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Monopole theories with massless and massive gauge fields

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We investigate magnetic-monopole-type theories, including those where the gauge field acquires a mass. The study is based on a modification of Zwanziger's local Lagrangian formulation of the usual (zero-mass) theory. The quantization is carried out by using Dirac's general method. For the mass-zero case the known results are recovered including the charge-quantization condition. The Hamiltonian and angular momentum for the massive case are derived and discussed. Further, it is shown how Nambu's static phenomenological Hamiltonian can be derived as a special case of the massive theory. Certain difficulties associated with the rotational invariance of such theories are pointed out.

I. INTRODUCTION

In a classic paper[1] Dirac pointed out the connection between the existence of magnetic monopoles in quantum theory and the quantization of electric charge. Besides this interesting feature such theories contain a natural mechanism for T-invariance violation. These attractive features prompted many authors[2] to investigate this theory in depth, both from formal and phenomenological points of view.

Recently, great progress has been made in understanding gauge theories where the gauge field acquires mass by spontaneous breakdown of an underlying symmetry. These developments provide a motivation for investigating theories with both electric and magnetic charges which are modified by the addition of a mass term for the vector field. Further, the addition of a mass term may solve some of the apparent experimental difficulties associated with such theories. Specifically the usual theories seem to predict an extremely large T violation as well as a superstrong coupling of the massless vector field to magnetic charges.[3] The addition of a mass term would be expected to suppress the order of magnitude of the effective coupling strength by a factor of $(M_p/\mu)^2$, where μ is the gauge boson mass and M_p, the proton mass, is a typical low-energy scaling factor. Thus for suitably large μ we may expect to overcome these difficulties.

For the treatment of gauge theories, it is most convenient to proceed from a local Lagrangian. Most previous workers on monopoles have not used this approach. However, Zwanziger[4] has recently given a local Lagrangian formulation of the problem. In the present paper we will start from Zwanziger's Lagrangian. First, for the $\mu=0$ case we will carry out the canonical quantization of the Lagrangian by a different method from that of Zwanziger. Since the Lagrangian of interest describes the vector field by two potentials, there are more than the usual number of redundant variables. To handle these we shall adopt a systematic method due to Dirac,[5] in which, among other things, the Poisson bracket in the classical theory is replaced by a new object called the Dirac bracket. Dirac's method is very powerful and has been used in general relativity, but does not seem to be widely appreciated in particle physics. Thus our treatment may also be useful as a nontrivial example of the method. (In particular, this is one of the few nontrivial examples containing the so-called second-class constraints). As a result of our treatment we recover Schwinger's Hamiltonian[6] for the problem and also find that his infinite antisymmetric form for the singularity line (the Dirac string) rather than Dirac's original form[1] is required for consistency. We also apply this formalism to study the angular momentum operator and rederive the charge-quantization condition, by generalizing a method due to Fierz.[7] This material is contained in Secs. II, III, and IV. Section II treats the noninteracting vector field, Sec. III treats the interacting case, and Sec. IV treats the angular momentum.

In Sec. V the quantization of the $\mu\neq 0$ system with interactions is carried out. The Hamiltonian and angular momentum operators are derived.

In Sec. VI we show that the static limit of the $\mu\neq 0$ system reproduces a phenomenological Hamiltonian recently proposed by Nambu[8] as a model of quark binding. We also explicitly show the lack of rotational invariance of this theory and suggest possible modifications to overcome this difficulty.

II. TWO-POTENTIAL DESCRIPTION OF FREE ELECTROMAGNETIC FIELD

The well-known modified Maxwell equations for electrodynamics when both electric and magnetic charges are present are

$$\partial_\mu F_{\mu\nu} = j_\nu^{(1)}, \quad \partial_\mu F^d_{\mu\nu} = -ij_\nu^{(2)}, \tag{1}$$

where $j_\nu^{(1)}$ and $j_\nu^{(2)}$ are the conserved electric and magnetic currents, respectively, and $F_{\mu\nu}$ is the electromagnetic field tensor, with

$$F^d_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}. \tag{2}$$

We are using the metric $g_{\mu\nu} = \delta_{\mu\nu}$. Zwangiger[4] has shown that these equations can be derived from the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}n_\alpha(\partial\wedge q^a)_{\alpha\mu}n_\beta(\partial\wedge q^a)_{\beta\mu} - (i/2)\epsilon_{ab}n_\alpha(\partial\wedge q^a)_{\alpha\mu}n_\beta(\partial\wedge q^b)^d_{\beta\mu} - j_\mu^{(a)}q_\mu^a, \tag{3}$$

with the identification

$$F_{\mu\nu} = n_\mu n_\alpha(\partial\wedge q^1)_{\alpha\nu} - n_\nu n_\alpha(\partial\wedge q^1)_{\alpha\mu} - i\epsilon_{\mu\nu\alpha\beta}n_\alpha n_\gamma(\partial\wedge q^2)_{\gamma\beta}. \tag{4}$$

Here the index a runs over 1 and 2, $\epsilon_{ab} = -\epsilon_{ba}$, with $\epsilon_{12} = +1$, while n_α is a fixed spacelike vector satisfying $n^2 = +1$. Also $(\partial\wedge q^b)^d$ denotes the dual of $(\partial\wedge q^b)$ as in (2) while the wedge product $(A\wedge B)$ is defined by $(A\wedge B)_{\mu\nu} = A_\mu B_\nu - A_\nu B_\mu$. Note that in this formulation the usual electromagnetic 4-potential has been replaced by two potentials q_μ^1 and q_μ^2.

Since the physical electromagnetic field has only two independent degrees of freedom while the two-potential formalism introduces eight degrees of freedom, it is clear that the system is very highly constrained. Therefore the problem of quantization is nontrivial. Zwanziger[4] has solved this problem by adding a gauge-dependent term to the Lagrangian. Here we shall carry out the quantization using Dirac's method,[5] which leads to a result analogous to the Coulomb gauge formulation of quantum electrodynamics. This method has the advantage of being systematic and generally applicable. Modifications of (3) can be handled in a routine way. Furthermore, the present Lagrangian leads, in Dirac's language, to second-class constraints and consequently is one of the few situations where the full power of his method is called into play.

First let us consider the free-field case, for which the last term of (3) is absent. The resulting Lagrangian $\mathcal{L}_\gamma$, as expected, turns out to be a complicated way of describing the noninteracting electromagnetic field. To proceed with Dirac's method for $\mathcal{L}_\gamma$ we first calculate the momenta π_μ^a canonically conjugate to q_μ^a. For definiteness we take $n_\mu = \delta_{\mu 3}$ in what follows. Instead of defining $\pi_\mu^a = \partial\mathcal{L}_\gamma/\partial\dot{q}_\mu^a$ we form the combinations

$$Z_1^a = \pi_1^a + \tfrac{1}{2}\epsilon_{ab}(\vec{\nabla}\times\vec{q}^b)_1, \tag{5a}$$

$$Z_2^a = \pi_2^a + \tfrac{1}{2}\epsilon_{ab}(\vec{\nabla}\times\vec{q}^b)_2, \tag{5b}$$

$$Z_4^a = \pi_4^a, \tag{5c}$$

$$\pi_3^a = \tfrac{1}{2}\epsilon_{ab}(\vec{\nabla}\times\vec{q}^b)_3 - i(\partial\wedge q^a)_{34}. \tag{5d}$$

The quantities π_μ^a and q_μ^a are here considered to obey the usual Poisson bracket (PB) relations

$$[q_\mu^a(\vec{x},t), \pi_\nu^b(\vec{y},t)]_{\text{PB}} = \delta_{ab}\delta_{\mu\nu}\delta^3(x-y) \quad (\text{others}=0), \tag{6}$$

as if no constraints were present. Note that the quantities π_μ^a would be given by $\partial\mathcal{L}_\gamma/\partial\dot{q}_\mu^a$ if we set

$$Z_1^a = Z_2^a = Z_4^a = 0. \tag{7}$$

Note the distinction between (5d) and (7). Thus (5d) defines *dynamical* momenta since it involves time derivatives of the fields q_μ^a. Equations (5a), (5b), and (5c), on the other hand, involve no time derivatives and imply that certain functions of the π's and q's (but not involving the $\dot{q}$'s) vanish, as expressed in (7). Equation (7) is thus a set of constraint equations called the *primary* constraints. In Dirac's method, constraint equations should be imposed only *after* evaluating all Poisson brackets. The Hamiltonian density gets modified to

$$\mathcal{H}_\gamma = -\mathcal{L}_\gamma + \pi_\mu^a\dot{q}_\mu^a + u_\alpha^a Z_\alpha^a, \tag{8}$$

where the π_μ^a are given by (5a)–(5d) with $Z_\alpha^a = 0$, and the u_α^a are six functions subject to certain (uninteresting) consistency requirements. Now to guarantee that the constraints (7) hold for all time, their time derivatives or Poisson brackets with $H_\gamma = \int d^3x\,\mathcal{H}_\gamma$ must vanish. In addition to determining some of the u's, this leads to some *secondary* constraints. Here, requiring $[H_\gamma, Z_4^a]_{\text{PB}} = 0$ gives the secondary constraints

$$Z_5^a = \vec{\nabla}\cdot\vec{\pi}^a = 0, \tag{9}$$

which again should only be considered to vanish in a Poisson bracket relation after the bracket has been evaluated. Continuing the process leads to no further constraints. Dirac points out that the division of the constraints into primary and secondary is not very important. The more meaningful division is into first-class and second-class constraints, the first-class ones being the constraints which have vanishing Poisson brackets with all constraints on the constraint surface.[9] The remaining ones are then said to be second class. First-class constraints can be thought of as generators of gauge transformations while the second-class constraints can be effectively eliminated from the theory by modifying the Poisson bracket to a new object called the Dirac bracket. In our case the first-class constraints are easily seen to be Z_4^a and Z_5^a. The others have the nonvanishing Poisson brackets:

$$\Delta_{ai,bj}(x-y)=[Z_i^a(\vec{x},t),Z_j^b(\vec{y},t)]_{PB}$$
$$=-\epsilon_{ab}\epsilon_{ij3}\frac{\partial}{\partial x_3}\delta^3(x-y),$$
$$\{i,j\}=\{1,2\}\,. \quad (10)$$

We need the inverse of (10), satisfying

$$\int d^3y\,\Delta_{ai,bj}(x-y)\Delta_{bj,ck}{}^{-1}(y-z)=\delta_{ac}\delta_{ik}\delta^3(x-z)\,.$$

The inverse is thus

$$\Delta_{ai,bj}{}^{-1}(x-y)=-\epsilon_{ab}\epsilon_{ij3}\left(\frac{\partial}{\partial x_3}\right)^{-1}\delta^3(x-y), \quad (11)$$

where the integral operator $(\partial/\partial x_3)^{-1}$ requires a boundary condition for its complete specification which we will discuss below.

Using (11), the Dirac bracket of 2 dynamical quantities A and B, $[A,B]^*$, is defined to be

$$[A,B]^*=[A,B]_{PB}$$
$$-\int d^3x\int d^3y[A,Z_i^a(x)]_{PB}$$
$$\times\,\Delta_{ai,bj}{}^{-1}(x-y)[Z_j^b(y),B]_{PB}\,. \quad (12)$$

In (12) the indices i and j take on the values 1 and 2. The equations of motion take the same form (on the constraint surface) whether we use Poisson or Dirac brackets of the Hamiltonian with the dynamical variables. In this formalism if we use Dirac brackets to write the equation of motion, the second-class constraints may be set equal to zero identically, since they have vanishing Dirac brackets with all variables (on the constraint surface). Straightforward computation with (12) gives the fundamental Dirac brackets for this theory:

$$[q_\mu^a(\vec{x},t),q_\nu^b(\vec{y},t)]^*=-\epsilon_{ab}\epsilon_{\mu\nu34}\left(\frac{\partial}{\partial x_3}\right)^{-1}\delta^3(x-y), \quad (13a)$$

$$[\pi_\mu^a(\vec{x},t),\pi_\nu^b(\vec{y},t)]^*=\tfrac{1}{4}\epsilon_{ab}\epsilon_{\mu\nu\rho4}\frac{\partial}{\partial x_\rho}\delta^3(x-y), \quad (13b)$$

$$[\pi_\mu^a(\vec{x},t),q_\nu^b(\vec{y},t)]^*=-\tfrac{1}{2}\delta_{ab}\left(\delta_{\mu\nu}+\delta_{\mu3}\frac{\partial}{\partial x_\nu}\left(\frac{\partial}{\partial x_3}\right)^{-1}\right)$$
$$\times\,\delta^3(x-y)\quad(\nu=1,2), \quad (13c)$$

$$[\pi_\mu^a(\vec{x},t),q_\nu^b(\vec{y},t)]^*=-\delta_{ab}\delta_{\mu\nu}\delta^3(x-y),\quad(\nu=3,4)\,. \quad (13d)$$

Now, the passage to quantum theory is made by replacing the fundamental Dirac brackets of Eqs. (13a) to (13d) by $(-i)$ times the commutators and by interpreting the first-class constraints as the following supplementary conditions on the allowed states $|\,\rangle$ of the theory:

$$\pi_4^a|\,\rangle=0, \quad (14a)$$

$$\vec{\nabla}\cdot\vec{\pi}^a|\,\rangle=0\,. \quad (14b)$$

The second-class constraints are to be set equal to zero as operator identities. The "observables" of the theory are those operator functions which commute with the four first-class constraints.

Since the quantization requires the replacement of the Dirac bracket $[A,B]^*$ by a commutator $[A,B]$ which is antisymmetric in A and B, we should require the Dirac bracket itself to be antisymmetric in A and B. The antisymmetry property in turn requires $\Delta_{ai,bj}{}^{-1}(x-y)$ to be antisymmetric in x and y. This leads to the following determination of $\partial_3{}^{-1}\equiv(\partial/\partial x_3)^{-1}$:

$$(\partial_3{}^{-1}f)(x_1,x_2,x_3)=\tfrac{1}{2}\int dx_3'\epsilon(x_3-x_3')f(x_1,x_2,x_3'), \quad (15)$$

where $\epsilon(x)=+1$ for $x>0$ and -1 for $x<0$. This is the determination of $\partial_3{}^{-1}$ adopted by Schwinger,[6] and is different from the one originally proposed by Dirac.[1] It is interesting that we are forced to adopt Schwinger's prescription for consistency.

To show the equivalence between the present theory and the theory of the free electromagnetic field, consider the Hamiltonian acting on states $|\,\rangle$ satisfying (14a) and (14b). We find, using the first- and second-class constraints,

$$H_\gamma|\,\rangle=\tfrac{1}{2}\int d^3x\left\{\sum_{i=1}^{2}(\vec{\nabla}\times\vec{q}^a)_i(\vec{\nabla}\times\vec{q}^a)_i+[\pi_3^a-\tfrac{1}{2}\epsilon_{ab}(\vec{\nabla}\times\vec{q}^b)_3]^2\right\}|\,\rangle\,.$$

Making use of the identity

$$(\vec{\nabla}\times\vec{q}^a)_3=-\partial_3{}^{-1}\sum_{i=1}^{2}\partial_i(\vec{\nabla}\times\vec{q}^a)_i$$

and the second-class constraints, we find

$$(\vec{\nabla}\times\vec{q}^a)_3=-2\epsilon_{ab}\partial_3{}^{-1}\sum_{i=1}^{2}\partial_i\pi_i^b\,. \quad (16)$$

If we define

$$P_i^a=2\pi_i^a-\delta_{i3}\partial_3{}^{-1}\vec{\nabla}\cdot\vec{\pi}^a \quad (17)$$

and use (7) and (16), we get simply

$$H_\gamma|\,\rangle=\tfrac{1}{2}\int d^3x\,\vec{P}^a\cdot\vec{P}^a|\,\rangle\,. \quad (18)$$

Equations (13) imply that the quantities P_i^a obey, in the quantized theory, the equal-time *commuta-*

tion relations:

$$[P_i^a(\vec{x}, t), P_j^b(\vec{y}, t)] = i\epsilon_{ab}\epsilon_{ijk} \frac{\partial}{\partial x_k} \delta^3(x-y). \quad (19)$$

These are the commutation relations of the electric and magnetic fields $\vec{E}$ and $\vec{H}$ when we identify either

$$\vec{E} = \vec{P}^2, \ \vec{H} = -\vec{P}^1 \ \text{or} \ \vec{E} = -\vec{P}^1, \ \vec{H} = \vec{P}^2.$$

Furthermore the definition (17) implies that

$$\vec{\nabla}\cdot\vec{P}^a = \vec{\nabla}\cdot\vec{\pi}^a, \quad (20)$$

which, being a first-class constraint, vanishes on the allowed states. Thus only the tranverse parts of $\vec{P}^a$ contribute to (18), which therefore describes the free electromagnetic field.

With the canonical Fourier decomposition for the transverse parts of $\vec{P}^a$ one can show that the commutation relations implied by (13) for all the field components can be fulfilled by introducing some additional operators.[10]

III. INTERACTING ELECTRICALLY AND MAGNETICALLY CHARGED PARTICLES

For definiteness we consider a system of spin-$\frac{1}{2}$ fields ψ_n carrying both electric and magnetic charges. The conserved currents in (1) take the form

$$j_\mu^{(a)} = -ie_n^a \bar{\psi}_n \gamma_\mu \psi_n, \quad (21)$$

where e_n^1 and e_n^2 are respectively the electric and magnetic charges of ψ_n. The total Lagrangian is thus

$$\mathcal{L} = \mathcal{L}_\gamma - j_\mu^{(a)} q_\mu^a - \bar{\psi}_n(\gamma_\mu \partial_\mu + m_n)\psi_n, \quad (22)$$

where m_n is the mass associated with ψ_n. For quantizing (22) we will treat the spin-$\frac{1}{2}$ field in the usual way. We can also proceed by using Dirac's method, but we shall not give details since it leads to nothing new. The quantization of the boson fields is almost exactly the same as in the last section. The canonical momenta are still given by Eqs. (5) and (7). The primary constraints (7) are thus unchanged. The only change is in the secondary constraint (9). In this case, requiring the Poisson bracket of the total Hamiltonian H and Z_4^a to vanish gives the secondary *first-class* constraint

$$Z_5^a = \vec{\nabla}\cdot\vec{\pi}^a + ij_4^{(a)} = 0. \quad (23)$$

The second term clearly arises from the interaction term in the total Hamiltonian. Since the second-class constraints are the same as in the preceding section the fundamental Dirac bracket relations (13) still hold.

Proceeding as before, the total Hamiltonian of the system on allowed states takes the form

$$H|\rangle = \int d^3x[\tfrac{1}{2}\vec{P}^a\cdot\vec{P}^a + \bar{\psi}_n(\vec{\gamma}\cdot\vec{\nabla} + m_n)\psi_n + \vec{j}^{(a)}\cdot\vec{q}^a]\,|\rangle,$$

where the second-class and the new first-class constraints have been freely used. Note that $\vec{P}^a$ is still given in terms of the $\vec{\pi}^a$ by (17) so that (20) is still valid. The Fermi field ψ_n is seen not to commute with the constraint Z_5^a given in (23). This means that ψ_n acting on allowed states may create unphysical states. This is just another way of saying that ψ_n is not gauge-invariant in the ordinary sense, since Z_5^a is a generator of gauge transformations. It is convenient to eliminate ψ_n in favor of the gauge-invariant field

$$\psi_n' = \exp\left(-ie_n^a \frac{1}{\nabla^2} \vec{\nabla}\cdot\vec{q}^a\right)\psi_n \qquad \text{(no sum on } n). \quad (24)$$

The Hamiltonian then becomes

$$H|\rangle = \int d^3x[\tfrac{1}{2}\vec{P}^a\cdot\vec{P}^a + \bar{\psi}_n'(\vec{\gamma}\cdot\vec{\nabla} + m_n)\psi_n' + \vec{j}^a\cdot\vec{q}^{aT}]\,|\rangle, \quad (25)$$

where $\vec{j}^a = -ie_n^a\bar{\psi}_n\vec{\gamma}\psi_n = -ie_n^a\bar{\psi}_n'\vec{\gamma}\psi_n'$ and $\vec{q}^{aT} = \vec{q}^a - (\vec{\nabla}/\nabla^2)\vec{\nabla}\cdot\vec{q}^a$ is the transverse part of $\vec{q}^a$. A more physically transparent form of (25) is obtained by separating the field strengths $\vec{P}^a$ into their transverse and longitudinal parts $\vec{P}^{aT}$ and $\vec{P}^{aL} = (\vec{\nabla}/\nabla^2)\vec{\nabla}\cdot\vec{P}^a$. Using (23) we find

$$H|\rangle = \int d^3x\Big[\tfrac{1}{2}\vec{P}^{aT}\cdot\vec{P}^{aT} + \bar{\psi}_n'(\vec{\gamma}\cdot\vec{\nabla} + m_n)\psi_n' + \vec{j}^a\cdot\vec{q}^{aT} + \tfrac{1}{2}j_4^{(a)}\frac{1}{\nabla^2}j_4^{(a)}\Big]\,|\rangle. \quad (26)$$

This is the Hamiltonian of Schwinger.[6] The last term in (26) represents the generalization of the static Coulomb interaction to the case where particles have both electric and magnetic charges.

Note that the constraint equations (5a), (5b), and (7) imply that

$$(\vec{\nabla}\times\vec{q}^{aT})_i = (\vec{\nabla}\times\vec{q}^a)_i = \epsilon_{ab}(P_i^b - \delta_{i3}\partial_3^{-1}\vec{\nabla}\cdot\vec{P}^b), \quad (27)$$

where the equation for $i=3$ is inferred from the identity $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{q}^{aT}) = 0$. Since $\vec{\nabla}\cdot\vec{q}^{aT} = 0$ we can solve (27) for $\vec{q}^{aT}$ to get

$$q_i^{aT} = -\epsilon_{ab}\epsilon_{ijk}\frac{1}{\nabla^2}\partial_j(P_k^b - \delta_{k3}\partial_3^{-1}\vec{\nabla}\cdot\vec{P}^b), \quad (28)$$

which agrees with previous work.[6] Perturbation theory for the Hamiltonian (26) has been discussed by Rabl.[6]

IV. ANGULAR MOMENTUM AND CHARGE QUANTIZATION

The most striking feature of theories containing both electrically and magnetically charged particles is the existence of a charge-quantization condition. The need for such a condition has been discussed by several authors in different ways. Here we would like to generalize the argument of Fierz,[7] which was made in a first-quantized framework, to the present field-theoretic framework. Zwanziger[7] has also treated this problem with a different method from the present one.

Fierz noticed that the Coulomb-type fields resulting from a classical point electric charge e^1 separated from a classical point magnetic charge e^2 give a nonvanishing net contribution $\int d^3x\,\vec{r}\times(\vec{E}\times\vec{H})=(e^1e^2/4\pi)(\vec{r}/r)$ to the angular momentum of the system. It is noteworthy that this contribution does not vanish even when e^1 and e^2 are infinitely separated from each other. In the passage to quantum theory the quantization of angular momentum then leads directly to the quantization of $e^1e^2/4\pi$ as well.

For the field-theoretic case we now consider the canonical expression for the angular momentum

$$J_i=(-i/2)\epsilon_{ijk}\times\int d^3x\{\pi_l^a[-i\delta_{lm}(x_j\partial_k-x_k\partial_j)+(S_{jk})_{lm}]q_m^a+\psi_n^\dagger[(x_j\partial_k-x_k\partial_j)+(i/2)\Sigma_{jk}]\psi_n\}\,, \tag{29}$$

where $(S_{jk})_{lm}=-i(\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl})$ and $\Sigma_{jk}=(1/2i)[\gamma_j,\gamma_k]$. Using (17) and (20) to solve for $\vec{\pi}^a$ in terms of $\vec{P}^a$ and using (27) for $\vec{\nabla}\times\vec{q}^a$ we find

$$J_i=\int d^3x\{-\tfrac{1}{2}\epsilon_{ab}[\vec{x}\times(\vec{P}^a\times\vec{P}^b)]_i+(\vec{\nabla}\cdot\vec{P}^a)(\vec{x}\times\vec{q}^a)_i-i\epsilon_{ijk}\psi_n^\dagger[x_j\partial_k+(i/4)\Sigma_{jk}]\psi_n\}\,. \tag{30}$$

Substituting for ψ_n in terms of ψ_n' from (24) and using (23) we find

$$J_i|\,\rangle=\int d^3x\{-\tfrac{1}{2}\epsilon_{ab}[\vec{x}\times(\vec{P}^a\times\vec{P}^b)]_i-i\epsilon_{ijk}\psi_n'^\dagger\times[x_j(\partial_k-ie_n^aq_k^{aT})+(i/4)\Sigma_{jk}]\psi_n'\}|\,\rangle\,, \tag{31}$$

where $|\,\rangle$ is an allowed state and the longitudinal part of $\vec{q}^a$ has been eliminated. To elucidate the meaning of (31) we will substitute for $\vec{q}^{aT}$ in terms of $\vec{P}^a$ from (28) and further separate $\vec{P}^a$ into its transverse and longitudinal parts. By (23) and (20),

$$\vec{P}^{aL}|\,\rangle=\frac{\vec{\nabla}}{\nabla^2}\vec{\nabla}\cdot\vec{P}^a|\,\rangle=-i\frac{\vec{\nabla}}{\nabla^2}j_4^{(a)}|\,\rangle\,.$$

Then (31) takes the form

$$J_i|\,\rangle=\int d^3x\Big\{-\tfrac{1}{2}\epsilon_{ab}[\vec{x}\times(\vec{P}^{aT}\times\vec{P}^{bT})]_i-i\epsilon_{ijk}\psi_n'^\dagger[x_j\partial_k+(i/4)\Sigma_{jk}]\psi_n'+\epsilon_{ab}\partial_i\frac{1}{\nabla^2}j_4^{(a)}\vec{x}\cdot\vec{\nabla}\frac{1}{\nabla^2}j_4^{(b)}-\epsilon_{ab}j_4^{(a)}(x_3\partial_i-\delta_{i3}\vec{x}\cdot\vec{\nabla})\frac{1}{\nabla^2}\partial_3^{-1}j_4^{(b)}\Big\}|\,\rangle\,. \tag{32}$$

In arriving at (32) we used integration by parts[11] [and (33) below] to find that all terms linear in $\vec{P}^{aT}$ cancel each other. The evaluation of the third term in (32) is analogous to that of Fierz. Introducing

$$\frac{1}{\nabla^2}F(x)=-\frac{1}{4\pi}\int d^3x'\frac{F(x')}{|\vec{x}-\vec{x}'|} \tag{33}$$

and carrying out some integrations results in the following net contribution of the third term in (32) to L_i:

$$-\frac{\epsilon_{ab}}{8\pi}\int d^3x\,d^3y\,j_4^{(a)}(x)j_4^{(b)}(y)\frac{(\vec{x}-\vec{y})_i}{|\vec{x}-\vec{y}|}\,. \tag{34}$$

Next consider the fourth term of (32). From the definition of ∂_3^{-1} given in (15) we get the result

$$\left(\frac{\partial}{\partial x_3}\right)^{-1}\frac{1}{|\vec{x}-\vec{y}|}=\tfrac{1}{2}\ln\frac{|\vec{x}-\vec{y}|+(x_3-y_3)}{|\vec{x}-\vec{y}|-(x_3-y_3)}\,. \tag{35}$$

Using (35) and (33), the fourth term of (32) becomes

$$+\frac{\epsilon_{ab}}{8\pi}\int d^3x\,d^3y\,j_4^{(a)}(x)j_4^{(b)}(y)\frac{\delta_{i3}(x_3-y_3)|\vec{x}-\vec{y}|^2-(x_3-y_3)^2(x_i-y_i)}{|\vec{x}-\vec{y}|[\,|\vec{x}-\vec{y}|^2-(x_3-y_3)^2]}\,. \tag{36}$$

Combining the above terms we get the final expression

$$J_i|\,\rangle=\int d^3x\{-\tfrac{1}{2}\epsilon_{ab}[\vec{x}\times(\vec{P}^{aT}\times\vec{P}^{bT})]_i-i\epsilon_{ijk}\psi_n'^{\dagger}[x_j\partial_k+(i/4)\Sigma_{jk}]\psi_n'\}\,|\,\rangle$$
$$-\frac{1}{8\pi}\epsilon_{ab}\int d^3x\,d^3y\,j_4^{(a)}(x)j_4^{(b)}(y)(1-\delta_{i3})\,\frac{x_i-y_i}{|\vec{x}-\vec{y}|}\,\frac{|\vec{x}-\vec{y}|^2}{|\vec{x}-\vec{y}|^2-(x_3-y_3)^2}\,|\,\rangle\,. \quad (37)$$

The first two terms of (37) are the usual expressions for the angular momenta of electromagnetic and Fermi fields. The last term does not vanish only when both electric and magnetic charges are present. Note that this term also vanishes when the ratio of e_n^1 to e_n^2 is the same for all particles n. We may see this by using the expression for $j_\mu^{(a)}$ given in (21) and the antisymmetry in x and y of the remaining integrand.

To see the need for charge quantization we will consider J_i operating on the allowed state

$$\psi_m'^{\dagger}(z_1)\psi_n'^{\dagger}(z_2)|0\rangle\,, \quad (38)$$

where $|0\rangle$ is the vacuum state. We find

$$J_i\psi_m'^{\dagger}(z_1)\psi_n'^{\dagger}(z_2)|0\rangle=[J_i,\psi_m'^{\dagger}(z_1)\psi_n'^{\dagger}(z_2)]|0\rangle$$
$$=-i\epsilon_{ijk}\left(z_{1j}\frac{\partial}{\partial z_{1k}}+z_{2j}\frac{\partial}{\partial z_{2k}}\right)\psi_m'^{\dagger}(z_1)\psi_n'^{\dagger}(z_2)|0\rangle$$
$$+\tfrac{1}{4}\epsilon_{ijk}\{[\Sigma_{jk}\psi_m'^{\dagger}(z_1)]\psi_n'^{\dagger}(z_2)+\psi_m'^{\dagger}(z_1)[\Sigma_{jk}\psi_n'^{\dagger}(z_2)]\}|0\rangle$$
$$+\frac{1}{4\pi}\epsilon_{ab}e_m^a e_n^b(1-\delta_{i3})\,\frac{z_{1i}-z_{2i}}{|\vec{z}_1-\vec{z}_2|}\,\frac{|\vec{z}_1-\vec{z}_2|^2}{|\vec{z}_1-\vec{z}_2|^2-(z_{13}-z_{23})^2}\,\psi_m'^{\dagger}(z_1)\psi_n'^{\dagger}(z_2)|0\rangle\,, \quad (39)$$

where we have used $j_4^{(a)}|0\rangle=0$, which corresponds to neglecting higher-order pair effects.[12]

On introducing relative and center-of-mass coordinates $\vec{z}=\vec{z}_1-\vec{z}_2$ and $\vec{X}=\vec{z}_1+\vec{z}_2$, the operator representing the ith component of the relative angular momentum on the state (38) is seen to be

$$\mathcal{J}_i=\epsilon_{ijk}\left(-iz_j\frac{\partial}{\partial z_k}+\tfrac{1}{4}\Sigma_{jk}^{(1)}+\tfrac{1}{4}\Sigma_{jk}^{(2)}\right)$$
$$+\frac{\epsilon_{ab}}{4\pi}e_m^a e_n^b(1-\delta_{i3})\,\frac{z_i}{|\vec{z}|}\,\frac{|\vec{z}|^2}{|\vec{z}|^2-(z_3)^2}\,, \quad (40)$$

where $\Sigma^{(i)}$ is the spin matrix for the field labeled by z_i. The preceding expression is precisely the one which has been shown by several authors to require the quantization relation

$$\mu_{mn}\equiv\epsilon_{ab}\left(\frac{e_m^a e_n^b}{4\pi}\right)=\frac{N_{mn}}{2}\,, \quad (41)$$

where for each pair (mn), N_{mn} is a positive or negative integer (including zero). A sketch of a simple proof of (41) follows. Construct the state f_j satisfying

$$\mathcal{J}_3 f_j=jf_j,\quad(\mathcal{J}_1+i\mathcal{J}_2)f_j=0 \quad (42)$$

(where for simplicity we have suppressed spin indices). f_j is the state of highest weight for a given angular momentum j. Since the irreducible representation of the rotation group associated with f_j must be $(2j+1)$-dimensional, it is clear that we need

$$(\mathcal{J}_1-i\mathcal{J}_2)^{2j+1}f_j=0\,. \quad (43)$$

Equation (43) implies (41). For instance, in the spinless case where the Σ's are set equal to zero, condition (43) requires[7]

$$\left(\frac{d}{d\cos\theta}\right)^{2j+1}[(1-\cos\theta)^{j+\mu_{mn}}(1+\cos\theta)^{j-\mu_{mn}}]=0\,, \quad (44)$$

where θ is the angle $\vec{z}$ makes with the third axis. It follows that the expression in square brackets must be a polynomial in $\cos\theta$ of degree $2j$. As j is non-negative and integral or half-integral, (41) must hold for μ_{mn}. Note that j may only take one of the values $|\mu_{mn}|$, $|\mu_{mn}|+1$, $|\mu_{mn}|+2,\ldots$.

V. MASSIVE VECTOR MESON WITH ELECTRIC AND MAGNETIC COUPLINGS

One of the most natural applications of the present type of theory would seem to be to the construction of a model for T violation in weak interactions. If one associates the massless vector field with the usual electromagnetic field, the quantization condition seems, at least naively, to imply a very large magnitude for the T violation in addition to a superstrong photon coupling. Thus it is interesting to investigate the possibility that the vector field is not the electromagnetic field, but a massive field like an intermediate boson. An attractive mechanism for generating a mass term for the vector meson is the Higgs mechanism.[13] For the Lagrangian (22), the simplest procedure is to add a scalar field with only type-2 charge in an appropriate way. If only one scalar particle is added, this choice of charge is essentially general because of the freedom we have to make rotations in the two-dimensional space of electric and magnetic charges.[14] For our present purposes, the degrees of freedom associated with the auxiliary scalar particle are not particularly interesting, and the net result is that we are led to consider a Lagrangian with an additional mass term:

$$\mathcal{L}' = \mathcal{L}_\gamma - \tfrac{1}{2}\mu^2 q_\mu^2 q_\mu^2 - j_\mu^{(a)} q_\mu^a - \bar{\psi}_n(\gamma_\mu \partial_\mu + m_n)\psi_n \; . \quad (45)$$

At the outset it is not obvious that the first two terms of (45) describe a free massive vector meson. However, we will see below that this is indeed the case.

We will now apply Dirac's method to the quantization of $\mathcal{L}'$. Since the mass term does not affect the definition of the canonical momenta, the primary constraints are the same as those of Secs. II and III. On the other hand, the secondary constraints become

$$Z_5^a = \vec{\nabla}\cdot\vec{\pi}^a + i\mu^2\delta_{a2}q_4^2 + ij_4^a \; . \quad (46)$$

The additional term in Z_5^a is due to the fact that Z_4^a does not commute with the term $\frac{1}{2}\mu^2 q_\mu^2 q_\mu^2$ in the Hamiltonian density. It is interesting that the division of the constraints into first and second classes differs somewhat from the $\mu = 0$ case. The first-class constraints are

$$Z_4^1 = \pi_4^1 , \qquad Z_5^1 = \vec{\nabla}\cdot\vec{\pi} + ij_4^{(1)} , \quad (47)$$

while the second-class constraints are

$$Z_i^a = \pi_i^a + \tfrac{1}{2}\epsilon_{ab}(\vec{\nabla}\times\vec{q}^b)_i \quad (i = 1, 2), \qquad Z_4^2 = \pi_4^2 , \qquad Z_5^2 = \vec{\nabla}\cdot\vec{\pi}^2 + i\mu^2 q_4^2 + ij_4^{(2)} \; . \quad (48)$$

As we saw earlier, to compute the Dirac brackets of the dynamical variables we need the matrix $D(x-y)$ of the commutators of the second-class constraints where

$$D_{a\mu,b\nu}(x-y) = [Z_\mu^a(\vec{x}, t), Z_\nu^b(\vec{y}, t)]_{\text{PB}} \; . \quad (49)$$

Here a and μ take on the values indicated in (48). This can be expressed as

$$D(x-y) = \begin{bmatrix} \Delta(x-y) & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \;\; 0 & 0 & -i\mu^2\delta^3(x-y) \\ 0 \;\; 0 & i\mu^2\delta^3(x-y) & 0 \end{bmatrix}, \quad (50)$$

where the 4×4 matrix $\Delta(x-y)$ is given in (10) and the fifth and sixth rows refer respectively to Z_4^2 and Z_5^2. The inverse $D^{-1}(x-y)$ is clearly

$$D^{-1}(x-y) = \begin{bmatrix} \Delta^{-1}(x-y) & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ 0 \;\; 0 & 0 & (-i/\mu^2)\delta^3(x-y) \\ 0 \;\; 0 & (i/\mu^2)\delta^3(x-y) & 0 \end{bmatrix}, \quad (51)$$

where $\Delta^{-1}(x-y)$ is given in (11). The Dirac bracket $[A, B]^*$ is then defined as in (12), but with $D^{-1}(x-y)$ replacing $\Delta^{-1}(x-y)$. Except for

$$[q_i^2(\vec{x}, t), q_4^2(\vec{y}, t)]^* = \frac{-i}{\mu^2}\frac{\partial}{\partial x_i}\delta^3(x-y), \quad i = 1, 2, 3 \quad (52a)$$

$$[\pi_4^2(\vec{x}, t), q_4^2(\vec{y}, t)]^* = 0 , \quad (52b)$$

the fundamental Dirac brackets are the same as in Eqs. (13).

To quantize the theory we replace the Dirac brackets by $(-i)$ times the commutators and consider the first-class constraints (47) as supplementary conditions defining the allowed states. The Fermi fields obey the normal anticommutation relations among themselves. Note, however, that

$$[q_4^2(\vec{x}, t), \psi_n(\vec{y}, t)] = -\frac{i}{\mu^2} e_n^2 \psi_n(\vec{x}, t)\delta^3(x-y) \; . \quad (53)$$

This follows by observing that $[q_4^2, \psi_n]^*$ involves a factor $[Z_5^2, \psi_n]_{\text{PB}}$ which by (48) is nonvanishing when we assume that ψ_n has the usual Poisson bracket with $j_4^{(2)}$ and vanishing Poisson brackets with Bose fields. The field ψ_n commutes with the Bose fields other than q_4^2.

It is convenient, as before, to introduce the modified Fermi field ψ_n' given in (24) which will commute with the first-class constraints (47). The boson variables which commute with the first-class constraints can be taken to be

$$P_i^a = 2\pi_i^a - \delta_{i3}\partial_3^{-1}\vec{\nabla}\cdot\vec{\pi}^a , \quad (54a)$$

$$q_i^{aT} = -\epsilon_{ab}\epsilon_{ijk}\frac{1}{\nabla^2}\partial_j(P_k^b - \delta_{k3}\partial_3{}^{-1}\vec{\nabla}\cdot\vec{P}^b), \qquad (54b)$$

and $\vec{\nabla}\cdot\vec{q}^2$. [The list does not include q_4^2 since it equals $(i/\mu^2)(\vec{\nabla}\cdot\vec{P}^2 + ij_4^{(2)})$ by the vanishing of the second-class constraint Z_5^2]. However, these boson variables do not commute with ψ_n'. For many purposes it is useful to define careted boson variables which do commute with ψ_n' and also the first-class constraints (47). We therefore define

$$\hat{P}_i^a = P_i^a + \frac{i}{\nabla^2}\partial_i j_4^{(a)}, \qquad (55)$$

which by (54b) leads to the definition

$$\hat{q}_i^{aT} = -\epsilon_{ab}\epsilon_{ijk}\frac{1}{\nabla^2}\partial_j(\hat{P}_k^b - \delta_{k3}\partial_3{}^{-1}\partial_l\hat{P}_l^b) \qquad (56a)$$

$$= q_i^{aT} + i\epsilon_{ab}\epsilon_{ij3}\frac{1}{\nabla^2}\partial_j\partial_3{}^{-1}j_4^{(b)}. \qquad (56b)$$

Since $\vec{\nabla}\cdot\vec{q}^2$ commutes with both ψ_n' and the first-class constraints we may set $\hat{q}_i^{2L} = q_i^{2L}$ for the longitudinal part of $\hat{q}_i^2$ and use the unified notation

$$\hat{q}_i^2 = \hat{q}_i^{2T} + \hat{q}_i^{2L}. \qquad (57)$$

The commutation relations among the careted variables themselves are seen to be the same as the commutation relations among the uncareted variables. The commutator $[\hat{P}_i^a, \hat{P}_j^b]$ is given by (19) with the replacement of P's by $\hat{P}$'s. In addition we have the usual commutation relation

$$[\hat{P}_i^a(\vec{x},t), \hat{q}_j^2(\vec{y},t)] = -i\delta_{a2}\delta_{ij}\delta^3(x-y). \qquad (58)$$

The remaining commutators between the careted variables follow from (54b) and (56a).

To understand the structure of the present theory we now examine its Hamiltonian. After the introduction of the careted variables and some manipulations similar to those of the $\mu = 0$ case we get

$$H|\rangle = H_0|\rangle + H_I|\rangle, \qquad (59)$$

where the free-field Hamiltonian H_0 acting on allowed states takes the form

$$H_0|\rangle = \int d^3x\left[\tfrac{1}{2}\hat{P}_i^2\hat{P}_i^2 + \tfrac{1}{2}\epsilon_{ijk}(\partial_j\hat{q}_k^2)\epsilon_{ilm}(\partial_l\hat{q}_m^2) + \frac{\mu^2}{2}\hat{q}_i^2\hat{q}_i^2 + \frac{1}{2\mu^2}(\partial_i\hat{P}_i^2)^2 + \bar{\psi}_n'(\vec{\gamma}\cdot\vec{\nabla} + m_n)\psi_n'\right]|\rangle \qquad (60)$$

and the interaction Hamiltonian is given by

$$H_I|\rangle = \int d^3x\Big[i\partial_i\hat{P}_i^2\frac{1}{\nabla^2}j_4^{(2)} + \tfrac{1}{2}j_4^{(a)}\frac{1}{\nabla^2}j_4^{(a)} - \frac{\mu^2}{2}\left(\epsilon_{ij3}\frac{1}{\nabla^2}\partial_j\partial_3{}^{-1}j_4^{(1)}\right)^2 + i\mu^2\epsilon_{ij3}\hat{q}_i^{2T}\frac{1}{\nabla^2}\partial_j\partial_3{}^{-1}j_4^{(1)}$$
$$+ j_i^{(a)}\hat{q}_i^{aT} - i\epsilon_{ab}\epsilon_{ij3}j_i^{(a)}\frac{1}{\nabla^2}\partial_j\partial_3{}^{-1}j_4^{(b)}\Big]|\rangle. \qquad (61)$$

We see that (60) contains the well-known expression[15] for the Hamiltonian of the free massive vector field in terms of the canonical variables $\hat{q}_i^2$ and $\hat{P}_i^2$. Since (58) shows that these two variables indeed obey canonical commutation relations we are entitled to conclude that the first two terms of (45) do describe a massive vector meson.

It is also of interest to compute the angular momentum operator for the present theory. Carrying through an analogous procedure to Sec. IV we find, starting from the canonical expression (29),

$$J_i|\rangle = -\epsilon_{ijk}\int d^3x\{\hat{P}_m^2 x_j\partial_k\hat{q}_m^2 + \hat{P}_j^2\hat{q}_k^2 + i\psi_n'^\dagger[x_j\partial_k + (i/4)\Sigma_{jk}]\psi_n'\}|\rangle$$
$$-\frac{1}{8\pi}\epsilon_{ab}\int d^3x\,d^3y[j_4^{(a)}(x)j_4^{(b)}(y) + 2i\delta_{a2}\partial_j\hat{P}_j^2(x)j_4^{(b)}(y)](1-\delta_{i3})\frac{x_i - y_i}{|\vec{x}-\vec{y}|}\frac{|\vec{x}-\vec{y}|^2}{|\vec{x}-\vec{y}|^2 - (x_3-y_3)^2}|\rangle. \qquad (62)$$

The first three terms comprise the normal form for the angular momentum operator of a system with a massive vector meson and fermions. The fourth term in (62) is the same as the last term in (37) while the last term is unique to the present case. Note that this term contains products of Bose and Fermi field operators. Thus even in lowest order, J_i acting on a state containing, for example, one Fermi particle, will produce among other things a state of one Fermi and one Bose particle. This appears to be an important difference from the $\mu = 0$ case.

Since H_0 given in (60) has the usual form, we can easily develop a perturbation theory to treat the effects of H_I in (61). Although the validity of the perturbation expansion is doubtful due to lack of manifest covariance (see also Sec. VI), we give for completeness the effective momentum-space propagators for the exchange of the massive vector mesons. Denoting these by $P_{\mu\nu}^{ab}(k)$ for the elastic scattering of a particle of type a and a particle of type b, we find[16]

$$P^{11}_{\mu\nu}(k)=\frac{-i}{k^2+\mu^2}\left\{\delta_{\mu\nu}+\frac{\mu^2}{(n\cdot k)^2}(\delta_{\mu\nu}-n_\mu n_\nu)\right\}, \quad (63a)$$

$$P^{22}_{\mu\nu}(k)=\frac{-i\delta_{\mu\nu}}{k^2+\mu^2}, \quad (63b)$$

$$P^{12}_{\mu\nu}(k)=\frac{-1}{k^2+\mu^2}\epsilon_{\mu\nu\rho\sigma}\frac{k_\rho n_\sigma}{n\cdot k}. \quad (63c)$$

VI. NAMBU'S HAMILTONIAN

Nambu[8] has recently proposed a static Hamiltonian for two interacting particles ("quarks") which contains a short-range Yukawa potential as well as a long-range "string" potential proportional to the separation between the particles. His motivation for considering this Hamiltonian was to retain some desirable features of both dual string and conventional field theory models. We now show that his Hamiltonian is the static limit of the Hamiltonian given in (60) and (61). We also briefly indicate some of the peculiarities of the angular momentum operators in such models.

We commute our Hamiltonian with the observable boson variables $\hat{P}_i^a$ and $\partial_i\hat{q}_i^2$ to find the equations of motion

$$\frac{d\hat{P}_i^a}{dt}=-\delta_{a1}\epsilon_{ijk}\partial_j\hat{P}_k^2+\delta_{a2}(\nabla^2-\mu^2)\hat{q}_i^2 -\delta_{a2}\partial_i\,\partial_l\hat{q}_l^2-j_i^{aT}-i\mu^2\delta_{a2}\epsilon_{ij3}\frac{1}{\nabla^2}\partial_j\partial_3^{-1}j_4^1, \quad (64)$$

$$\frac{d}{dt}\partial_i\hat{q}_i^2=\left(1-\frac{\nabla^2}{\mu^2}\right)\partial_i\hat{P}_i^2-ij_4^2. \quad (65)$$

Note that, due to (56a), these equations determine the time evolution of the conventional variables $\hat{q}_i^2$ and $\hat{P}_i^2$.

Nambu's Hamiltonian can be obtained by first considering the situation where only charge densities are present ($\vec{j}^a=0$) and $\hat{P}_i^a$ and $\hat{q}_i^a$ are time-independent. Equations (64) and (65) then give

$$\hat{P}_i^2-\frac{\partial_i}{\nabla^2}\partial_j\hat{P}_j^2=0, \quad (66)$$

$$\partial_i\hat{q}_i^2=0, \quad (67)$$

$$\partial_i\hat{P}_i^2=\frac{i\mu^2}{\mu^2-\nabla^2}j_4^2, \quad (68)$$

$$\hat{q}_i^2=\frac{-i\mu^2}{\mu^2-\nabla^2}\frac{1}{\nabla^2}\epsilon_{ij3}\partial_j\partial_3^{-1}j_4^1. \quad (69)$$

Substitution of these expressions into the total Hamiltonian yields, after some straightforward algebra,

$$H=\frac{1}{2}\int d^3x\left(j_4^a\frac{1}{\nabla^2-\mu^2}j_4^a-\mu^2 j_4^1\frac{1}{\nabla^2-\mu^2}\partial_3^{-2}j_4^1\right), \quad (70)$$

where we have omitted the kinetic energy terms for the charged particles. The operator ∂_3^{-2} is familiar in two-dimensional quantum electrodynamics. It is defined by[17]

$$(\partial_3^{-2}f)(x_3)=\tfrac{1}{2}\int dx_3'\,|x_3-x_3'|f(x_3'). \quad (71)$$

Equation (70) contains Nambu's results. The first term evidently represents a Yukawa interaction between the static charges, while the second term contains the "stringlike" interaction that he finds. To see the latter, consider the situation when there are only two particles present. If one is at the origin and the other is at $\vec{\xi}$, the magnetic charge density j_4^1 is given by

$$j_4^1(\vec{x})=ig\delta^3(x)+ig'\delta^3(x-\xi), \quad (72)$$

where g and g' are coupling constants. If self-interactions of the particles are ignored, the second term of (70) becomes

$$-\frac{1}{8\pi}\mu^2gg'\int dx_3'\,|x_3'-\xi_3| \times\frac{\exp[-\mu(\xi_1^2+\xi_2^2+x_3'^2)^{1/2}]}{(\xi_1^2+\xi_2^2+x_3'^2)^{1/2}}. \quad (73)$$

Here we have used (71), (72) and the identity

$$\left(\frac{1}{\nabla^2-\mu^2}f\right)(\vec{x})=-\frac{1}{4\pi}\int d^3x'\,\frac{\exp(-\mu|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}\times f(\vec{x}'). \quad (74)$$

For large $|\xi_3|$ (which implies a large separation between the particles), the interaction (73) is evidently proportional to $|\xi_3|$. Nambu considers the special configuration $\xi_1=\xi_2=0$ so that both the particles are located on the third axis. In this case as well, (73) is proportional to $|\xi_3|$ for large $|\xi_3|$ (the "stringlike" interaction), although the integral multiplying $|\xi_3|$ is infinite. These are the same as Nambu's results.

The angular momentum operators J_i in the static approximation can be obtained from Eq. (62) by using the preceding expression for the fields and currents. As in the usual magnetic-monopole theory, one of the subtle questions concerns the conservation of angular momentum. The commutators $[J_i,H]$ can be worked out directly, but a more convenient approach is to use the expression for the divergence of the angular momentum tensor density $M_{\alpha\mu\nu}$. For our purposes, it is sufficient to consider the equations

$$\partial_\alpha M_{\alpha 3a} = i\epsilon_{3a\alpha\beta}[\partial_3^{-1}(j_\alpha^2 + \mu^2 q_\alpha^2)][\partial_3^{-1} j_\beta^1], \quad a=1,2 \tag{75}$$

$$\partial_\alpha M_{\alpha 12} = 0 \,. \tag{76}$$

Here we have not made the static approximation. Recall that the field q_α^2 is given by

$$q_i^2 = \hat{q}_i^2 + i\epsilon_{ij3}\frac{1}{\nabla^2}\partial_j \partial_3^{-1} j_4^1 \,,$$
$$q_4^2 = \frac{+i}{\mu^2}\partial_i \hat{P}_i^2 \,. \tag{77}$$

Equations (75) and (76) can be obtained from Zwanziger's Eqs. (4.18) and (4.19) (Ref. 18) if it is observed that the 4-divergences of the gauge-field tensor $F_{\mu\nu}$ and its dual $F_{\mu\nu}^d$ [cf. Eq. (4)] fulfill the equations

$$\partial_\mu F_{\mu\nu} = j_\nu^1 \,,$$
$$\partial_\mu F_{\mu\nu}^d = -i(j_\nu^2 + \mu^2 q_\nu^2) \,. \tag{78}$$

Equation (16) differs from Zwanziger's field equations for $F_{\mu\nu}$ by the addition of the term $\mu^2 q_\nu^2$ to the current j_ν^2. This is reflected directly in the difference between Eq. (75) and Zwanziger's result.

In the static approximation (75) gives

$$\partial_\alpha M_{\alpha 3a} = \left[\partial_a \partial_3^{-2}\left(\frac{\mu^2}{\nabla^2 - \mu^2}\right) j_4^1\right](\partial_3^{-1} j_4^1), \quad a=1,2 \tag{79}$$

where we have used (69) and (77). The time evolution of J_1 and J_2 is thus given by

$$-i\frac{\partial}{\partial t}\int d^3x\, M_{43a} = \int d^3x\left[\partial_a \partial_3^{-2}\left(\frac{\mu^2}{\nabla^2 - \mu^2}\right) j_4^1\right] \times (\partial_3^{-1} j_4^1), \quad a=1,2 \tag{80}$$

and the identifications

$$J_a = -\frac{i}{2}\epsilon_{aij}\int d^3x\, M_{4ij}, \quad a=1,2 \,. \tag{81}$$

It follows that if μ is zero, angular momentum is conserved in the static approximation, while if μ is not zero, $[J_a, H]$ is not zero for $a=1,2$. One may, at this point, hope to recover angular momentum conservation for $\mu \neq 0$ by requiring that $[J_a, H]$ vanish on the allowed states of the theory. Unfortunately, however, it seems difficult to impose such constraints. For example, when the charge density is given by (72), we find from (79)–(81) that $[J_a, H]$ is

$$-i\epsilon_{ab3}\frac{1}{8\pi}\mu^2 g g' \int dx_3\, \epsilon(x_3 - \xi_3)\frac{\partial}{\partial \xi_b}\int dx_3' |x_3'| \frac{\exp\{-\mu[\xi_1^2 + \xi_2^2 + (x_3 - x_3')^2]^{1/2}\}}{[\xi_1^2 + \xi_2^2 + (x_3 - x_3')^2]^{1/2}} \tag{82}$$

if self-interactions are ignored and the definitions of the operator inverses are used.[17] This expression does not vanish for any value of $\vec{\xi}$ when $\mu \neq 0$. Essentially this is because the operator $\mu^2/(\mu^2 - \nabla^2)$ in (79) "spreads" the point charges in (72) into charge distributions filling all three-dimensional space. Thus the constraint that (82) vanish on allowed wave functions will imply that these wave functions are identically zero.

In the usual nonrelativistic magnetic-monopole theories it is well known that one requires a constraint on allowed wave functions to insure angular momentum conservation. To see the need for this constraint, however, one must go beyond the static approximation and retain the contribution to the current $\vec{j}^1$ (say) from at least one of the particles. In this ($\mu = 0$) case no operators like $\mu^2/(\mu^2 - \nabla^2)$ are present and operators like ∂_3^{-1} which are present "spread" the point current $\vec{j}^1$ out along a line joining this particle to a particle carrying type-2 charge.[19] Thus the constraint only amounts to requiring that the allowed wave functions vanish sufficiently fast along this line and hence allows a viable theory. We may note here, for completeness, that if the current $\vec{j}^1$ is retained when $\mu \neq 0$, then, besides (82), there are additional terms in $[J_a, H]$ which are nonvanishing everywhere.[20] The presence of these terms may be shown by calculations similar to those indicated above and are due to the effect of the operator $\mu^2/(\mu^2 - \nabla^2)$ on the current.

There seem to be at least two different ways of overcoming these difficulties connected with rotational invariance. The first is to note that the mass term for the gauge field can be thought of as arising from the Higgs mechanism. Since the gauge field is massless and the theory is rotationally invariant before spontaneous breakdown (cf. Ref. 4), a more careful treatment of the Higgs mechanism may give a rotationally invariant theory after spontaneous breakdown as well (cf. Ref. 21). Alternatively, since the difficulties with angular momentum seem to be connected to the fact that the "Dirac string" is regarded as a given fixed object and not as a dynamical variable, another approach to the problem may be to treat the Dirac string as a dynamical variable, (cf. Ref. 8). It is suggestive, in this context, that (82) vanishes when both particles are aligned on the third axis, that is, when $\xi_a = 0$. Both these approaches are currently under study.

Since the completion of this paper, it has been brought to our attention that monopole theories with massive gauge fields have been previously considered[22] by Taylor and by Acharya and Horvath. The last-named authors, in particular, point out that the Dirac-Schwinger charge-quantization condition is not expected to hold when the gauge field is massive. Further new reports by Creutz[23] and by Jevicki and Senjanovic[21] on very similar models have appeared. The last-named authors have also discussed questions of rotational invariance.

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[9]More precisely, what we mean is that the Poisson bracket of any first-class constraint with any other constraint is proportional to the constraints and hence vanishes when the constraint equations are imposed. Note that the term "constraint" is being used to denote either a function like Z_b^a or an equation like $Z_b^a=0$. This should cause no confusion.

[10]To find a realization of the commutation relations implied by (13) we choose to expand three sets of canonically conjugate operators in terms of ordinary free-field creation and destruction operators. These three sets are specified by

(a) $[q_i^2(\vec{x},t),\ P_j^2(\vec{y},t)]=i\delta_{ij}\delta^3(x-y)$,

(b) $[q_4^a(\vec{x},t),\ \pi_4^b(\vec{y},t)]=i\delta_{ab}\delta^3(x-y)$,

and

(c) $[\phi(\vec{x},t),\ \pi_\phi(\vec{y},t)]=i\delta^3(x-y)$.

ϕ and π_ϕ are auxiliary fields. Note that all variables referring to one set are assumed to commute with the variables of the other sets. Then, using the second-class constraints and (13) we find that the remaining quantities P_i^1 and $q_i^1=q_i^{1T}+q_i^{1L}$ are given by

$$P_i^1=-(\vec{\nabla}\times\vec{q}^2)_i-\delta_{i3}\partial_3^{-1}\pi_\phi,$$

$$q_i^{1T}=-\epsilon_{ijk}\frac{1}{\nabla 2}\partial_j(P_k^2-\delta_{k3}\partial_3^{-1}\vec{\nabla}\cdot\vec{P}^2),$$

and $q_i^{1L}=\partial_i\phi$. Finally, we may use (17) and (20) to find the π_i^a from the more convenient P_i^a; namely $\pi_i^a=\frac{1}{2}(P_i^a+\delta_{i3}\partial_3^{-1}\vec{\nabla}\cdot\vec{P}^a)$. Similar constructions can be carried out for the massive case as well.

[11]Note however, that integration by parts is not allowed for the last two terms in (32). This may be seen by inserting (33) and (35) in (32) and examining the surface terms.

[12]Here it is understood that the current operator is appropriately normal ordered.

[13]P. W. Anderson, Phys. Rev. 130, 439 (1963); F. Englert and R. Brout, Phys. Rev. Lett. 13, 321 (1964); G. S. Guralink, C. R. Hagen, and T. W. B. Kibble, *ibid.* 13, 585 (1964); P. W. Higgs, Phys. Rev. 145, 1156 (1966); T. W. B. Kibble, *ibid.* 155, 1554 (1967).

[14]This is the symmetry under the so-called duality transformation discussed in Ref. 2, for example.

[15]See, for example, D. Lurié, *Particles and Fields* (Interscience, New York, 1968), Chap. 4.

[16]Here we assume that the particle of type 1 is characterized by $e_1^1\neq 0$ and $e_1^2=0$, while the particle of type 2 is characterized by $e_2^1=0$ and $e_2^2\neq 0$.

[17]This definition is consistent with Zwanziger's definition of ∂_3^{-2} [cf. Eq. (6.15) of Ref. 4].

[18]As before we have set $n_\mu=\delta_{\mu 3}$ in arriving at (75) and (76).

[19]Cf. Ref. 4, Eq. (6.5).

[20]Similarly $[J_1, J_2]=iJ_3+\phi$, where ϕ has support along a line if $\mu=0$ [cf. Eq. (6.6) of Ref. 4] and is nonvanishing everywhere if $\mu\neq 0$.

[21]A. Jevicki and P. Senjanovic, Phys. Rev. D 11, 860

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[22]J. G. Taylor, Phys. Rev. Lett. 18, 713 (1967);
R. Acharya and Z. Horvath, Nuovo Cimento Lett. 8, 513 (1973).
[23]M. Creutz, Phys. Rev. D 10, 2696 (1974).
[24]See in this context, N. Murai, Prog. Theor. Phys. 47, 678 (1972).