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Crossing multiparticle amplitudes: Pole singularities

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Canonical variables suitable for crossing multiparticle amplitudes are discussed and then used to show that crossing a multiparticle amplitude involves analytic continuation in only one variable. A representation for analytically continuing multiparticle amplitudes in this variable is then developed for simple types of singularities.

I. INTRODUCTION

One of the major difficulties confronting an S-matrix theory of the strongly interacting particles involves the analysis of multiparticle reactions. Such reactions are tied to elastic reactions by the unitarity equations, but much less is known about the structure of multiparticle reactions as compared with $2\to 2$ reactions. This is not only because of the larger number of variables needed in the amplitudes that describe multiparticle reactions (along with the concomitant increase in difficulty in analyzing multiparticle reactions), but also because most multiparticle reactions cannot even be probed experimentally in the laboratory. This means that models of multiparticle amplitudes can only be tested by relating them to experimentally accessible reactions.

Here it is necessary to distinguish between $2\to N$ reactions and $N'\to N$ reactions $(N,N'>2)$. One is able to learn something about $2\to N$ reactions from high-energy accelerators and cosmic-ray data, but there is no way at present of directly probing the $N'\to N$ $(N',N>2)$ reactions. Since the amplitude for any $N'\to N$ reaction is connected via crossing with the amplitude for a $2\to N'+N-2$ reaction, it seems natural to probe into the structure of an $N'\to N$ reaction by crossing from the experimentally accessible $2\to N'+N-2$ reaction and checking the resulting amplitude through the unitarity equations. But the problem is with the analytic continuation involved in such multiparticle amplitudes. There are many different sets of variables that can be used, and for a given set the singularity structure is not known; in particular this means that when an awkward set of variables is chosen, a simple kind of singularity may give rise to very complicated and difficult-to-manage amplitude behavior.

What we wish to show in this paper is how a canonical choice of variables reduces the analytic continuation of a multiparticle amplitude to continuation in one variable, a boost variable related to the total energy. It will be shown that this boost variable has as its physical domain the real axis, while in the crossed channel the physical domain lies on the $i\pi$ axis. Thus, if *one* particle is crossed, the only analytic continuation for *any* multiparticle amplitude involves the boost variable of the particle being crossed in the strip between 0 and $i\pi$.

The canonical set of variables is discussed in Ref. 1 and will be reviewed for completeness in Sec. II. Now, given such a canonical set in which analytic continuation is required in only one variable, one naturally inquires as to what sorts of singularities are possible in the strip. In this paper the main emphasis will be on pole singularities, and only a few remarks will be made about cuts in the strip. For pole singularities it is clear that analytic continuation in the strip is possible, if the function is explicitly given. But in general, even restricting oneself to pole singularities, the location and residues of the poles in a scattering amplitude are not known, and one would like to analyze classes of functions for which poles could be scattered anywhere in the strip. While a path of analytic continuation for any one such function would exist, there would not in general be a single path which would suffice for all functions in the class. Thus one would like a representation which would allow for any function in the class to be analytically continued. This topic will be pursued in Sec. IV with the help of a Paley-Wiener theorem[2]; it will be shown how knowledge of the Fourier transform may be used to compute the function anywhere in the strip and in particular on the $i\pi$ axis.

The basic idea underlying all of this analysis is to exploit properties of the substitution rule[3] for crossing a single particle and in particular to make use of the discrete nature of the substitution rule. The emphasis is thus on the discrete operation of replacing p_μ^C, the four-momentum of the particle C to be crossed, with $-p_\mu^{\bar{C}}$, the four-momentum of the antiparticle $\bar{C}$. When the substitution rule is combined with the group-theoretical structure of relativistic multiparticle states, the canonical variables arise as the natural variables to use in multiparticle amplitudes. Section II will review the meaning and use of these variables by way of preparation for Sec. III, where two-dimensional space-time amplitudes are introduced to simplify the study of the analytic properties of multiparticle amplitudes in the boost variable of the particle being crossed. Finally, Sec. IV will begin a discussion of how to represent functions so that their behavior under the substitution rule can be given meaning.

II. REVIEW OF KINEMATICS AND CANONICAL VARIABLES FOR MULTIPARTICLE AMPLITUDES

In this paper general multiparticle reactions will be considered in which a cluster A of incoming particles [labeled $1', \ldots, N'$ ($N' \geq 2$)] react to produce a cluster B of outgoing particles [labeled $1, \ldots, N$ ($N \geq 2$)] and a particle C, the particle which is to be crossed. Such reactions will be written $A \to B + C$, where A has invariant "mass" $s_A = (p_1' + \cdots + p_N')^2 \geq M_A^2$, B has invariant "mass" $s_B = (p_1 + \cdots + p_N)^2 \geq M_B^2$, and C is the single particle of mass M_C, which is to be crossed to the reaction $A + \overline{C} \to B$. All particles are assumed to be spinless, with known nonzero masses. (Spin and isospin will be treated in a subsequent paper.)

As shown in Ref. 1, canonical variables needed in the amplitude describing the $A \to B + C$ reaction can be broken into three distinct types:

(i) variables within clusters A and B which do not change when particle C is crossed (the choice of these variables will depend on how the amplitude is being used; for example, if another particle from cluster A is subsequently to be crossed, to give crossing as the term is generally used in S-matrix theory, then canonical variables within the A cluster would be appropriate),

(ii) angles specifying the direction of particle C relative to other momenta, and

(iii) two energies s_A and s_B, which will be involved in some sort of analytic continuation, as can be seen by noting that the physical domain for the $A \to B + C$ reaction is partially characterized by $\sqrt{s_A} \geq \sqrt{s_B} + M_C$, while the physical domain for the $A + \overline{C} \to B$ reaction satisfies the opposite inequality $\sqrt{s_B} \geq \sqrt{s_A} + M_C$.

The angles appearing under (ii) are defined relative to two frames of reference: the frame in which $\vec{p}_A \equiv \vec{p}_{1'} + \cdots + \vec{p}_{N'} = \vec{0}$ (the c.m. frame for the $A \to B + C$ reaction) and the frame $\vec{p}_B \equiv \vec{p}_1 + \cdots + \vec{p}_N = \vec{0}$, which is called the helicity frame for the $A \to B + C$ reaction. Clearly the two frames interchange roles for the $A + \overline{C} \to B$ reaction, so that $\vec{p}_A = \vec{0}$ becomes the helicity frame and $\vec{p}_B = \vec{0}$ becomes the c.m. frame. The notation $\theta_{ij(A)}$ will denote the (polar) angle between the momentum vectors $\vec{p}_i$ and $\vec{p}_j$ in the frame $\vec{p}_A = \vec{0}$, with a similar definition for $\theta_{ij(B)}$. Since azimuthal angles are angles between planes, the notation $\varphi_{ij-kl(A)}$ means the angle between the plane formed by $\vec{p}_i - \vec{p}_j$ and the plane $\vec{p}_k - \vec{p}_l$, all evaluated in the frame $\vec{p}_A = \vec{0}$. Reference 1 shows that variables for the amplitude describing the $A \to B + C$ reaction can be chosen to be of the form

$$F^{A \to B+C} = F^{A \to B+C}[s_A, \cos\theta_{1'C(A)}; \varphi_{1'C-C1(A)}, s_B, \cos\theta_{1C(B)}, \varphi_{C1-12(B)}; \varphi_{C1'-1'2'(A)}] \tag{2.1}$$

plus subenergy variables within the A and B clusters which are suppressed here because they do not change under the crossing of particle C. These variables are chosen because of their behavior with respect to a similar set chosen for the crossed reaction $A + \overline{C} \to B$:

$$F^{A+\overline{C} \to B} = F^{A+\overline{C} \to B}[s_B, \cos\theta_{1\overline{C}(B)}; \varphi_{\overline{C}1-12(B)}; \varphi_{1\overline{C}-\overline{C}1'(A)}, s_A, \cos\theta_{1'\overline{C}(A)}, \varphi_{\overline{C}1'-1'2'(A)}]\,. \tag{2.2}$$

Reference 1 shows that when the four-momentum p_μ^C of particle C goes to $-p_\mu^{\overline{C}}$ of the antiparticle $\overline{C}$ under the substitution rule, all the variables in Eqs. (2.1) and (2.2) except s_A and s_B stay in their physical regions, but take on new physical meanings:

$$\begin{array}{lcl}
A \to B + C & \longrightarrow & A + \overline{C} \to B \\
p_\mu^C & \longrightarrow & -p_\mu^{\overline{C}} \\
\cos\theta_{1'C(A)} \text{ (direct-channel scattering angle)} & \longrightarrow & -\cos\theta_{1'\overline{C}(A)} \text{ (crossed-channel helicity angle)} \\
\cos\theta_{1C(B)} \text{ (direct-channel helicity angle)} & \longrightarrow & -\cos\theta_{1\overline{C}(B)} \text{ (crossed-channel scattering angle)} \\
\varphi_{1'C-C1(A)} & \longrightarrow & \varphi_{1'\overline{C}-\overline{C}1(A)} \text{ (also true in frame } B) \\
\varphi_{C1-12(B)} & \longrightarrow & \varphi_{\overline{C}1-12(B)} + \pi \\
\varphi_{C1'-1'2'(A)} & \longrightarrow & \varphi_{\overline{C}1'-1'2'(A)} + \pi
\end{array} \tag{2.3}$$

The arrows connecting angles mean that when the angles are written explicitly as relativistic invariants, the form they have in the $A \to B+C$ reaction becomes transformed to the relativistically invariant form they should have in the $A+\overline{C} \to B$ reaction [see Eqs. (5) and (7) of Ref. 1].

It is not hard to show that the variables appearing in Eqs. (2.1) and (2.2) form complete independent sets, so that any function of relativistic invariants describing the $A \to B+C$ or $A+\overline{C} \to B$ reactions can be transformed to the above canonical set. Reference 4 shows how the simplest production Feynman diagrams can be expressed in these canonical variables. It should be noted that the partial-wave amplitudes for both reactions are obtained by transforming the appropriate polar and azimuthal angles to an angular momentum variable and two spin projection variables using O(3) D functions; under the substitution rule the angular momentum of cluster A simply exchanges its role with the angular momentum of cluster B.

Since under the transformation $p_\mu^C \to -p_\mu^{\overline{C}}$ all the angles remain in their physical regions, it is only necessary to analyze the behavior of the two remaining variables s_A and s_B. Now s_B is a subenergy for the $A \to B+C$ reaction, but since it becomes the total energy for the $A+\overline{C} \to B$ reaction it is sufficient to choose $\sqrt{s_B} \geq M_A + M_C$ ($M_A = M_{1'} + \cdots + M_N$) in order that, when the substitution rule is used, s_B also stays in its physical region. Thus, we have reduced the problem of connecting the amplitudes for the two reactions to a study of the behavior of the amplitude $F^{A\to B+C}$ as a function of s_A, with all other variables held fixed and in particular with $\sqrt{s_B}$ fixed but greater than $M_A + M_C$.

To make a change of variable to a boost variable, we orient the momenta of all outgoing particles so that $\vec{p}_C$ forms the z axis of a coordinate system in the outgoing particles. Let β be the boost parameter which takes C from its rest frame to the frame B, where its four-momentum is

$$p_\mu^C = \begin{bmatrix} E \\ 0 \\ 0 \\ p_z \end{bmatrix}, \quad E = \frac{s_A - s_B - M_C^2}{2\sqrt{s_B}\, M_C}$$

$$= M_C \begin{bmatrix} \cosh\beta \\ 0 \\ 0 \\ \sinh\beta \end{bmatrix}, \quad \beta \geq 0 . \tag{2.4}$$

That is, in frame B where $\vec{p}_B = \vec{0}$,

$$s_A = (p_B + p_C)^2$$
$$= s_B + M_C^2 + 2\sqrt{s_B}\, M_C \cosh\beta . \tag{2.5}$$

Since s_B is fixed, Eq. (2.5) is a change of variable from s_A to β. The amplitude $F^{A\to B+C}(s_A, s_B)$ then becomes

$$f_{s_B}^{A\to B+C}(\beta) \equiv F^{A\to B+C}(s_A, s_B), \tag{2.6}$$

where

$$0 \leq \beta \leq \infty, \quad M_B + M_C, M_A \leq \sqrt{s_A} \leq \infty,$$

so that it is necessary to investigate the behavior of β as $p_\mu^C \to -p_\mu^{\overline{C}}$. From Eq. (2.4) it is clear that sending the four-vector momentum of C into its antiparticle $\overline{C}$ forces $\beta \to \beta + i\pi$; that is, $p_\mu^C \to -p_\mu^C$ implies $\beta \to \beta + i\pi$. Then

$$s_A = s_B + M_C^2 - 2\sqrt{s_B}\, M_C \cosh\beta , \tag{2.7}$$

implying

$$\sqrt{s_A} \leq \sqrt{s_B} - M_C ,$$

which is the correct inequality for the crossed channel. Since s_A is a subenergy in the $A+\overline{C} \to B$ reaction, bounded by the fixed value of s_B, it also satisfies the inequality $\sqrt{s_A} \geq M_A$, which means

$$M_A^2 \leq s_B + M_C^2 - 2\sqrt{s_B}\, M_C \cosh\beta , \tag{2.8}$$

where

$$\cosh\beta \leq \frac{s_B + M_C^2 - M_A^2}{2\sqrt{s_B}\, M_C}, \quad \sqrt{s_B} \geq M_A + M_C .$$

Thus, when β is real, $f_{s_B}(\beta)$ is the amplitude for the $A \to B+C$ reaction [$f_{s_B}(\beta)$ is even in β, as can be seen from Eq. (2.5)], while when β moves to the upper boundary of the strip, $f_{s_B}(\beta + i\pi)$ is the amplitude for the crossed reaction. So for any multiparticle amplitude, crossing one particle requires knowledge of the analytic properties of that amplitude in the total energy variable only, or, equivalently, requires knowledge of the analytic properties in the boost variable β in the strip between 0 and $i\pi$. If one wishes to cross more than one particle, one proceeds, as noted in Ref. 1, in a stepwise fashion, always choosing canonical variables relative to the particle being crossed, which then determines how a boost variable β is related to the relevant total energy. (Transformation from one canonical set to another involves the Racah coefficients of the Poincaré group.[5])

It is now necessary to ask how one might make sense of $f_{s_B}(\beta + i\pi)$, when f_{s_B} is originally defined only for β on the real line. To try and answer this question the next section introduces a simplification, namely, all four-momenta are reduced to two-momenta, which is equivalent to dealing with multiparticle amplitudes in only two space-time dimensions.

III. SIMPLE FEYNMAN DIAGRAMS IN TWO-DIMENSIONAL SPACE-TIME

It is possible to analyze multiparticle scattering amplitudes with respect to crossing in two-dimensional space-time because, as discussed in Sec. II, all canonical variables except s_A and s_B remain in their physical regions (but with a possibly new physical meaning). That the angles discussed in Sec. II stay in their physical regions under crossing for simple Feynman diagrams is shown in Ref. 4. In dropping two spatial dimensions, every angle is replaced by a sgn, indicating whether a particle has a positive or a negative momentum Thus, the analog of angle dependence for amplitudes in four-dimensional space-time is a sgn dependence for amplitudes in two-dimensional space-time (group theoretically this corresponds to the full three-dimensional rotation group being restricted to the "one-dimensional" rotation group, which means only parity transformations). But the important point in this reduction is that the energy-subenergy dependence remains and can be explored more simply. Also, since the underlying invariance group is now the two-dimensional Poincaré group, with the O(1, 3) Lorentz transformations restricted to O(1, 1) transformations, the Racah coefficient structure is also much simpler; this is convenient when one is discussing the crossing of a sequence of particles (rather than just one as in this paper).

The goal of this section is to examine the simplest production Feynman diagrams in order to see what their analyticity properties are in the β variable between 0 and $i\pi$. The simplest diagrams are given in Fig. 1. Each particle can be thought of as being boosted from its rest frame to the frame $\vec{p}_B=\vec{0}$, where it has its appropriate energy and momentum:

$$E_i = M_i \cosh\beta_i, \quad \beta_i \geq 0$$
$$p_i = \pm M_i \sinh\beta_i \,. \tag{3.1}$$

The $\pm$ sign fixes the direction of a particle relative to the z axis. Further, each β_i can be written as a function of s_A and s_B. For example, in the B cluster

$$\sqrt{s_B} = M_1 \cosh\beta_1 + M_2 \cosh\beta_2 \,,$$
$$0 = M_1 \sinh\beta_1 + M_2 \sinh\beta_2 \,, \tag{3.2}$$

which for fixed masses M_1, M_2 gives β_1 and β_2 as functions of s_B.

To simplify matters we assume that the particles represented by solid lines have mass M, while those with the wavy lines have mass m. Then two of the diagrams of Fig. 1 have amplitudes

$$F^{A\to B+C}_{(a)+(b)} \sim \frac{1}{t_{1'1}-m^2}\left[\frac{1}{s_{2C}-M^2}+\frac{1}{t_{2'C}-M^2}\right]$$
$$=\frac{1}{2M^2-m^2-2M^2\cosh(\beta_1'-\beta_1)}\left[\frac{1}{m^2+2mM\cosh(\beta_2-\beta)}+\frac{1}{m^2-2mM\cosh(\beta_2'-\beta)}\right]. \tag{3.3}$$

It is clear that the only types of singularities for these simple graphs will be poles and thus the question is where they are located. For the $s_{2C}-M^2$ propagator, we see that

$$\cosh(\beta_2-\beta) = -\frac{m}{2M}\,,$$
$$\beta=\beta_2+i(\theta_0+2\pi n), \quad \cos\theta_0 = -\frac{m}{2M}\,, \tag{3.4}$$

so that there is a pole in the strip between 0 and $i\pi$, located at $\beta=\beta_2(s_B)+i\theta_0$. As shown in Eq. (3.2), β_2 is a function of s_B only and therefore the pole is fixed within the strip. The other two poles can be handled in an analogous manner, except that $\beta_{1'}$ and $\beta_{2'}$ are also functions of s_A and therefore of β. The location of these poles is given implicitly by

$$\cosh(\beta_{1'}-\beta_1) = 1-\frac{m^2}{2M^2}\,,$$
$$\beta_{1'}=\beta_1+i\cos^{-1}\left(1-\frac{m^2}{2M^2}\right) \tag{3.5}$$

and

$$\cosh(\beta_{2'}-\beta)=\frac{m}{2M}\,,$$
$$\beta_{2'}-\beta=i\cos^{-1}\left(\frac{m}{2M}\right). \tag{3.6}$$

To locate the poles explicitly it is necessary to determine the dependence of $\beta_{1'}$ and $\beta_{2'}$ on s_A and hence on β. But as far as crossing is concerned, only analyticity in the strip is of interest and it is clear that poles will occur in the strip because it is possible to add factors of $2\pi n$ to the angles $\cos^{-1}(1-m^2/2M^2)$ and $\cos^{-1}(m/2M)$, which means that there is some β in the strip satisfying Eqs. (3.5) and (3.6). There will in general also be graphs of the kind given in Figs. 1(c) and 1(d), which again generate pole singularities; in particular, Fig. 1(c) has a pole of the form $(s_A - m^2)^{-1}$ which locates the pole as a function of β at $\cosh^{-1}(\sqrt{s_B}/2m) + i\pi$. Now if one knows explicitly where the poles are, it is always possible to continue analytically $f_{s_B}(\beta)$ to $f_{s_B}(\beta+i\pi)$. The goal of the next section is to find a way of computing $f(\beta+i\pi)$ via an integral representation which expresses the content of analytic continuation.

To conclude this section we briefly describe what happens when cuts also occur in the strip That cuts will occur is clear from the fact that poles occurred in the simplest Feynman production diagrams of Fig. 1. It is not so useful to analyze two-dimensional Feynman diagrams that generate cuts, as it is not clear how one generally represents their discontinuities. This will be seen in the next section. But there will be branch points whenever new channels open up, which, in the $A \to B + C$ reaction, occurs at $\sqrt{s_A} = km$ ($k = 2, 3, 4, \ldots$); in the β variable this translates to

$$s_B + M_C^2 + 2\sqrt{s_B}\, M_C \cosh\beta_{\mathrm{BP}} = (km)^2 . \qquad (3.7)$$

Because $\sqrt{s_A} \geq \sqrt{s_B} + M_C$ this means the branch points β_{BP} will occur along the real β axis with branch cuts chosen to go to $+\infty$. Similarly, the amplitude for the $A + \bar{C} \to B$ reaction will indicate the existence of new channels by branch points in the *subenergy* s_A. These branch points, when translated to the β variable, are located along the $i\pi$ line, with cuts going to ∞. Finally, branch points not in the physical s_A-s_B region will become branch points in the interior of the strip.

In the next section a general representation for functions meromorphic in the strip will be given. Such a representation seems difficult to generalize to functions with cuts in the strip, so it would be highly desirable to map away the cuts conformally. Such a conformal map should leave the real β axis invariant, so that it is still possible to Fourier transform the conformally mapped function. It is at this point that we make use of the fact that the physical region of the $A + \bar{C} \to B$ reaction extends only between

$$0 + i\pi \leq \beta + i\pi \leq \cosh^{-1}[(s_B + M_C^2 - M_A^2)/2\sqrt{s_B}\, M_C] + i\pi ,$$

because s_A is a subenergy in the crossed reaction bounded by the fixed s_B. The conformal map should move the cuts onto the $i\pi$ boundary while also leaving the physical subenergy region on the $i\pi$ boundary. If such a conformal map could be found, it would have the effect of moving all the cuts to the $i\pi$ line, leaving the interior of the strip with only pole singularities.

We assume here that all cuts go in a straight line in the strip from the branch points to $+\infty$ (possibly overlapping other branch points). This means the region in the strip is simply connected and there exists a Schwarz-Christoffel transformation[6] carrying the strip to the upper half plane. The boundary of the strip plus the cuts get mapped

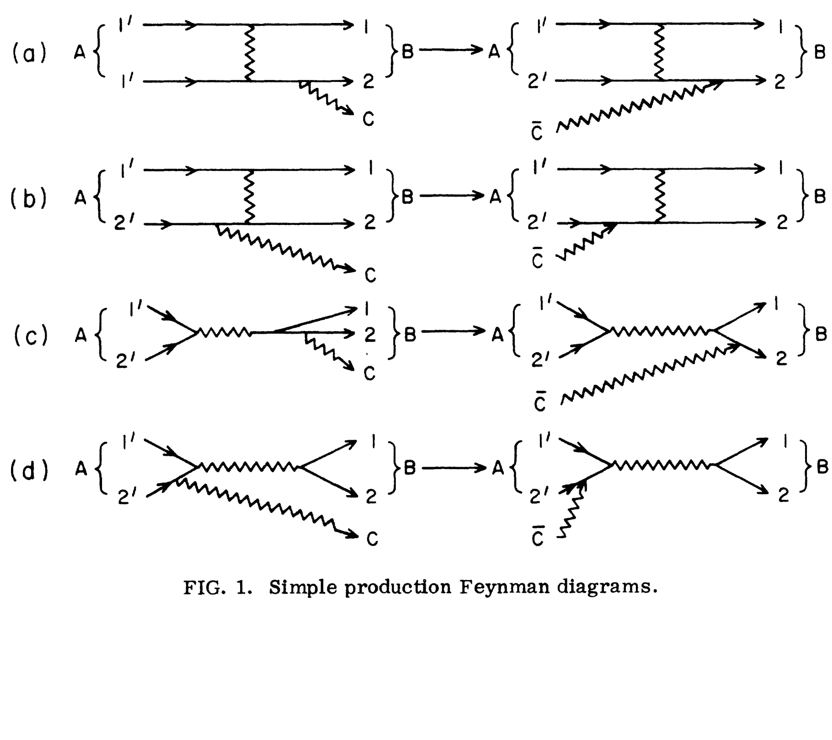

FIG. 1. Simple production Feynman diagrams.

onto the real-line boundary of the upper half plane. By translating the half plane to the left or the right it is possible to adjust the points on the real line, so that the conformal map $w=-e^{-z}$ carries the upper half plane to the strip with all the cuts lying on the $i\pi$ line. Thus, in general it should be possible to map all the cuts conformally to the upper boundary of the strip.

As an example of this procedure consider a cut starting at $i\pi/2$ and going to $+\infty$. Then a conformal map which transforms the cut to the upper boundary of the strip is given by the sequence of maps illustrated in Fig. 2, and given functionally by

$$\begin{aligned} w &\equiv u+iv \\ &= w_0 - \tfrac{1}{2}\ln[z_0+1-e^{-(z+z_1)}] \\ &\quad - \tfrac{1}{2}\ln[z_0-1-e^{-(z+z_1)}], \\ z &= x+iy\,. \end{aligned} \tag{3.8}$$

The pertinent map is the Schwarz-Christoffel map from region (2) to region (3) in Fig. 2, given in the appendix of Ref. 6, p. 210. The map from (1) to (2) simply allows for the possibility that the branch point does not start at $\mathrm{Re}\beta=0$. The map from (3) to (4) is again a translation which brings the point B to the origin so that upon going to (5) the segment AB gets mapped onto the whole real axis. Finally, (6) is again a translation which puts H at $i\pi$. The two wavy lines indicate the physical regions for the direct and crossed reactions. Thus, it is clear that if the locations of the branch points are known it is possible to map away the cuts onto the line $i\pi$ of the strip. Implicit in this discussion, of course, is that there is some path of analytic continuation to the $i\pi$ boundary. We defer to a future paper the question of what sorts of cuts are generated by production Feynman diagrams or from general S-matrix considerations using the unitarity discontinuity equations. At this point we simply want to make use of the fact that if there are cuts in the strip, they can be mapped away conformally.

IV. A REPRESENTATION FOR MULTIPARTICLE AMPLITUDES IN THE STRIP

In this section we assume that only pole or cut singularities of the type discussed in Sec. III occur in the complex β strip, so that analytic continuation is possible. The question to be raised here is how the function $f_{s_B}(\beta+i\pi)$ can be computed from a known $f_{s_B}(\beta)$. Clearly, for a given $f_{s_B}(\beta)$ whose singularities in the strip are known, analytic continuation will uniquely define $f_{s_B}(\beta+i\pi)$. What we wish to do is find an integral representation for $f_{s_B}(\beta+i\pi)$ which expresses the content of analytic continuation. To obtain a representation only the weak assumption that $f_{s_B}(\beta)$ have a Fourier transform is made (including Fourier transforms in the distribution sense[7]). For such a class of functions singularities can occur everywhere in the strip and there is in general no one path of analytic continuation that can be used to compute $f_{s_B}(\beta+i\pi)$.

The integral representation for $f_{s_B}(\beta+i\pi)$ is obtained in two steps. First $f_{s_B}(\beta)$ is assumed to be *analytic* in the strip, in which case the Paley-Wiener theorem[2] guarantees that if

$$\hat{f}_{s_B}(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}d\beta\, e^{-ik\beta}f_{s_B}(\beta) \tag{4.1}$$

then

$$f_{s_B}(\beta+i\pi)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}dk\, e^{-k(\beta+i\pi)}\hat{f}(k)\,. \tag{4.2}$$

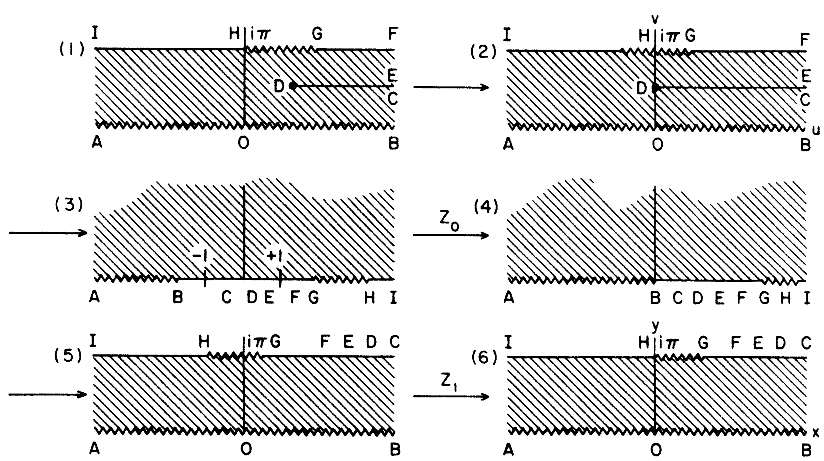

FIG. 2. Sequence of conformal maps moving the cut CDE to the $i\pi$ boundary.

It is clear that the Paley-Wiener theorem allows one to compute explicitly the analytic continuation of $f_{s_B}(\beta)$ to $f_{s_B}(\beta+i\pi)$ when f_{s_B} is analytic in the strip. To generalize let $f_{s_B}(\beta)$ be meromorphic in the strip. Then $f_{s_B}(z)-\sum_j \Gamma_j/(z-z_j)$ is analytic in the strip (Γ_j is the residue of the jth pole located at z_j), which means that the Paley-Wiener theorem can again be used:

$$\begin{aligned} f_{s_B}(z)-\sum_j \frac{\Gamma_j}{z-z_j} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dk\, e^{ikz}\left[\hat{f}_{s_B}(k)-\sum_j \Gamma_j \hat{f}_j(k)\right], \\ f_{s_B}(\beta+i\pi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} dk \left[2\cosh k(\beta+i\pi)\hat{f}_{s_B}(k)-e^{ik(\beta+i\pi)}\sum_j \Gamma_j \hat{f}_j(k)\right]+\sum_j \frac{\Gamma_j}{\beta+i\pi-z_j}\,. \end{aligned} \tag{4.3}$$

It is noted that the upper limit of integration of the Fourier transform is 0 rather than $+\infty$; use has been made of the fact that $\hat{f}_{s_B}(k)$ is even, while $\hat{f}_j(k)$, the Fourier transform of the jth pole, vanishes for $k>0$. Now the two integrands in (4.3) cannot be split since they are both singular at $-\infty$. Also, the pole integrals evaluated at 0 just cancel the $\sum_j \Gamma_j/(\beta+i\pi-z_j)$ terms. Thus, it is possible to write

$$f_{s_B}(\beta=i\pi)=\lim_{a\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-a}^{0} dk \left[2\cosh k(\beta+i\pi)\hat{f}_{s_B}(k)-\sum_j \Gamma_j \frac{e^{-ia(\beta+i\pi-z_j)}}{\beta+i\pi-z_j}\right], \tag{4.4}$$

where the Paley-Wiener theorem guarantees that the limit as $a\to\infty$ will exist when f_{s_B} is meromorphic in the strip.

To see how this works consider the simple example where

$$f=\frac{1}{\beta^2-z_0}, \quad z_0\in \text{strip}. \tag{4.5}$$

Then

$$\begin{aligned} \hat{f} &= \left(\frac{2\pi}{z_0}\right)^{1/2} e^{i|k|z_0}, \\ f(\beta+i\pi) &= \lim_{a\to\infty}\frac{1}{\sqrt{2\pi}} \\ &\quad\times\int_{-a}^{0}\left[2\cosh k(\beta+i\pi)e^{-ikz_0} \right. \\ &\qquad \left. -\frac{1}{2\sqrt{z_0}}\,\frac{e^{-ia(\beta+i\pi-\sqrt{z_0})}}{\beta+i\pi-\sqrt{z_0}}\right]dk \\ &= \frac{1}{(\beta+i\pi)^2-z_0}\,. \end{aligned} \tag{4.6}$$

More instructive would be computing the Fourier transform of the amplitudes of Eq. (3.3). Here the hyperbolic functions generate an infinite number of poles. Because of the complicated dependence of β_1' on β it has not been possible to work out the Fourier transforms for Figs. 1(a), 1(b), and 1(d). For Fig. 1(c) we have

$$f_{s_B}(\beta)\propto\left[\frac{1}{\cosh\beta+(\sqrt{s_B}/2m)}\right]\left[\frac{1}{\cosh(\beta-\beta_2)+(m/2M)}\right], \tag{4.7}$$

so even here the Fourier transform is complicated; if the pole at $\cosh\beta=-\sqrt{s_B}/2m$ is simply ignored it is possible to compute the Fourier transform of Eq. (4.7), but it has not been possible to evaluate the indefinite integral needed for the limits in Eq. (4.3).

To conclude we turn to cuts. Here the procedure is somewhat different. If the locations of the branch points are known, with the cuts going to $+\infty$, it is possible to map the cuts onto the $i\pi$ line of the z plane of Fig. 2. We have

$$\begin{aligned} f_{s_B}(\beta) &= \tilde{f}_{s_B}(x), \\ \hat{f}_{s_B} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dx\, e^{-ikx}\tilde{f}_{s_B}(x), \end{aligned} \tag{4.8}$$

with the change of variable from β to x given by the conformal map (see Fig. 2). Then

$$\begin{aligned} \tilde{f}_{s_B}(x+i\pi) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dk\, e^{ik(x+i\pi)}\hat{f}_{s_B}(k), \\ f_{s_B}(\beta+i\pi) &= \tilde{f}_{s_B}(x+i\pi), \end{aligned} \tag{4.9}$$

where for simplicity we have assumed that there are no poles in the strip. If there are poles, Eq. (4.3) replaces Eq. (4.9). Thus, by conformally mapping the cuts away, Fourier-analyzing in the cut-free strip, analytically continuing via the Paley-Wiener theorem, and then changing variables back to $\beta+i\pi$, it is possible to give meaning to $f_{s_B}(\beta+i\pi)$ via an integral representation. Whether all possible cuts can be dealt with in this manner requires further analysis into the cut structure of Feynman graphs and the use of general S-matrix considerations.

V. CONCLUSION

We have shown how to choose a set of canonical variables for any multiparticle amplitude with the property that only analytic continuation in the total energy s_A is required to connect with the amplitude for the crossed reaction. And the analytic continuation can be brought to a standard form by changing variables to the boost variable β of the particle being crossed (the form is standard in the sense that it is possible to write the crossed amplitude via an integral representation expressing the content of analytic continuation). This should be useful in two ways. First, in phenomenological applications, where one knows some approximate properties of an amplitude over a limited region, it should be possible to check how this behavior is manifested in the crossed channel; in particular it should be possible to check the influence of cuts by seeing how they "interact" with the direct-channel approximate amplitude in the crossed channel. That is, by providing a means to connect direct-channel and crossed-channel amplitudes without knowing the detailed singularity structure, it should be possible to work out interesting constraints in phenomenological analyses.

For example, in a $2 \rightarrow 4$ reaction, with the production of two π mesons, one of which is to be crossed to the $3 \rightarrow 3$ reaction, one has eight variables. By performing an approximate angular distribution in $\cos\theta_{1'C(B)}$, while holding s_B fixed, it is possible to obtain the angular momentum dependence J_B of the amplitude in the crossed channel. If something is known about the cluster of three outgoing particles, this fixes the behavior of the three incoming particles (in the crossed channel) because of time reversal. But inelastic unitarity[8] fixes the associated $2 \rightarrow 3$ reaction and the inelasticity parameter of the $2 \rightarrow 2$ reaction, so that one has a check on the phenomenological form used for the B cluster of the $2 \rightarrow 4$ reaction. In this way crossing can be used in conjunction with unitarity to provide tight constraints on phenomenological fits of production processes.

In a more theoretical vein one might hope to use the representations of Sec. IV when more channels were open and find a representation which "diagonalized" the simultaneous constraints of inelastic unitarity and crossing in all the channels open up to a certain energy. This should provide a way of truncating infinite sets of amplitudes coupled by unitarity and crossing. Of course, it is not at all clear what other assumptions, such as the high-energy behavior of multiparticle amplitudes, would be needed in order to get a unique solution.

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