to include powerful expansion methods which bypass ordinary perturbation theory. For example, we have neglected the fermion self-energy contributions which are connected with the solutions of spinor-vector BS equations. The ladder-type Green's functions of spinor-spinor and spinor-vector systems can be constructed from the solutions of the corresponding homogeneous BS equations which include marginally singular interaction kernels. We suggest expansion methods involving ladder-type Green's functions instead of the free propagators of the standard perturbation theory. These Green's functions depend, of course, on the explicit form of the interaction kernels, which are to be fixed by requiring optimal convergence of the higher approximations. It is an interesting possibility that, in the region $R\to 0$, boundary conditions prescribe the leading terms of a realistic marginally singular interaction which governs the dominant short-distance properties of the BS amplitudes.

---

[1]E. E. Salpeter and H. A. Bethe, Phys. Rev. 84, 1232 (1951); M. Gell-Mann and F. Low, *ibid.* 84, 350 (1951); J. Schwinger, Proc. Nat. Acad. Sci. U.S.A. 37, 452 (1951).

[2]N. Nakanishi, Prog. Theor. Phys. Suppl. 43, 1 (1969).

[3]M. Böhm, H. Joos, and M. Krammer, Acta Phys. Austriaca 11, 3 (1973).

[4]R. Delbourgo, A. Salam, and J. Strathdee, Nuovo Cimento 50A, 193 (1967).

[5]P. Narayanaswamy and A. Pagnamenta, Nuovo Cimento 53A, 635 (1968); Phys. Rev. 172, 1750 (1968).

[6]M. K. Sundaresan and P. J. S. Watson, Ann. Phys. (N.Y.) 59, 375 (1970).

[7]M. Böhm, H. Joos, and M. Krammer, Nucl. Phys. B51, 397 (1973).

[8]M. Krammer, Acta Phys. Austriaca 40, 187 (1974).

[9]D. Amati, S. Fubini, A. Stanghellini, and M. Tonin, Nuovo Cimento 22, 569 (1961); L. Bertocchi, S. Fubini, and M. Tonin, *ibid.* 25, 626 (1962).

[10]K. G. Wilson, Phys. Rev. 179, 1499 (1969); H. Fritzsch and M. Gell-Mann, in *Proceedings of the International Conference on Duality and Symmetry in Hadron Physics, Tel-Aviv, 1971,* edited by E. Gotsman (Weizmann Science Press, Jerusalem, 1971).

[11]M. L. Goldberger, talk presented at the IX Balaton Symposium on Particle Physics, Balatonfüred, Hungary, 1974 (unpublished).

[12]R. Jackiw and K. Johnson, Phys. Rev. D 8, 2386 (1973).

[13]F. Englert and R. Brout, Phys. Rev. Lett. 13, 321 (1964); H. Pagels, Phys. Rev. D 7, 3689 (1973); J. M. Cornwall and R. E. Norton, *ibid.* 8, 3338 (1973).

[14]P. W. Higgs, Phys. Rev. 145, 1156 (1966).

[15]M. Gourdin, Nuovo Cimento 7, 338 (1958).

[16]R. F. Keam, J. Math. Phys. 9, 1462 (1968); 10, 594 (1969); 11, 394 (1970); 12, 515 (1971).

[17]B. W. Lee, Phys. Rev. D 5, 823 (1972); N. Nakanishi, Prog. Theor. Phys. 49, 640 (1973).

[18]G. Domokos and P. Surányi, Nucl. Phys. 54, 529 (1964).

[19]K. Ladányi, Ann. Phys. (N.Y.) 77, 471 (1973).

[20]W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for Special Functions of Mathematical Physics* (Springer, New York, 1966).

[21]A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton Univ. Press, Princeton, N. J., 1957).

[22]K. Ladányi, ITP Report No. 327, 1973 (unpublished).

[23]J. S. Goldstein, Phys. Rev. 91, 1516 (1953).

[24]S. Mandelstam, Proc. R. Soc. A233, 248 (1955).