Poles with both magnetic and electric charges in non-Abelian gauge theory*

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We show that a non-Abelian gauge theory with Higgs fields exhibits classical solutions which are both electrically and magnetically charged. This represents a specific realization of the dyons discussed some years ago by Schwinger. At the classical level the electric charge of the dyon does not appear to be quantized. We present some remarks in this connection.

I. INTRODUCTION

The possible existence of magnetic monopoles has fascinated physicists since Dirac's classic work[1] of more than forty years ago. The interest in this subject[2] has been enhanced by a recent observation of 't Hooft[3] that classical solutions having the properties of magnetic monopoles may be found in a Yang-Mills theory[4] with spontaneous symmetry breaking and with a suitable identification of the electromagnetic field.

The magnetic charge is quantized according to Dirac's condition.[5] In particular, for an SU(2) gauge theory with a Higgs field[6] in the vector representation the magnetic charge g is quantized to have the value

$$g = 1/e . \tag{1.1}$$

In this paper, we would like to extend the discussion of 't Hooft by showing that it is also possible to construct classical solutions having both magnetic and electric charges. Particles with both magnetic and electric charges had been previously discussed by Schwinger[7] and by Zwanziger.[8] Following Schwinger, we will refer to such particles as dyons. In Sec. II we discuss the Yang-Mills equations of motion. In Sec. III the boundary conditions necessary are introduced. In Sec. IV the calculation of the energy of these dyon solutions is presented. In Sec. V we address ourselves to the question of whether the dyons are stable. We show that (at least for a range of parameter such that the charged vector meson mass is comparable to the Higgs meson mass) a dyon is energetically not allowed to decay into a magnetic monopole by emitting charged vector mesons. If there are fermions in the theory the dyon may capture a fermion to become a fermionic dyon. At the classical level the electric charge of the dyon is not quantized; the connection with the quantization of angular momentum is remarked upon. Appendix A contains some details of our numerical analysis. The calculation of the energy is further clarified in Appendix B. In Appendix C we discuss the introduction of an electric charge for the string.

II. EQUATIONS OF MOTION

Following 't Hooft we will consider an SU(2) gauge theory with a Higgs triplet. The equations of motion are

$$D^\mu \vec{F}_{\mu i} + e\vec{\phi}\times D_i\vec{\phi} = 0 , \tag{2.1}$$

$$D^\mu D_\mu \vec{\phi} - \mu^2\vec{\phi} + \lambda\vec{\phi}^2\vec{\phi} = 0 , \tag{2.2}$$

where the covariant derivative D_μ is given by

$$D_\mu = \partial_\mu + e\vec{A}_\mu \times .$$

This is to be supplemented by the constraint equation

$$D^\mu \vec{F}_{\mu 0} + e\vec{\phi}\times D_0\vec{\phi} = 0 . \tag{2.3}$$

These equations correspond to the Lagrangian

$$\begin{aligned}\mathcal{L} &= -\tfrac{1}{4}\vec{F}_{\mu\nu}\cdot\vec{F}^{\mu\nu} + \tfrac{1}{2}D_\mu\vec{\phi}\cdot D^\mu\vec{\phi} - V(\phi)\\ &= -\tfrac{1}{4}\vec{F}_{ij}\cdot\vec{F}_{ij} + \tfrac{1}{2}\vec{F}_{0i}\cdot\vec{F}_{0i}\\ &\quad - \tfrac{1}{2}D_i\vec{\phi}\cdot D_i\vec{\phi} + \tfrac{1}{2}D_0\vec{\phi}\cdot D_0\vec{\phi} - V(\phi) ,\end{aligned} \tag{2.4}$$

with

$$V(\phi) = -\frac{\mu^2}{2}\vec{\phi}^2 + \frac{\lambda}{4}(\vec{\phi}^2)^2$$

and

$$\mu^2 > 0 .$$

We consider a time-independent (in the sense that the gauge-covariant fields do not depend on time) solution of the form[9]

$$\begin{aligned}A_i^a &= \epsilon_{abi}\hat{x}_b\left(\frac{K(r)-1}{er}\right),\\ A_0^a &= \hat{x}_a J(r)/er ,\\ \phi^a &= \hat{x}_a H(r)/er .\end{aligned} \tag{2.5}$$

$\hat{x}_a$ is the unit vector $\vec{x}_a/r$.

This form solves the equations of motion and of constraint if the radial "wave functions" satisfy the coupled differential equations

$$r^2J'' = 2JK^2\,, \tag{2.6}$$

$$r^2H'' = 2HK^2 - \mu^2 r^2 H\left(1 - \frac{\lambda}{e^2\mu^2 r^2}H^2\right)\,, \tag{2.7}$$

$$r^2K'' = K(K^2 - J^2 + H^2 - 1)\,. \tag{2.8}$$

The case discussed by 't Hooft[3] corresponds to $J = 0$.

A few remarks are now appropriate. Firstly, notice that the form chosen satisfies the transversality conditions

$$\partial_i \vec{A}_i = 0\,.$$

Secondly, since $D_0\vec{\phi} = 0$ the Higgs field and A_0^a do not directly influence each other as is evidenced in Eq. (2.6) and Eq. (2.7). Thirdly, since the spatial dependence of A_0^a has the same form as that of ϕ^a and since $F_{i0}^a = (D_i\vec{A}_0)^a$ we see that the A_0^a components of the gauge field almost act like another isotriplet Higgs field in addition to ϕ^a [aside from the potential $V(\phi)$] had it not been for the fact that $(\vec{F}_{0i})^2$ and $(D_i\vec{\phi})^2$ appear with opposite signs in the Lagrangian. This particular circumstance is reflected in the fact that the terms J^2 and H^2 contribute with opposite signs in Eq. (2.8). This will be discussed further below. Fourthly, we have scaled out e so that the functions J, K, and H depend on the coupling constants only through the combination $\beta^2 \equiv e^2/\lambda$ as is evidenced by Eqs. (2.6), (2.7), and (2.8). Furthermore, the mass parameter μ may be absorbed into r.

In a local neighborhood of a point in space (not including the origin) the $\vec{\phi}$ field will point in a definite direction in isospin space and one of the three gauge fields will remain massless and hence is to be identified as the photon field $\vec{F}_{\mu\nu}$. A gauge-invariant definition[3] is

$$\mathfrak{F}_{\mu\nu} = \frac{1}{|\phi|}\,\phi^a F_{\mu\nu}^a - \frac{1}{e|\phi|^3}\epsilon_{abc}\,\phi_a D_\mu\phi_b D_\nu\phi_c\,, \tag{2.9}$$

where $|\phi| = (\vec{\phi}\cdot\vec{\phi})^{1/2}$. It is easily verified that if $\vec{\phi} = (0, 0, 1)$, say, then $\mathfrak{F}_{\mu\nu} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$. In the present case we find that the electric field is

$$\mathfrak{F}_{0i} = -\hat{x}_i\frac{d}{dr}\left[J(r)/er\right]. \tag{2.10}$$

The electric charge can be written as

$$\begin{aligned} Q &= \int dS_i\,\mathfrak{F}_{0i} \\ &= \int d^3x\,\partial_i\mathfrak{F}_{0i} \\ &= -\int d^3x\,\partial_i\left[\hat{x}_i\frac{d}{dr}\left(\frac{J(r)}{er}\right)\right] \\ &= -\frac{8\pi}{e}\int_0^\infty dr\frac{JK^2}{r}\,, \end{aligned} \tag{2.11}$$

where we have used the constraint equation (2.3) and Eq. (2.6). Thus, the charge is given by an expression of the form

$$Q = \left(\frac{\hbar c}{e^2}\right) e\,\zeta\left(\frac{\lambda}{e^2}, \frac{M}{\mu}\right), \tag{2.12}$$

where M is the scale of A_0^a and has the dimension of a mass. (This scale will be introduced below.) The function ζ can be determined only numerically. In contrast, the magnetic charge g is quantized exactly as in 't Hooft's solution since $K(r)\to 0$ at infinity and

$$\mathfrak{F}_{ij} \to \epsilon_{ija}\,\hat{x}_a\left(-\frac{1}{er^2}\right) \quad \text{as } r\to\infty\,.$$

By Gauss's theorem, $g = \hbar c/e$.

III. BOUNDARY CONDITIONS

As $r\to 0$ we impose $J\to 0$, $H\to 0$, $K\to 1$. Using Eqs. (2.6), (2.7), and (2.8) we find $J\to \text{const}\times r^2$, $H \to \text{const}\times r^2$, $K\to 1 + \text{const}\times r^2$; the fields are differentiable at the origin ensuring a finite-energy solution. As $r\to\infty$ the Higgs field approaches its "vacuum expectation value," that is, $H(r)_{r \simeq \infty}\beta\mu r + \cdots$. From Eq. (2.6) we see that if $K_{r\simeq\infty}0$ exponentially we may have $J(r)_{r\simeq\infty}Mr + b + \cdots$. M is a parameter with the dimension of a mass and sets the scale for J. The parameter b determines the charge. As can be seen from Eq. (2.11) the asymptotic form of Eq. (2.8) is then

$$r^2K'' = \left[(\beta\mu)^2 - M^2\right] r^2 K + \cdots$$

and admits one particular solution decreasing like e^{-ar}, where $a = [(\beta\mu)^2 - M^2]^{1/2}$. Thus we require $\beta\mu > M$. In particular, the hope that the Higgs field Φ could have been omitted cannot be realized for this type of solution. The A_0^a components of the gauge field act like an isotriplet Higgs field with negative metric, and by themselves would cause the other components of the gauge field A_i^a to oscillate rather than decrease exponentially as $r\to\infty$.

We have studied the nature of the solutions to Eqs. (2.6), (2.7), and (2.8). We have also integrated them numerically. The result is discussed in Appendix A, and the general form of the solutions is displayed in Fig. 1. We will just note here that the electric charge can assume a nonzero value. From Eq. (2.6), we see that starting with $J''(0) > 0$, J stays positive, and the electric charge given by Eq. (2.11) is nonzero. In the limit of small J on the effect of J on the equations is small, and thanks to the Higgs field, the solution should exist as in 't Hooft's example with $J = 0$.

The Higgs fields ensure that our solution has a finite energy but introduce two parameters λ and

μ which enter in the electric charge $\zeta(\lambda/e^2, M/\mu)$ Eq. (2.12). The electric charge is then apparently continuous; we will return to this problem.

IV. MASS OF THE DYON

The mass of the dyon is given by $\int d^3x T_{00}$ according to the principle of equivalence, where $T_{\mu\nu}$ is the covariant energy-momentum tensor coupled to the gravitational field. The "gravitational" Hamiltonian density in this case has the form

$$\mathcal{H}_g = \vec{F}_{0i}\cdot\vec{F}_{0i} + D_0\vec{\phi}\cdot D_0\vec{\phi} - \mathcal{L}\,.$$

This expression is gauge-invariant, of course. Explicitly, the mass of the dyon is

$$E = \int d^3x \mathcal{H}_g$$
$$= (4\pi M_W/e^2)\int_0^\infty dx\left[(K')^2 + (K^2-1)^2/2x^2 + \tfrac{1}{2}x^2(\Phi')^2 + \tfrac{1}{2}x^2(\Psi')^2 + K^2(\Phi^2+\Psi^2) + (x^2/4\beta^2)(1-\Phi^2)^2\right]\cdots, \qquad (4.1)$$

where $x = M_W r$, $M_W = \beta\mu$, $\Phi = H/M_W r$, $\Psi = J/M_W r$, and $f' = df/dx$. Some numerical examples are given in Appendix A. Dyons are not much more massive than magnetic monopoles. A dyon with an electric charge $Q \sim 137e$ is only about 40% heavier than the magnetic monopole (with $g = 1/e$).

In Appendix B we give some clarifying remarks on the relationship between the "gravitational" Hamiltonian and the "canonical" Hamiltonian.

V. STABILITY CONSIDERATIONS AND CONCLUDING REMARKS

't Hooft's monopole is guaranteed to be stable by topological arguments.[10] There is no such argument for our solution. We find apparently a solution for each M in a range of values $0 \leq M < \beta\mu$. To answer the question of stability we should treat the quantum problem. But that appears to be a complex task, given the presently available methods, without knowing the explicit form of the solution. We will only give a heuristic discussion based on simple energetic considerations. The electric field of the dyon will polarize the vacuum, producing W^+W^- pairs. Here W denotes the charged massive vector meson in the theory. (For ordinary electrodynamics the intensity of the field needed[11] to create a pair of mass m is enormous, of the order $e|\vec{E}| \sim m^2$.) One member of such a pair created far from the dyon will fall in towards the dyon giving up an amount of energy equal to $\sim Qe/R \sim \zeta m$ since the size of the dyon R is of order $1/m$, where m is a combination of M_W and μ; ζ was introduced in Eq. (2.12). On the other hand, it costs $2M_W$ to create the pair at infinity. As one member of the pair falls in towards the dyon it effectively loses its mass. Thus, we expect the dyon to be stable provided $\zeta m < M_W$. If the charge of the dyon measured in units of $1/e$ is small the dyon should be stable, at least against emission of W's. A dyon with a very high charge (in units of $1/e$) may indeed radiate off W mesons to lower its charge. This process cannot terminate in the pure monopole since the numerical computation (based on the choice of parameters $\beta^2 = 2$) presented in Appendix A shows that the mass of the dyon (in units of M_W) increases slowly with charge (in units of e). For example, the mass difference between a dyon with $Q \sim 137e$ and the magnetic monopole is only about $60\,M_W$. Thus, the decay processes

$$\text{dyon}(Q) \to \text{dyon}(Q-1) + W$$

and

$$\text{dyon}(Q) \to \text{magnetic monopoles} + (Q/e)W\text{'s}$$

are kinematically forbidden (at least for $\beta^2 = 2$). Our argument does not preclude the dyon from decaying via

$$\text{dyon}(Q) \to (\text{? bound state with } Q-1) + W\,,$$

where (? bound state with $Q-1$) represents some possible solution with electric and magnetic charge which we have not studied and which is unknown at present. In any case, there must be at least one state which has both electric and magnetic charge and which is stable. We thus suggest that the

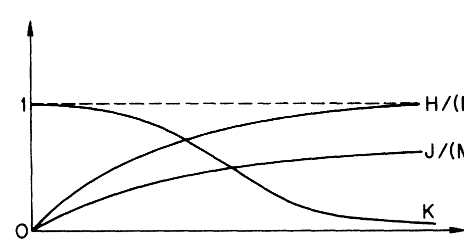

FIG. 1. Schematic representation of the behavior of $H(r)$, $J(r)$, and $K(r)$. The functions $H(r)$, $J(r)$, and $K(r)$ correspond, respectively, to the Higgs field, the time components, and the space components of the Yang-Mills field.

magnetic monopole may like to capture W mesons to convert itself into a dyon.

We may endow our solution with a fermionic quantum number in the following way. A fermion may be introduced into the theory in the standard manner. This fermion may then be put into orbit around the dyon and be trapped by the electric field and the Higgs field. If the fermion mass is large, then the dyon will be energetically stable against emission of antifermions.

Schwinger[5] and Zwanziger[8] had argued that if particle i has electric charge e_i and magnetic charge g_i the quantization condition of Dirac should be generalized to read

$$e_i g_j - e_j g_i = n = \text{an integer}.$$

This would imply that the electric charge of our solution should be

$$Q = ne,$$

or, equivalently,

$$\frac{1}{e^2}\zeta\left(\frac{\lambda}{e^2}, \frac{M}{\mu}\right) = n.$$

We see no evidence at the level of our calculation that the quantity on the left-hand side should be equal to an integer with the three parameters e^2, λ/e^2, and M/μ taking on arbitrary values. In fact, if for some reason $\zeta(\lambda/e^2, M/\mu)$ can only take on discrete values for any choice of λ/e^2 and M/μ and if the argument of Schwinger and Zwanziger is relevant for the present case, then one would apparently be faced with a mysterious condition saying that the theory will only make sense for some definite value of e^2. Of course, it may well be possible that some such quantization condition will emerge when one extends the present classical discussion to a quantum discussion.

In the argument[12] of Sahd and Wilson, the quantization condition emerges when the angular momentum of a magnetic monopole in motion around a dyon (or an electric charge) is quantized. In our case the electric charge distribution and the magnetic charge distribution are not separated. The angular momentum of the solution is clearly zero. While it is possible to envisage classical solutions with an electric charge in orbit around the dyon we do not yet know how to write down a solution representing a magnetic monopole and a dyon. Another suggestive connection between the angular momentum and the quantization of charge is described in Appendix C for the case of the Abelian Higgs model. In conclusion, the price for the introduction of the Higgs fields, which are necessary for the convergence of the energy, is a nonquantized electric charge. The nonlinearity of the Yang-Mills equations was not sufficient for our very symmetrical solution to ensure discrete values for the charge at the classical level. Besides the possibility of a quantization at the quantum level, it is of interest to look for less symmetrical solutions which could not be reduced to an "Abelian Dirac string" form in a specific gauge but about which very little is known at the present time.

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APPENDIX A

We did the numerical computation choosing the parameter $\beta^2 \equiv e^2/\lambda = 2$; all the parameters are then of order 1 and $M_W = \sqrt{2}\mu$. We have to solve a system of three coupled second-order differential equations. We may impose six boundary conditions

$$H(0) = J(0) = K(0) - 1 = 0, \tag{A1}$$

$$K(r) \to 0, \tag{A2}$$

$$\frac{J(r)}{r} - M \to 0, \tag{A3}$$

$$\frac{H(r)}{r} - \beta\mu \to 0, \tag{A4}$$

as $r \to \infty$. In practice we will impose $J''(0) = C$ and not Eq. (A3). We use a very elementary approximation procedure: Starting from the origin with a set of parameters $C = J''(0)$ fixed, $D = K''(0)$, $E = H''(0)$ arbitrary, we adjust them to fit the boundary conditions at infinity. The asymptotic form of Eqs. (2.7) and (2.8) gives the general solution $Ae^{-br} + Be^{+br}$ + const for H/r and K. Adjusting the parameters with a precision e^{-6} should give us a coefficient B of that order of magnitude, and in fact we find good approximate solutions for r up to $6/b$.

We give here two solutions ($\alpha \equiv e^2/4\pi\hbar c$ is the fine-structure constant):

(i) $Q = (0.324e)\alpha^{-1} \simeq 44e$, $E = 1.25 M_W \alpha^{-1} \simeq 171 M_W$,

(ii) $Q = (1.235e)\alpha^{-1} \simeq 169e$, $E = 1.85 M_W \alpha^{-1} \simeq 253 M_W$.

For comparison, note that for the monopole

(iii) $Q = 0$, $E = 1.18 M_W \alpha^{-1} \simeq 162 M_W$.

We may make two remarks:

(a) The convergence of our approximation procedure is about the same as for 't Hooft's monopole solution.

(b) The dependence of the dyon mass E on the charge is regular.

APPENDIX B

In the text we appealed to the principle of equivalence and used the time-time component of the energy-momentum tensor to compute the mass of the dyon. One may, however, also insist on considering the canonical Hamiltonian (in Coulomb gauge $\partial_i \vec{A}_i = 0$):

$$\mathcal{H}_c = \vec{F}^T_{0i} \cdot \partial_0 \vec{A}_i + D_0 \vec{\phi} \cdot \partial_0 \vec{\phi} - \mathcal{L} ,$$

where $\vec{F}_{0i} = \vec{F}^T_{0i} + \vec{F}^L_{0i}$ with $\partial_i \vec{F}^T_{0i} = 0$ and $\epsilon_{ijk} \partial_i \vec{F}^L_{0j} = 0$. (For a static solution $\mathcal{H}_c$ is equal to the negative of the Lagrangian.) The longitudinal components $\vec{F}^L_{0i}$ are not independent. Writing them in the gradient form $\vec{F}^L_{0i} = -\partial_i \vec{f}$, one may determine[13] $\vec{f}$ in terms of $\vec{A}_i \times \vec{F}^T_{0i}$. By using the constraint equation (2.3) one may show that the two Hamiltonians differ by the surface integral

$$\int d^3x (\mathcal{H}_g - \mathcal{H}_c) = - \int dS_i (\vec{F}_{0i} \cdot \vec{A}_0 + \vec{f} \cdot \partial_0 \vec{A}_i)$$

and thus generate the same dynamics (in our case).

In our solution $\partial_0 \vec{A}_i = \vec{0}$, but

$$\vec{F}_{0i} \cdot \vec{A}_0 = -\frac{1}{e^2} \hat{x}_i \frac{J(r)}{r} \frac{d}{dr}\left(\frac{J(r)}{r}\right) \to \frac{1}{e^2} \hat{x}_i \frac{Mb}{r^2} \text{ for } r \to \infty$$

and the surface term does not vanish. But it is gauge-dependent and unphysical; if we compute the work necessary to bring a small charge δQ from infinity to a finite distance, at constant potential A_∞ at infinity, the surface term cancels out. It is worth remarking that the surface term here is just QM.

Let us consider the dyon with electric charge Q and described by a classical solution of the form given by Eq. (2.5) with $A^a_0 \to M \hat{x}_a$ as $r \to \infty$. We will refer to this equilibrium state as (1). By a gauge transformation with the group parameters $\epsilon_a = e \delta M t \hat{x}_a \omega(r)$ [where $\omega(r)$ is a smooth function which $\to 1$ as $r \to \infty$] one may effect the following changes in the potentials at $r = \infty$:

$$\delta A^a_0 = \delta M \hat{x}_a , \quad \delta A^b_i = 0, \text{ and } \delta \phi^a = 0 .$$

This of course describes the same dyon; in this form $A^a_0 \to (M + \delta M) x_a$ as $r \to \infty$. Let us consider another state, to be referred to as (1′), which consists of this dyon plus an infinitesimal charge δQ at infinity (spread out over a thin spherical shell of infinite radius, say). We wish to compute the work necessary to bring the charge δQ in from infinity to form a dyon with charge $Q + \delta Q$, a state to be referred to as (2). This work is equal to the difference between the values of the Hamiltonian for the final state (2) and the initial state (1′), *keeping the boundary conditions fixed.* Using the Hamiltonian H_g we find the work W,

$$W = H_g(2) - H_g(1') = H_g(2) - H_g(1) . \tag{B1}$$

The last equality follows because the charge δQ does not contribute to the energy of the field. To argue this we picture the situation physically as a dyon located at the origin and surrounded by a thin shell of radius R and charge δQ. The infinitesimal increase in the field energy is $4\pi Q \delta Q / R \to 0$ as $R \to \infty$.

On the other hand, using the canonical but gauge-dependent Hamiltonian we find that the work necessary

$$W' = H_c(2) - H_c(1') , \tag{B2}$$

but $H_c(1') = H_g(1') + (Q + \delta Q)(M + \delta M)$, where we have used the previously mentioned fact that the difference between the two Hamiltonians is equal to the product of the total charge and the value of the potential at infinity. Thus, using $H_c(2) = H_g(2) + (Q + \delta Q)(M + \delta M)$ we find $W' = W$ as desired.

The prescription to keep the boundary conditions fixed is crucial in the derivation of the equations of motion, and also in the elimination of the superfluous degrees of freedom, when we express A^a_0 in terms of $\vec{A}_k \times \vec{E}_j$. In the present case the boundary values are nonvanishing and the necessity of keeping them fixed is especially apparent.

APPENDIX C

In this appendix we present some remarks similar to those in the text but for the case of ar Abelian gauge theory. The theory to be considered is scalar electrodynamics:

$$\mathcal{L} = (D_\mu \phi)(D^\mu \phi)^\dagger - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mu^2 \phi \phi^\dagger - \frac{\lambda}{2} (\phi \phi^\dagger)^2 \cdots , \tag{C1}$$

where $D_\mu \phi = (\partial_\mu - ieA_\mu)\phi$ and $\mu^2 > 0$. Nielsen and Olesen[14] have pointed out that this theory admits a stringlike classical solution with finite energy per unit length. We would like to ask if a solution of the form

$$A_\mu = (h(r), -y f(r), x f(r), 0)$$

and

$$\phi = \rho(r) e^{in\theta}$$

(r = radius and θ = azimuthal angle in cylindrical coordinates) can exist. [In the solution of Nielsen and Olesen[14] $h(r) = 0$.] With a nonzero A_0 the solution contains an "electric" field $E_i = F_{0i} = -\partial_i A_0 = -\hat{r}_i h'$. (Here $\hat{r}$ is the radial unit vector and $h' = dh/dr$.) Of course, in this theory all the vector mesons are massive and E_i is not an electric field in the usual sense. Nevertheless, the angular momentum density is still given by $\vec{E}\times\vec{B}$ and the solution would have a spin. We will conclude that this situation implies infinite energy per unit length. The covariant energy-momentum tensor is

$$T_{\mu\nu} = (D_\mu\phi)(D_\nu\phi)^\dagger + (D_\mu\phi)^\dagger(D_\nu\phi) - F_{\mu\lambda}F_\nu{}^\lambda - g_{\mu\nu}\mathcal{L}\,.$$

The angular momentum is given by

$$J_k = \epsilon_{ijk}\int d^3x(x_i T_{0j} - x_j T_{0i})\,.$$

By using the equation of constraint

$$\partial_m F_{0m} + ie\phi^\dagger D_0\phi - ie(D_0\phi)^\dagger\phi = 0 \tag{C2}$$

and by integrating by parts, we find that

$$J_k = \epsilon_{ijk}\int d^3x\{x_i[D_0\phi\partial_j\phi + (D_0\phi)^\dagger\partial_j\phi] + x_i F_{0m}\partial_j A_m + F_{0i}A_j\} - \epsilon_{ijk}\int d^3x\partial_m(x_i A_j F_{0m})\,. \tag{C3}$$

The last term in Eq. (C3) is a surface term and for $k = 3$ has the explicit form

$$-\int d^3x\partial_m(\hat{r}_m r^2 f h')\,.$$

This may be dropped since the boundary conditions are $f\to$ constant, $h\to$ constant, $\rho\sim$ constant r^n as $r\to 0$, and $\rho\to(\mu^2/\lambda)^{1/2}$, $h\to(\pi/2\zeta)^{1/2}e^{-\zeta}$, $\zeta\equiv(2\mu^2/\lambda)^{1/2}er$, $f\sim n/er^2$ as $r\to\infty$. The first term in Eq. (C2) is just the canonical expression for the angular momentum and for $k = 3$ has the explicit form

$$J_z = 2\pi\int dz\int_0^\infty dr\, r(-2enh\rho^2)\,. \tag{C4}$$

The combination $(-2e^2h\rho^2)$ may be recognized as precisely the "charge" density, however:

$$\partial^\mu F_{\mu 0} = -2e^2h\rho^2\,. \tag{C5}$$

Thus, we find a relation between the angular momentum and the charge (per unit length if one prefers):

$$J/Q = n/e\,. \tag{C6}$$

This would connect the quantization of charge and angular momentum. We note, however, that a solution with nonzero charge must necessarily have infinite energy (per unit length) since $Q = \int d^3x\vec{\nabla}\cdot\vec{E} = \int_{S(\infty)}d\vec{S}\cdot\vec{E} - \int_{S(0)}d\vec{S}\cdot\vec{E}$, where $S(\infty)$ and $S(0)$ denote cylindrical surfaces at $r=\infty$ and at $r=\epsilon$ $(\epsilon\to 0)$, respectively. Since $\vec{E}\to 0$ exponentially as $r\to\infty$, the charge is nonzero only if $|\vec{E}|\sim 1/r$ as $r\to 0$, i.e., $h\sim\log r$ as $r\to 0$. In that case the string has infinite energy per unit length.

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