Renormalization-group equation for high-energy wide-angle scattering in ϕ^4 theory*

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It is shown that the scattering amplitude in ϕ^4 theory obeys the renormalization-group equation in the limit of high energy for fixed wide scattering angle. It is conjectured that this feature is common to all renormalizable field theories involving spinor and/or scalar fundamental fields but no vector fields.

I. INTRODUCTION

The Green's functions of renormalizable quantum field theories obey certain linear homogeneous partial differential equations, known as renormalization-group equations[1] for large Euclidean momenta provided the squares of all nontrivial partial sums of momenta approach infinity at the same rate *(nonexceptional momenta)*. Although the asymptotic limit of Green's functions for nonexceptional momenta is of no direct physical interest it nevertheless provides information on the asymptotic behavior of the coefficient functions in Wilson's operator-product expansion.[2] Thus, it is related to the short-distance behavior of electroproduction structure functions in the Bjorken limit.[3]

In this paper it is shown that, in certain field theories, renormalization-group equations are valid for certain exceptional momenta as well. Specifically in a theory of scalar particles coupled by a ϕ^4 interaction, the *scattering amplitude* $T(p, \theta, m_r, \lambda_r)$ as a function of the center-of-mass momentum p, the scattering angle θ, the renormalized mass m_r, and the renormalized coupling constant λ_r obeys the asymptotic equation

$$\left[m_r \frac{\partial}{\partial m_r} + \beta(\lambda_r) \frac{\partial}{\partial \lambda_r} + 4\gamma(\lambda_r)\right] T(p, \theta, m_r, \lambda_r) = O\left(\frac{1}{p^2}\right) \tag{1}$$

in the limit $p \to \infty$ for $\theta \neq 0, \pi$. The differential operator on the left-hand of Eq. (1) coincides with the one which appears in the renormalization-group equation for the off-shell four-point Green's function. It is likely that *all* multiparticle S-matrix elements satisfy the renormalization-group equations in this theory (in the limit where all energies and momentum transfers grow at the same rate) but the relevant analysis has not yet been carried out in detail.

In a technical sense Eq. (1) follows from the fact that in a perturbation expansion of the scattering amplitude the Feynman integrals for $p \to \infty$ behave like p^0 times some polynomial in $\ln(p/\Lambda)$ (where Λ is the ultraviolet cutoff), the remainder being of order $1/p^2$. The essential point is that, just as in the case of nonexceptional momenta, the unrenormalized mass does not appear in the leading asymptotic part of the amplitude.

The study of asymptotic behavior in perturbation theory is most convenient in the framework of Speer's analytic renormalization method[4] which is equivalent to the more widely used Bogoliubov-Parasiuk-Hepp scheme.[5] In Sec. II the Callan-Symanzik equations[2] and the renormalization group equations are derived for the ϕ^4 theory following the Speer method. Section III focuses on the physical scattering amplitude and the proof of Eq. (1). The main ingredients of the proof are the asymptotic estimates obtained in Sec. IV.

In Sec. V the validity of renormalization-group equations for S-matrix elements is explored for other field theories. It is conjectured that they should hold for all renormalizable theories involving as fundamental objects scalar and/or spinor fields but no vector fields. In theories with vector fields only amplitudes for the scattering of "neutral" particles are likely to obey renormalization-group equations.

II. ANALYTIC RENORMALIZATION AND RENORMALIZATION-GROUP EQUATIONS FOR ϕ^4 THEORY

Let $\phi(x)$ be a Hermitian scalar field whose dynamics arises from the classical Lagrangian density:

$$\mathcal{L} = -\tfrac{1}{2}(\partial_\mu \phi)^2 - \tfrac{1}{2} m^2\phi^2 - \frac{\lambda}{4!} \phi^4 .$$

As mentioned in Sec. I, the renormalized Green's functions will be obtained in perturbation theory according to Speer's method of analytic renormalization. A specific choice of analytic regularization will be employed which consists in replacing the bare propagator $(q^2 + m^2 - i\epsilon)^{-1}$ by

$$\frac{(\sigma^2)^\rho}{(q^2 + m^2 - i\epsilon)^{1+\rho}} .$$

A complex variable ρ is thus introduced for each internal line of a Feynman graph. The real positive parameter σ which has the dimensions of a mass is common to all propagators and ensures that the regularized integrals have the same naive mass dimension as the original unregularized ones independently of the values of ρ parameters. It can be verified (e.g., by checking the naive degree of divergence of subgraphs and using Weinberg's theorem[6]) that if $\mathrm{Re}\rho_i > \frac{1}{3}$ for all ρ_i of a graph, the regularized integral is convergent. The so-defined regularized Feynman integral depends on the external momenta p_i, the (finite) unrenormalized mass m, the (finite) unrenormalized constant λ, and the parameter σ; furthermore, it is a meromorphic function of the set of complex variables $\{\rho_1, \rho_2, \ldots\}$. In fact, there is, in general, a multiple-pole singularity at $\rho_1 = \rho_2 = \cdots = 0$. Following Speer, one removes this singularity and defines a *finite* unrenormalized integral at $\rho_i = 0$ by applying the "evaluator" W which is the operation defined by

$$Wf = \frac{1}{(2\pi i)^n} \oint_{C_1} \frac{d\rho_1}{\rho_1} \oint_{C_2} \frac{d\rho_2}{\rho_2} \cdots \times \oint_{C_n} \frac{d\rho_n}{\rho_n} \frac{1}{n!} \sum_{\text{perm}} f(\rho_1, \rho_2, \ldots, \rho_n).$$

The sum runs over all $n!$ permutations of $\rho_1, \rho_2, \ldots, \rho_n$ and C_i is the contour $|\rho_i| = R_i$ with $R_i > \sum_{j=1}^{i-1} R_j$.

Let $\Gamma^{(n)}(p_i, m, \sigma, \lambda)$ denote the finite, unrenormalized, truncated, one-particle irreducible, n-point Green's function obtained by the above procedure and $\Delta_F'(p^2, m, \sigma, \lambda)$ the full unrenormalized propagator. In terms of the renormalized m_r and the renormalized coupling constant λ_r, the conditions

$$\Delta_F'^{-1}(p^2, m, \sigma, \lambda)|_{p^2 = -m_r^2} = 0,$$

$$\frac{\partial}{\partial (p^2)} \Delta_F'^{-1}(p^2, m, \sigma, \lambda)|_{p^2 = -m_r^2} = Z^{-1},$$

$$\Gamma^{(4)}(p_i, m, \sigma, \lambda)|_{p_i = 0} = -\lambda_r Z^{-2}$$

express m_r, λ_r, and Z as functions of m, λ, and σ.

The *renormalized* Green's functions $\Gamma_r^{(n)}$ are given by

$$\Gamma^{(n)}(p_i, m, \sigma, \lambda) = Z^{-n/2}(m, \sigma, \lambda)\Gamma_r^{(n)}(p_i, m_r(m, \sigma, \lambda), \lambda_r(m, \sigma, \lambda)). \tag{2}$$

Equation (2) essentially asserts that the theory is multiplicatively renormalizable. The differential equations of dimensional analysis for $\Gamma^{(n)}$ and $\Gamma_r^{(n)}$ read

$$\left(\sum_{i=1}^{n-1} p_{i\mu} \frac{\partial}{\partial p_{i\mu}} + \sigma \frac{\partial}{\partial \sigma} + m \frac{\partial}{\partial m} + n - 4\right)\Gamma^{(n)}(p_i, m, \sigma, \lambda) = 0, \tag{3}$$

$$\left(\sum_{i=1}^{n-1} p_{i\mu} \frac{\partial}{\partial p_{i\mu}} + m_r \frac{\partial}{\partial m_r} + n - 4\right)\Gamma_r^{(n)}(p_i, m_r, \lambda_r) = 0. \tag{4}$$

A straightforward combination of Eqs. (2), (3), and (4) leads to the Callan-Symanzik equation

$$\left(m_r \frac{\partial}{\partial m_r} + \beta(\lambda_r) \frac{\partial}{\partial \lambda_r} + n\gamma(\lambda_r)\right)\Gamma_r^{(n)} = Z^{n/2}\left(1 - \frac{\sigma}{m_r}\frac{\partial m_r}{\partial \sigma}\right)^{-1} m \frac{\partial}{\partial m}\Gamma^{(n)}, \tag{5}$$

where

$$\beta(\lambda_r) = -\left(1 - \frac{\sigma}{m_r}\frac{\partial m_r}{\partial \sigma}\right)^{-1} \sigma \frac{\partial \lambda_r}{\partial \sigma},$$

$$\gamma(\lambda_r) = \frac{1}{2}\left(1 - \frac{\sigma}{m_r}\frac{\partial m_r}{\partial \sigma}\right)^{-1} \sigma \frac{\partial \ln Z}{\partial \sigma}.$$

The fact that the dimensionless quantities β and γ depend only on λ_r (and not also on Z) can be established by looking at the asymptotic behavior of $\Gamma^{(n)}$ and $m(\partial/\partial m)\Gamma^{(n)}$ for large nonexceptional momenta: $p_i = \eta p_i'$, $\eta \to \infty$. As will be shown in Sec. IV, in every order of perturbation theory

$$\Gamma^{(n)}, \Gamma_r^{(n)} \sim \eta^{4-n}, \qquad m \frac{\partial}{\partial m}\Gamma^{(n)} \sim \eta^{3-n} \tag{6}$$

within powers of $\ln\eta$. Dropping the right-hand side of Eq. (5) for $\eta \to \infty$ one obtains

$$\left(m_r \frac{\partial}{\partial m_r} + \beta(\lambda_r)\frac{\partial}{\partial \lambda_r} + n\gamma(\lambda_r)\right)\Gamma_{r,\mathrm{AS}}^{(n)} = 0, \tag{7}$$

where $\Gamma_{r,\mathrm{AS}}^{(n)}$ is the asymptotic part of $\Gamma_r^{(n)}$ which clearly does not depend on Z. Thus, β and γ cannot depend on Z. Equation (7) is the familiar renormalization-group equation for the ϕ^4 theory.

III. RENORMALIZATION-GROUP EQUATION FOR THE SCATTERING AMPLITUDE

Equation (5) and its analogs in other renormalizable field theories are useful whenever the right-hand side can be neglected (as in the case of large nonexceptional momenta), in which case it leads to a homogeneous linear differential equation for the asymptotic Green's function, i.e., a renormalization-group equation like Eq. (7). The purpose of this paper is to show that in ϕ^4 theory an equation like Eq. (7) is valid for the scattering amplitude.

Consider the amplitude $\Gamma^{(4)}$ for a process $p_1 + p_2$

$\to p_3+p_4$ where all external masses are taken to be equal:

$$p_1^2=p_2^2=p_3^2=p_4^2=-m_e^2.$$

Thus, $\Gamma^{(4)}$ may be considered a function of m_e, m, σ, λ, the center-of-mass momentum p, and the scattering angle θ defined by $-(p_1+p_2)^2=4p^2+4m_e^2$, $-(p_1-p_3)^2=-2p^2(1-\cos\theta)$. The relation between $\Gamma^{(4)}$ and the renormalized amplitude $\Gamma_r^{(4)}$ is

$$\Gamma^{(4)}(p,\theta,m_e,m,\sigma,\lambda)=Z^{-2}(m,\sigma,\lambda)\Gamma_r^{(4)}(p,\theta,m_e,m_r(m,\sigma,\lambda),\lambda_r(m,\sigma,\lambda)).$$

From dimensional analysis one has

$$\left(p\frac{\partial}{\partial p}+m_e\frac{\partial}{\partial m_e}+m\frac{\partial}{\partial m}+\sigma\frac{\partial}{\partial\sigma}\right)\Gamma^{(4)}=0,$$

$$\left(p\frac{\partial}{\partial p}+m_e\frac{\partial}{\partial m_e}+m_r\frac{\partial}{\partial m_r}\right)\Gamma_r^{(4)}=0.$$

From the preceding three equations one obtains the differential relation

$$\left(-p\frac{\partial}{\partial p}+\beta(\lambda_r)\frac{\partial}{\partial\lambda_r}+4\gamma(\lambda_r)\right)\Gamma_r^{(4)}=\frac{Z^2m(\partial/\partial m)Z^{-2}\Gamma_r^{(4)}}{(m/m_r)\partial m_r/\partial m}+m_e\frac{\partial}{\partial m_e}\Gamma_r^{(4)}. \quad (8)$$

In Sec. IV it is shown that to every order in perturbation theory for large p and $\theta\neq 0$ or π we have

$$\Gamma_r^{(4)}\sim p^0,\qquad \frac{\partial}{\partial m}\Gamma_r^{(4)},\ \frac{\partial}{\partial m_e}\Gamma_r^{(4)}\sim\frac{1}{p^2} \quad (6')$$

within powers of $\ln p$. Thus, one may drop the right-hand side of Eq. (8) in this limit and write

$$\left(-p\frac{\partial}{\partial p}+\beta(\lambda_r)\frac{\partial}{\partial\lambda_r}+4\gamma(\lambda_r)\right)\Gamma_{r,\mathrm{AS}}^{(4)}=0.$$

This equation is valid for arbitrary but finite m_e. In particular, it is valid for the scattering amplitude

$$T(p,\theta,m_r,\lambda_r)\equiv\Gamma_r^{(4)}(p,\theta,m_e,m_r,\lambda_r)|_{m_e=m_r}.$$

Since dimensional analysis for $T(p,\theta,m_r,\lambda_r)$ implies

$$\left(p\frac{\partial}{\partial p}+m_r\frac{\partial}{\partial m_r}\right)T=0,$$

the differential equation for T can also be written as

$$\left(m_r\frac{\partial}{\partial m_r}+\beta(\lambda_r)\frac{\partial}{\partial\lambda_r}+4\gamma(\lambda_r)\right)T_{\mathrm{AS}}(p,\theta,m_r,\lambda_r)=0. \quad (9)$$

IV. ASYMPTOTIC ESTIMATES IN PERTURBATION THEORY

The derivation of the renormalization-group equations in Secs. II and III relies crucially on the asymptotic estimates of Eqs. (6) and (6′). In this section these results are derived for four-point functions from a study of the asymptotic behavior of Feynman integrals.

Let G be a Feynman graph with four external lines, I internal lines, and v vertices. The number of independent loops is $L=I-v+1$. Note that all vertices are of the ϕ^4 type since no explicit mass counterterm is used in the analytic renormalization method. It follows that $I=2L$ and $v=L+1$.

To construct the analytically regularized amplitude associated with G, a (bare) propagator of the form

$$(\sigma^2)^{\rho_i}(q_i^2+m^2-i\epsilon)^{-1-\rho_i}$$

is used for the ith internal line carrying momentum q_i. The resulting integral is convergent if $\mathrm{Re}\rho_i>\frac{1}{3}$ for all i. If the I denominators are combined via the "fractional-power" Feynman identity

$$\prod_{j=1}^{I}D_j^{-1-\rho_j}=\frac{\Gamma(n+\sum\rho_j)}{\prod_j\Gamma(1+\rho_j)}\times\int\left(\prod x_j^{\rho_j}dx_j\right)\delta(\textstyle\sum x_j-1)(\sum x_jD_j)^{-n-\Sigma\rho_j},$$

the L four-momentum integrations can be explicitly carried out to obtain the parametric form of the regularized integral:

$$F(G,\rho_i)=\frac{\gamma_G\lambda^{L+1}}{(4\pi)^{2L}}\frac{\Gamma(\sum\rho_i)}{\prod\Gamma(1+\rho_i)}(\sigma^2)^{\Sigma\rho_i}\times\int\frac{(\prod x_j^{\rho_j}dx_j)\delta(\sum x_j-1)}{U^2(f/U+m^2-i\epsilon)^{\Sigma\rho_i}}. \quad (10)$$

Here γ_G is a numerical factor which depends on the topology of G and is of no interest for this discussion. The Symanzik functions U and f are given by well-known topological formulas:

$$U=\sum_T\left(\prod_{j\neq T}x_j\right), \quad (11)$$

$$f=\sum_{T_2}\left(\prod_{j\neq T_2}x_j\right)P_\mu^2(T_2). \quad (12)$$

In Eq. (11) the sum runs over all *tree* graphs T of G, i.e., over all sets of $v-1$ lines which form no loops. In Eq. (12) the sum runs over all two-tree graphs T_2 of G, i.e., over all sets of

$v-2$ lines of G which form no loops. A two-tree graph divides the vertices of G into two disjoint sets. In Eq. (12) $P_\mu^2(T_2)$ stands for the Lorentz square of the sum of the external momenta flowing into one of these sets associated with T_2. The momentum $P_\mu(T_2)$ shall be referred to as the two-tree momentum of T_2.

In carrying out the analytic continuation of $F(G,\rho_i)$ to the neighborhood of $\rho_i=0$, Speer makes use of certain topological concepts which will be recorded here for the convenience of the reader.

Definition 1. A graph is *2-connected* if it cannot be disconnected by removing a vertex. Every graph is the union of its maximal 2-connected subgraphs and single lines; these are called the *pieces* of the graph.

From now on, it will be assumed that G is 2-connected without loss of generality since the amplitude for a general graph is simply the product of the amplitudes of its pieces.

Definition 2. Two subgraphs of G are *disjoint* if they have no common lines; subgraphs are *nonoverlapping* if they are either disjoint or one is a subgraph of the other.

Definition 3. A *singularity family* of G is a maximal family E of nonoverlapping subgraphs of G, each 2-connected or consisting of a single line with the property that no union of two or more disjoint elements of E is 2-connected.

From definition 3 it follows that (i) G belongs to E and (ii) for each $H\in E$, there is precisely one line of H, called $\sigma(H)$, which lies in no subgraph of H in E. Thus, there is a one-to-one correspondence between lines of G and graphs in E.

It is helpful to visualize the following hierarchical construction of a singularity family E along with the corresponding function $\sigma(H)$. Begin with G which, of course, belongs to all singularity families. Choose as $\sigma(G)$ any one of its lines. If $\sigma(G)$ is removed, the pieces of $G-\sigma(G)$ will be members of E. From each such piece H which is not a single line choose as $\sigma(H)$ any one of its lines. Then the pieces of $H-\sigma(H)$ are also members of E. Next choose σ for each of the new pieces, etc. Clearly, at the end of this process of removal of lines one is left with a (tree) graph whose pieces are all single lines—they are precisely those members of E which consist of single lines.

Consider now the analytically regularized integral of Eq. (10) for a 2-connected Feynman graph G. For each singularity family E of G define the domain $D(E)$ in Feynman parameter space by

$$D(E)\equiv\{x_{\sigma(H)}\geqslant x_i \text{ if } i\in H\}.$$

It is easily seen that (i) if $E_1\neq E_2$ then $D(E_1)\cap D(E_2)$ is of zero measure and (ii) $\cup D(E)=\{x_i\geqslant 0\}$, so that

$$F(G,\rho_i)=\sum_E F(G,E,\rho_i),$$

where $F(G,E,\rho_i)$ is given by the integral of Eq. (10) when the domain of integration is restricted to $D(E)$. One may thus focus on the contribution of a single singularity family E. It is convenient to introduce new "scaling variables" t_H (one for each member H of E) according to

$$x_i=\prod_{i\in H} t_H .$$

Let I_H and L_H denote the number of lines and the number of loops, respectively, for the subgraph H and set $\sum_{i\in H}\rho_i=\Lambda_H$. The following relations are easily established:

$$\prod dx_i=\prod_{H\in E} dt_H\, t_H{}^{I_H-1},$$

$$\prod x_i{}^{\rho_i}=\prod_{H\in E} t_H{}^{\Lambda_H},$$

$$U=\prod_{H\in E} t_H{}^{L_H}\tilde{U}(t),$$

$$f=t_G\Big(\prod_{H\in E} t_H{}^{L_H}\Big)\tilde{f}.$$

Here $\tilde{U}$ and $\tilde{f}$ are polynomials in the t variables. In fact, they are *linear* in each t separately. Moreover, $\tilde{U}\geqslant 1$ and $\tilde{U}=1+O(t)$ for small t.

The t_G integration can be explicitly carried out and yields

$$F(G,E,\rho_i)=\frac{\gamma_G\lambda^{L+1}}{(4\pi)^{2L}}\frac{\Gamma(\Lambda_G)}{\prod\Gamma(1+\rho_i)}(\sigma^2)^{\Lambda_G}\times\int_0^1\frac{\prod_{H\in E} dt_H\, t_H{}^{\Lambda_H-(d_H/2)-1}}{\tilde{U}^2(\tilde{f}/\tilde{U}+m^2\sum\beta_j-i\epsilon)^{\Lambda_G}}, \tag{13}$$

where $\beta_j\equiv x_j(t)/t_G$ and $d_H\equiv 4L_H-2I_H=$ "degree of divergence" of the subgraph H.

$F(G,E,\rho_i)$ must now be analytically continued to the neighborhood of $\{\rho_i=0\}$. It is clear that if $d_H<0$ the t_H integration is convergent at $\rho_i=0$. In ϕ^4 theory there exist subgraphs with $d_H=2$ (self-energy parts) and $d_H=0$ (subgraphs with four and two external vertices). The required analytic continuation will be carried out essentially by appropriately redefining the integration with respect to the t variables of all such "divergent" subgraphs of E.

For subgraphs H with $d_H=0$, it will suffice to make the replacement

$$t^{\Lambda_H-1} \to \frac{\delta(t_H)}{\Lambda_H} + t^{\Lambda_H-1}\Delta_H \,, \tag{14}$$

where Δ_H is the "difference operator":

$$\Delta_H\psi(\ldots, t_H, \ldots) = \psi(\ldots, t_H, \ldots) - \psi(\ldots, 0, \ldots)\,. \tag{15}$$

For subgraphs with $d_H = 2$, it would be adequate, for the analytic continuation, to replace t^{Λ_H-2} by

$$\frac{\delta(t_H)}{\Lambda_H - 1} + \frac{\delta'(t_H)}{\Lambda_H} + t_H{}^{\Lambda_H-2}\Delta'_H \,, \tag{16}$$

where

$$\Delta'_H\,\psi(\ldots, t_H, \ldots) = \psi(\ldots, t_H, \ldots) - \psi(\ldots, 0, \ldots) - t_H \frac{\partial\psi}{\partial t_H}(\ldots, 0, \ldots)\,. \tag{17}$$

However, such a replacement would result in the appearance, in the integrand, of multiple derivatives of the quantity $(\tilde{f}/\tilde{U} + \sum \beta_j - i\epsilon)^{-\Lambda_G}$ and thus in several terms of the type

$$P_N(\tilde{f}/\tilde{U} + \sum \beta_j m^2 - i\epsilon)^{-\Lambda_G - N}\,,$$

with P_N an Nth-degree polynomial in the invariants $p_i \cdot p_j$. As a result the asymptotic estimate would require a detailed study of the structure of $\tilde{f}$. The following procedure avoids explicit derivatives. Let $d \in E$ be a self-energy part and let x_1, x_2 be the Feynman parameters of the lines joining d to the rest of G. Let U_d and $f_d q^2$ be the Symanzik functions of d as a Feynman graph with external momentum q. From the topological definition of the Symanzik functions [Eqs. (11) and (12)] it follows that the dependence of f and U on x_1, x_2 and the parameters of d is given by the relations

$$f = U_d f_s + [f_d + (x_1 + x_2)U_d]f_r\,,$$

$$U = U_d U_s + [f_d + (x_1 + x_2)U_d]U_r\,,$$

where f_r, U_r are the Symanzik functions for the graph obtained from G by removing the lines 1, 2, and all of d, and f_s, U_s are the Symanzik functions for the graph obtained from G by shrinking 1, 2, and d to a point. In terms of the t variables one obtains

$$\frac{\tilde{f}}{\tilde{U}} = \frac{\tilde{f}_s + (t_1 + t_2 + t_d\tilde{f}_d/\tilde{U}_d)\tilde{f}_r}{\tilde{U}_s + (t_1 + t_2 + t_d\tilde{f}_d/\tilde{U}_d)\tilde{U}_r}\,,$$

$$\tilde{U} = \tilde{U}_\pi[\tilde{U}_s + (t_1 + t_2 + t_d\tilde{f}_d/\tilde{U}_d)\tilde{U}_r]\,.$$

Note that $\tilde{f}/\tilde{U}$ and $\tilde{U}$ depend on the variables t_1, t_2, and t_d only via the combination $t_1 + t_2 + t_d\tilde{f}_d/\tilde{U}_d$. Actually, either 1 or 2 may be $\sigma(H)$ with $H \supset d$ in which case the corresponding t variable in the above formulas should be replaced by one. In any case, it may be assumed, without loss of generality, that

$$\frac{d}{\partial t_d} = \frac{\tilde{f}_d}{\tilde{U}_d}\frac{\partial}{\partial t_1}\,. \tag{18}$$

Thus, the derivative in Eq. (17) can be replaced according to Eq. (18) ($H = d$). After elimination of the derivative with respect to t_1 by an integration by parts one obtains the relation

$$t_1{}^{\rho_1} t_d{}^{\Lambda_d-2} = t_1{}^{\rho_1}\left[\frac{\delta(t_d)}{\Lambda_d - 1} + \frac{\delta'(t_d)}{\Lambda_d} + t_d{}^{\Lambda_d-2}\Delta_d\right] - t_d{}^{\Lambda_d-1}\frac{\tilde{f}_d}{\tilde{U}_d} \times[\delta(t_1 - 1) - \delta(t_1) - \rho_1 t_1{}^{\rho_1-1}\Delta_1]\,, \tag{19}$$

in which explicit derivatives do not appear.

As a result of the replacements indicated by Eqs. (14), (15), and (19) (carried out for all $H \in E$ with $d_H \geq 0$) the integral in Eq. (13) takes the form

$$F(G, E, \rho_i) = \int \prod_{H\,E} dt_H \times \sum_N \frac{a_n^{(E)}(t)}{P_n^{(E)}(\rho)}\left[U^{-2}\left(\frac{\tilde{f}}{\tilde{U}\sigma^2} + \sum \beta_j \frac{m^2}{\sigma^2}\right)^{-\Lambda_G}\right]_{(n)}\,, \tag{20}$$

where $a_n^{(E)}(t)$ is a product of factors like $t_H{}^{\Lambda_H}$, $t_H{}^{\Lambda_H-1}$, $t_H{}^{\Lambda_H-2}$, $\delta(t_H)$, $\delta(t_H - 1)$, $\tilde{f}_d/\tilde{U}_d$, and $P_n^{(E)}$ is a meroporphic function of the ρ's having zeros at $\Lambda_H = 0$ for H in some subset E_n of "divergent" subgraphs in E. The subscript n indicates that some t variables have been set equal to 0 or 1 inside the bracket. Note that the summation sign cannot be taken out of the integral because there are nonintegrable singularities in individual terms which cancel only in the sum.

The application of Speer's evaluator on $F(G, E, \rho_i)$ in the form of Eq. (20) results in a *finite* amplitude at $\rho_i = 0$ given by an integral of the form

$$F(G, E) = \int \prod_{H \in E} dt_H \times \sum_j b_j^{(E)}(t)\left[\ln\left(\frac{f}{U\sigma^2} + \frac{m^2}{\sigma^2}\sum \beta_j\right)\right]^{N_j(E)}. \tag{21}$$

The general term in the sum of Eq. (20) gives rise to one or more terms of the sum in Eq. (21). The highest power of the logarithm is equal to the number of "divergent" members of E:

$N(E) \equiv \max_{(j)} N_j^E =$ number of subgraphs $H \in E$ with $d_H \geq 0$.

Equation (21) is the starting point in the derivation of asymptotic estimates. Consider first the case of Euclidean, nonexceptional external momenta (i.e., such that no 2-tree momentum vanishes). Set $p_i = \eta p_i'$ so that $f(x, p) = \eta^2 f(x, p')$ and

$$\frac{\tilde{f}(t,p)}{\tilde{U}\sigma^2} = \left(\frac{\eta}{\sigma}\right)^2 \frac{\tilde{f}(t,p')}{\tilde{U}}.$$

It will be shown later that the polynomial $\tilde{f}(t, p')$ does not vanish identically when certain t's are set equal to zero or one as specified by the δ functions in b_j^E. One then concludes from Eq. (21) that $F(G, E)$ is *finite* for $m^2 = 0$ since (i) the factors $\ln(\tilde{f}/\tilde{U})$ are integrable and (ii) the poles in $b_j^{(E)}$ are still canceled in the sum as they are for all values of m^2. It follows that, as $\eta \to \infty$,

$$F(G,E) \to F(G,E)|_{m^2=0} \sim \left[\ln\left(\frac{\eta}{\sigma}\right)\right]^{N(E)}.$$

Furthermore, differentiation of Eq. (21) with respect to m^2 yields

$$\frac{\partial}{\partial m^2} F(G,E) = \int \prod_{H\in E} dt_H \sum_j b_j^{(E)}(t) \left(\frac{\sum \beta_j}{\sigma^2}\right) \frac{(\ln D)^{N_j(E)-1}}{D}, \tag{22}$$

where

$$D = \frac{\eta^2}{\sigma^2}\frac{\tilde{f}}{\tilde{U}} + \frac{m^2}{\sigma^2}\sum \beta_j.$$

As $\eta \to \infty$ the contribution to the integral of Eq. (22) from any open t-space region in which $\tilde{f} \neq 0$ behaves at most like $(\ln \eta)^{N(E)-1}/\eta^2$. On the other hand, because of the fact that $\tilde{f}$ is *linear* in each t variable separately, the hypersurfaces $\tilde{f} = 0$ are of a special kind. In the neighborhood of some point $t_H = \hat{t}_H$, at which $\tilde{f}$ vanishes, $\tilde{f}$ must be of the form

$$\tilde{f} \approx g\lambda_1\lambda_2 \cdots \lambda_\kappa,$$

where $g \neq 0$ and the λ_i's are linear combinations of *disjoint* sets of t_H's:

$$\lambda_i = \sum_{H\in E_i} c_H(t_H - \hat{t}_H),$$

$$E_i \cap E_j = 0 \text{ for } i \neq j.$$

By introducing the λ_i's as new integration variables, the asymptotic contribution of the neighborhood of the point $\{t_H = \hat{t}_H\}$ can be easily found to be stronger than that of the $\hat{f} \neq 0$ regions only by a factor of $(\ln\eta)^\kappa$. This establishes that

$$\frac{\partial}{\partial m^2} F(G,E) \sim \frac{1}{\eta^2} \text{ times some power of } \ln\eta.$$

Consider now the wide-angle high-energy limit of the scattering amplitude. In the notation of Sec. III, let $F(G, E, p, \theta, m_e, m, \sigma, \lambda)$ be the contribution to $\Gamma^{(4)}$ associated with the graph G and a particular singularity family E of G. The Symanzik function f has the form

$$f(x,p) = p^2 f_0(x,\theta) + m_e^2 g(x),$$

so that

$$\frac{\tilde{f}}{\tilde{U}\sigma^2} + \frac{m^2}{\sigma^2}\sum \beta_j = \frac{p^2}{\sigma^2}\frac{\tilde{f}_0}{\tilde{U}} + \frac{m_e^2}{\sigma^2} g(x) + \frac{m^2}{\sigma^2}\sum \beta_j,$$

where again the tilde denotes that the product $\prod t_H^{L_H}$ has been factored out.

The asymptotic behavior in this case is obtained the same way as in the Euclidean case except that it is now based on $\tilde{f}_0$ not vanishing identically. The result is that as p approaches infinity,

$$F(G,E) \to F(G,E)|_{m^2 = m_e^2 = 0} \sim \left[\ln\left(\frac{p}{\sigma}\right)\right]^{N(E)},$$

$$\frac{\partial}{\partial m^2} F(G,E) \sim \frac{1}{p^2}\left[\ln\left(\frac{p}{\sigma}\right)\right]^n \quad (n = \text{integer}).$$

It must now be shown that $\tilde{f}$ and $\tilde{f}_0$ do not vanish identically when certain t variables are set equal to zero or one as specified by the δ functions in $b_j^{(E)}$. It is clearly sufficient to carry out the proof for $\tilde{f}_0$.

Recall that $\tilde{f}_0$ depends on the variables of a subgraph H with $d(H) = 2$ only via the expression $t_1 + t_2 + t_d\tilde{f}_d/\tilde{U}_d$ which, as is obvious from Eq. (19), is never forced to vanish by the δ functions in b_j. Therefore, insofar as one is exploring the question of whether $\tilde{f}_0$ vanishes identically or not, one may replace each such expression by a single new variable, e.g., $\bar{t}_d$. This means, effectively, that each self-energy part in E together with the two lines connecting it to the rest of G may be replaced by a single line. Thus, it suffices to see whether $\tilde{f}_0$ vanishes identically when $t_H = 0$ for all $H \in E$ with $d(H) = 0$ $(H \neq G)$.

From the definition of the Symanzik functions it is easily seen that when the t variables of the above subgraphs are all set equal to zero, $\tilde{f}_0$ coincides with the corresponding Symanzik function of the "reduced" graph G_E obtained from G by shrinking these subgraphs to points. But G_E is still a 2-connected four-point graph with only ϕ^4-type vertices and has at least one loop. Therefore, G_E has at least one 2-tree graph with 2-tree momentum other than p_1, p_2, p_3, or p_4. Thus, its Symanzik function $\tilde{f}$ for $p_1^2 = p_2^2 = p_3^2 = p_4^2 = 0$, namely, $\tilde{f}_0$ does not vanish identically.

It should be pointed out that the above proof breaks down in the fixed-momentum-transfer limit: $s = -(p_1+p_2)^2 \to \infty$, $t = -(p_1 - p_3)^2 =$ fixed. The reason is this: the dependence of the Symanzik function on s, t, and m_e^2 is given by

$$f = f_1 s + f_2 t + g m_e^2,$$

where f_1, f_2, and g are multilinear polynomials in the Feynman parameters. For this asymptotic limit of the Feynman integral to be mass independent, $\bar{f}_1$ must not vanish identically. But this is simply not true for all graphs—there are reduced graphs G_B which do not have p_1+p_2 as a 2-tree momentum. (The t-dependent one-loop graph is the simplest example of such a graph.)

V. OTHER FIELD THEORIES

The question naturally arises whether the validity of renormalization-group equations for S-matrix elements is a general feature of renormalizable field theories or perhaps just an accident for the φ^4 theory. The following remarks are speculative and are only intended to provide a conjectural answer to this question.

Consider a general renormalizable quantum field theory arising from a Lagrangian density of mass dimension four or less. Such a Lagrangian may involve fundamental fields of spin 0, $\frac{1}{2}$, or 1 (see Ref. 7). At the tree-graph level the T-matrix element for any two-particle process like $a+b\to a'+b'$ behaves at most like a constant at high energy and fixed wide angle. Moreover, this "constant" depends on the scattering angle and the helicities but not on the scale of the unrenormalized masses (the perturbation expansion is considered here as an explicit function of the unrenormalized quantities and the ultraviolet cutoff). In order to explore the possibility that this feature (i.e., the mass independence of the high-energy limit) continues to be true at the loop level, consider the *imaginary part* of the one-loop approximation which, by unitarity, is given by angular integrals over products of the form

$$\langle a'', b'' | T | a', b' \rangle^*_{\text{tree}} \langle a'', b'' | T | a, b \rangle_{\text{tree}}. \qquad (23)$$

It is clear that the asymptotic contribution of the angular range in which *both* factors represent wide-angle scatterings is again mass independent because each factor may be replaced by its wide-angle limit.

However, the situation may be different in the angular range in which one of the two factors is a small-angle process; for example, when the angle between the momentum of the initial particle a and the momentum of the intermediate particle a'' is of order $1/\sqrt{s}$ (where $\sqrt{s}$ is the center-of-mass energy squared), then the large-s behavior of the second factor in Eq. (23) is no longer a mass-independent constant. It depends on (unrenormalized) masses and it generally behaves like s^J where J is the spin of the "exchanged" particle in the tree approximation. These "unwanted" mass-dependent contributions to the unitarity integral are of order s^{J-1} and they are negligible[8] if $J<1$. One is thus led to the conjecture[9] that *if the Lagrangian involves only scalar or spinor fundamental fields the renormalization-group equations are valid for S-matrix elements at high energy and wide angle. In theories with vector fields the renormalization-group equations should hold only for the scattering of "neutral" particles, i.e., particles which carry no "charge" or "color" to which vector mesons couple.* As a consequence, in such cases S-matrix elements will have a power behavior (in the wide-angle limit) determined by the anomalous dimensions of the relevant fields—provided they approach ultraviolet stability.

In the presence of vector fields the situation is considerably more complicated for S-matrix elements with "charged" particles in the initial and/or final states. For example, in a theory of fermions coupled to vectors (e.g., massive QED) the fermion-fermion amplitude in fourth order in the coupling constant behaves, at high energy and wide angle, like the Born approximation times $\ln^2(s/M_V{}^2)$. Such "mass logarithms" are to be distinguished from the cutoff logarithms like $\ln(s/\Lambda^2)$ which are "organized" by the renormalization-group equations.[10]

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[1]M. Gell-Mann and F. E. Low, Phys. Rev. 95, 1300 (1954).

[2]K. Wilson, Phys. Rev. 179, 1499 (1969); K. Symanzik, Commun. Math. Phys. 23, 491 (1971); C. Callan, Phys. Rev. D 5, 3203 (1972).

[3]N. Christ, B. Hasslacher, and A. Mueller, Phys. Rev. D 6, 3543 (1972).

[4]E. Speer, J. Math. Phys. 9, 1404 (1968); E. Speer, *Generalized Feynman Amplitudes* (Princeton Univ. Press, Princeton, 1969).

[5]K. Hepp, Commun. Math. Phys. 2, 301 (1966); N. N. Bogoliubov and O. S. Parasiuk, Acta Math. 97, 227 (1957); O. S. Parasiuk, Ukr. Mat. Zh. 12, 287 (1960).

[6]S. Weinberg, Phys. Rev. 118, 838 (1960).

[7]A complete classification of all such theories which obey unitarity bounds at high energy in tree approximation was given by J. M. Cornwall, D. N. Levin, and G. Tiktopoulos, Phys. Rev. D 10, 1145 (1974); 11, 972(E) (1975).

[8]In ϕ^4 theory there is no particle-exchange tree graph,

but anyway the tree amplitude behaves like s^0.

[9]This conjecture is supported by calculations in lowest nontrivial order of perturbation theory.

[10]The problem of high-energy wide-angle scattering for vector theories and the related question of "organizing" the mass logarithms will be dealt with in a forthcoming paper in collaboration with J. M. Cornwall.