Non-Abelian gauge theories of strong interactions and spectral-function sum rules*

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We present a diagram technique to isolate the leading singularities in the vacuum expectation values of current products, thereby making it possible to study the question of convergence of spectral-function sum rules associated with the currents. Working within asymptotically free theories, we derive certain sum rules, notable among them being Weinberg's first and second sum rules, which are shown to be valid if the strong interaction is globally SU(2) × SU(2) invariant.

I. INTRODUCTION

Sum rules[1,2] involving integrals over the current spectral functions have, in recent years, provided a useful tool in studies of low-energy hadronic processes, most notable among them being the successful calculation of the π^+-π^0 mass difference by Das *et al.*[3] and the prediction of the A_1 mass[1] by Weinberg. Derivations of the sum rules,[4] however, have always been based on phenomenological assumptions such as asymptotic symmetry or the possible nature of Schwinger terms, etc., and have never been altogether satisfying. This is, of course, largely due to our ignorance about the nature of strong-interaction physics. Some years ago, using the method of short-distance expansion of operator products, Wilson[5] discussed the question of convergence of these sum rules and their derivation and, in particular, showed that the spectral sum rules involving the ρ-A_1 combination used in the π^+-π^0 mass difference calculation can be derived in the exact SU(2)×SU(2) limit of strong interactions,[6] independent of its detailed nature. With the recent progress in gauge theories, it has been suggested[7] that asymptotically free "color" gauge theories may be used to describe strong-interaction processes. It is therefore of interest to investigate the validity of the spectral sum rules in these theories. For this purpose, we will use Wilson's method to isolate the "geometrical" structure of the most singular terms in the short-distance expansion of current products. We will present a diagram technique which helps us to write down the geometrical structure in asymptotically free theories. We will then be able to examine the validity of the various existing sum rules and derive some new ones.

II. THE WILSON PRESCRIPTION AND THE DIAGRAM TECHNIQUE

Let us define the current propagator function as follows:

$$\Delta_{\mu\nu}^{ij}(q,V)=i\int e^{iq\cdot x}d^4x\langle 0|\,T\{V_\mu^i(x)V_\nu^j(0)\}|\,0\rangle$$
$$=\int_0^\infty \frac{dm^2}{q^2+m^2}\left[\delta_{\mu\nu}\rho_1^{ij}(m^2,V)+\frac{q_\mu q_\nu}{m^2}\rho_2^{ij}(m^2,V)\right]$$
$$-\delta_{\mu 4}\delta_{\nu 4}\int dm^2\frac{\rho_2^{ij}(m^2,V)}{m^2}. \qquad (1)$$

$\Delta_{\mu\nu}^{ij}(q,A)$ is defined in a similar manner.

Derivation of the so-called first and second sum rules require that the suitable linear combination of Δ's vanish respectiv^ly as follows for large q^2:

$$\lim_{q^2\to\infty}\sum_{\substack{i,j\\ a=V,A}} C_{ij}^{(a)}\Delta_{\mu\nu}^{ij}(q,a)=0 \qquad (2)$$

and

$$\lim_{q^2\to\infty} q^2\sum_{\substack{i,j\\ a=V,A}} C_{ij}^{(a)}\Delta_{\mu\nu}^{ij}(q,a)=0\,. \qquad (3)$$

To estimate the asymptotic behavior of Δ's, observe that in the short-distance expansion of the T product only I, U_0, U_8, and U_{15} [since we are working within an SU(4) framework] will contribute, i.e.,

$$\langle 0|\,T\{V_\mu^i(x)V_\nu^j(0)\}|\,0\rangle$$
$$\underset{x\sim 0}{\simeq} H_{\mu\nu}(x)+H_{\mu\nu}^{(1)}(x)\langle 0|\,U_0|\,0\rangle$$
$$+H_{\mu\nu}^{(2)}(x)\langle 0|\,U_8|\,0\rangle+H_{\mu\nu}^{(3)}(x)\langle 0|\,U_{15}|\,0\rangle\,. \qquad (4)$$

Next, we observe that since the V_μ^i transform as (1, 15) + (15, 1) under SU(4)×SU(4), and also since we are interested in the vacuum expectation values, the representations that survive in $\Delta_{\mu\nu}$ are (1, 1), (1, 15) + (15, 1), (15, 15), and (84, 1) + (1, 84). The other representations contained in this product are (1, 20″) + (20″, 1) and (1, 45) + (45, 1), which, of course, have zero vacuum expectation value. On the other hand, I, U_0, U_8, and U_{15} belong to the (1, 1) and (4, 4*) + (4*, 4) representations of SU(4)×SU(4).

Notice, first of all, that if SU(4)×SU(4) were an exact symmetry of the Hamiltonian (not necessarily preserved by the vacuum), then the operator-product expansion (OPE) would have to preserve this symmetry and therefore U_0, U_8, and U_{15} would never appear in the OPE. Then the difference of any two Δ's would go asymptotically faster than $1/q^{2+\epsilon}$ and one would obtain the first and second spectral sum rules for all of them. SU(4) $\times$SU(4) is, however, not a symmetry of the strong-interaction Hamiltonian, and is broken by quark mass terms, which transform like $(4, 4^*)+(4^*, 4)$ under it, i.e.,

$$\mathcal{H}=\mathcal{H}_{\rm inv}+\epsilon_0 U_0+\epsilon_8 U_8+\epsilon_{15}U_{15}\equiv\mathcal{H}_{\rm inv}+\bar{q}\mathfrak{M}q\ . \quad (5)$$

Therefore, to get the singularity of the $H_{\mu\nu}$'s, we must do a spurion analysis in the manner of Wilson. ($\bar{q}\mathfrak{M}q$ denotes the quark mass term.)

A. First sum rules

Note that since currents have dimension three and U's have dimension three in asymptotically free theories for small x^2, $H_{\mu\nu}^{(i)}$ $(i=1, 2, 3)$ have singularity x^{-3} and therefore their contribution to Δ's is convergent enough to yield the first sum rule. (Their contribution to Δ's goes like q^{-1}.) We will therefore concentrate our attention on $H_{\mu\nu}(x)$ to see which of the first sum rules are convergent and which are not. $H_{\mu\nu}(x)$ is the coefficient of I in the OPE and therefore for a linear combination of Δ's which transforms like $(15, 1)+(1, 15)$ or $(15, 15)$ under SU(4)$\times$SU(4) [denoted by $\Delta_{\mu\nu}^{ij;15}(q, V)$] one must take the mass operator twice along with the currents. Therefore, to isolate the "geometrical" structure of the singularity we have to look at the diagrams listed in Figs. 1 and 2, and we find

$$\begin{aligned}\Delta_{\mu\nu}^{ij;15}(q^2; V) \underset{q^2\to\infty}{\sim} & K_{\mu\nu}^{(a)}(q)\,{\rm Tr}(\lambda_i\mathfrak{M}\lambda_j\mathfrak{M})\\ &+K_{\mu\nu}^{(b)}(q)\,{\rm Tr}(\lambda_i\mathfrak{M}^2\lambda_j)\\ &+K_{\mu\nu}^{(c)}(q)\,{\rm Tr}(\lambda_i\lambda_j\mathfrak{M}^2)\\ &+L_{\mu\nu}^{(a)}(q)\delta_{i0}\,{\rm Tr}(\lambda_j\mathfrak{M}^2)\\ &+L_{\mu\nu}^{(b)}\delta_{j0}\,{\rm Tr}(\lambda_i\mathfrak{M}^2)\ ,\end{aligned}$$

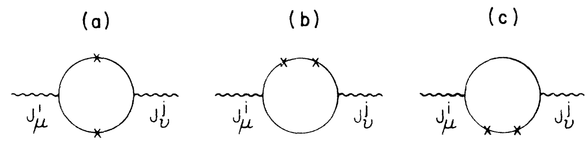

FIG. 1. Lowest-order diagrams contributing to the leading singularity of H. The crosses stand for the mass operators.

$$\begin{aligned}\Delta_{\mu\nu}^{ij;15}(q; A) \underset{q^2\to\infty}{\sim} & -K_{\mu\nu}^{(a)}(q)\,{\rm Tr}(\lambda_i\mathfrak{M}\lambda_j\mathfrak{M})\\ &+K_{\mu\nu}^{(b)}(q)\,{\rm Tr}(\lambda_i\mathfrak{M}^2\lambda_j)\\ &+K_{\mu\nu}^{(c)}(q)\,{\rm Tr}(\lambda_i\lambda_j\mathfrak{M}^2)\\ &+L_{\mu\nu}^{(c)}(q)\delta_{i0}\,{\rm Tr}(\lambda_j\mathfrak{M}^2)\\ &+L_{\mu\nu}^{(d)}\delta_{j0}\,{\rm Tr}(\lambda_i\mathfrak{M}^2)\ ,\end{aligned} \quad (6)$$

where $\bar{q}\mathfrak{M}q=\epsilon_0U_0+\epsilon_8U_8+\epsilon_{15}U_{15}$.

$K^{(i)}(q)$ are polynomials in $\ln q^2$ and denote the contributions of diagrams (a), (b), and (c) of Fig. 1. Only the lowest-order diagrams are listed and one must include all the radiative corrections (see Fig. 2 for some typical low-order graphs). It is easy to see that since radiative correction involves only the "color" direction, they do not alter the geometric structure of the graphs. The $L_{\mu\nu}^{(i)}$ terms, which have different geometric structure than the $K_{\mu\nu}^{(i)}$ terms, arise purely as a result of radiative correction [see Fig. 2(c)] and therefore vanish as $1/(\ln q^2)^\epsilon$ as $q^2\to\infty$. These terms, therefore, do not affect the first sum rules written below. The $\ln q^2$ dependence comes from the radiative correction after renormalization. Equation (6) therefore clearly displays the group structures associated with $\Delta_{\mu\nu}$ and will have to be analyzed so we can see which of the first sum rules can be derived. Without any further assumptions, we can derive only the following sum rules:

$$\int\frac{\rho_2^{8,15}(m^2, V)-\rho_2^{8,0}(m^2, V)/\sqrt{3}}{m^2}dm^2=0\ , \quad (7a)$$

$$\int\frac{\rho_2^{88}(m^2, V)-\rho_2^{15,15}(m^2, V)+\sqrt{2}\,\rho_2^{8,15}(m^2, V)-\sqrt{3}\,\rho_2^{0,15}(m^2, V)}{m^2}dm^2=0\ . \quad (7b)$$

We also have two more equations by replacing V and A in Eqs. (7a) and (7b). If we assume that chiral SU(2)$\times$SU(2) is an exact symmetry of strong interactions, then we have for the quark mass matrix

$$\mathfrak{M}=\begin{bmatrix}0 & & & \\ & 0 & & \\ & & m_\lambda & \\ & & & m_{\wp'}\end{bmatrix}\ .$$

So, clearly, we can obtain [as can be easily checked using Eq. (6)]

$$\int \frac{\rho^{11}(m^2, V) - \rho_2^{11}(m^2, A)}{m^2} dm^2 = 0 \tag{8a}$$

and also

$$\int \frac{\rho_2^{44}(m^2, V) - \rho_2^{44}(m^2, A)}{m^2} dm^2 = 0 . \tag{8b}$$

We also see that the first sum rule often used in the literature,[2]

$$\int \frac{\rho_2^{ij}(m^2, V)}{m^2} dm^2 = a\delta_{ij} , \tag{9}$$

is not valid in asymptotically free theories.[8]

B. Second sum rule

Now, it is easy to convince oneself without much difficulty that to obtain the second sum rules one has to analyze both the sets of diagrams listed in Figs. 1–4. Figures 1–3 give us the next leading singularity of $H_{\mu\nu}(x)$ and Fig. 4 provides us with the relevant singularity of $H_{\mu\nu}^{(i)}(x)$ $(i = 1, 2, 3)$. It is easy to convince oneself that the diagrams of Figs. 3 and 4 go asymptotically like $(\ln q^2)^\gamma/q^2$ and therefore if from their geometrical structure we find that for certain combinations of Δ's the diagrams of Figs. 3 and 4 give zero then they will certainly satisfy the second spectral sum rule, i.e.,

$$\int dm^2 \sum_{\substack{i,j \\ a=V,A}} C_{ij}^a \rho_1^{ij}(m^2, a) = 0 . \tag{10}$$

We give some examples: If we take[9] $J_\mu^i = \bar{q}\lambda_8\gamma_\mu q$ and $J_\nu^i = \bar{\mathcal{P}}'\gamma_\nu\mathcal{P}'$, then it is clear that only Figs. 2(c) and 4(c) will contribute in giving the leading singularity. If we ignore Fig. 2(c) for a moment, we find from Fig. 4(c) that it is proportional to the strong gauge coupling $\bar{g}(q^2)$ and therefore vanishes for large q^2 as $1/q^2(\ln q^2)^\gamma$ in asymptotically free theories, giving us the following sum rule:

$$\int [\rho_1^{8,15}(m^2, V) - \rho_1^{8,0}(m^2, V)/\sqrt{3}\,]dm^2 = 0 . \tag{11a}$$

However, if we do not ignore Fig. 2(c), Eq. (11a) will not be valid.

Similarly, we find that in the exact SU(2)×SU(2) limit (i.e., $m_{\mathcal{P}} = m_{\mathfrak{N}} = 0$), we obtain the second Weinberg sum rule, i.e.,

$$\int [\rho_1^{11}(m^2, V) - \rho_1^{11}(m^2, A)]dm^2 = 0 . \tag{11b}$$

C. Sum rules for spectral functions and pseudoscalar densities

The above techniques can of course be applied to spectral functions for scalar and pseudoscalar

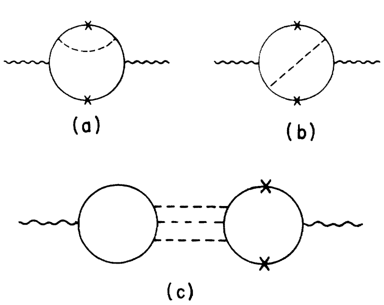

FIG. 2. Examples of diagrams contributing to the next leading singularity of H. We exhibit only some typical radiative corrections. The dashed line denotes the "color" gluon.

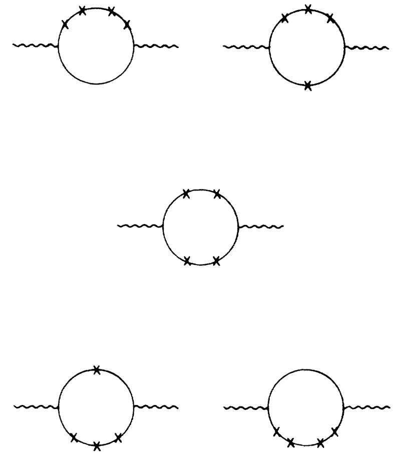

FIG. 3. Lowest-order diagrams relevant to the validity of the second sum rule.

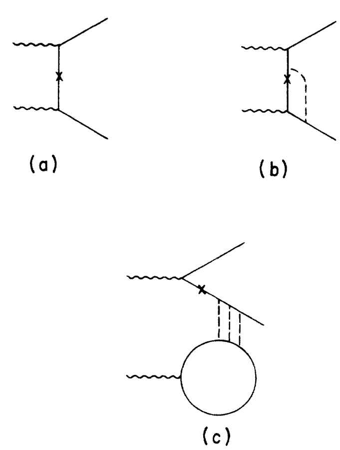

FIG. 4. Lowest-order diagrams giving the leading singularity of $H_{\mu\nu}^{(i)}$ $(i = 1, 2, 3)$.

(S or P) densities[10] defined and given by

$$\Delta^{ij}(q,S) = i\int e^{iq\cdot x}d^4x\langle 0|\,T\{U^i(x)U^j(0)\}|\,0\rangle\ , \tag{12}$$

and similarly for $\Delta^{ij}(q,P)$ for P densities $V^i(x)$. Here, however, more diagrams will contribute, since U^i and V^i transform as $(4,4^*)+(4^*,4)$ under $SU(4)\times SU(4)$. The conditions for deriving useful sum rules in this case are much more stringent, i.e., the suitable linear combination of Δ^{ij}'s must satisfy both the following constraints:

$$\lim_{q^2\to\infty}\sum_{\substack{i,j\\ b=S,P}} C^b_{ij}\Delta^{ij}(q,b)=0\ , \tag{13a}$$

$$\lim_{q^2\to\infty} q^2\sum_{\substack{i,j\\ b=S,P}} C^b_{ij}\Delta_{ij}(q,b)=0\ . \tag{13b}$$

Therefore, one must consider diagrams with one, two, three, and four mass operators as well as diagrams of the type shown in Fig. 4, with currents replaced by densities. It is easy to convince oneself that, in the exact $SU(2)\times SU(2)$ limit, owing to the special form of the mass operator one has to consider only the diagrams shown in Fig. 5 (all other diagrams being zero) if we look at $\Delta^{33}(q, S \text{ or } P)$ and $\Delta^{-1,-1}(q, S \text{ or } P)$ where

$$\lambda_{-1}=\begin{bmatrix}1&&&\\&1&&\\&&0&\\&&&0\end{bmatrix},$$

and we find

$$\lim_{q^2\to\infty} q^2[\Delta^{33}(q,S)-\Delta^{33}(q,P)]=0 \tag{14a}$$

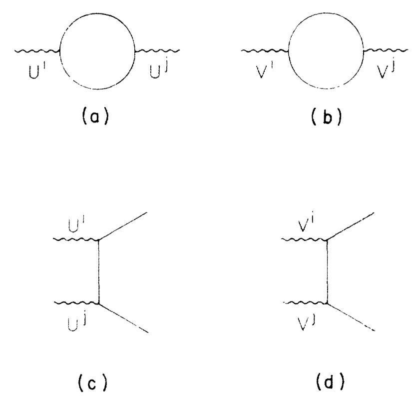

FIG. 5. The lowest-order diagrams contributing to the leading singularity in $\Delta^{33}(q,S)$ and $\Delta^{-1,-1}(q,P)$.

and

$$\lim_{q^2\to\infty} q^2[\Delta^{-1,-1}(q,S)-\Delta^{-1,-1}(q,P)]=0\ , \tag{14b}$$

from which we get the sum rules for the corresponding spectral functions.

III. CONCLUSION

In conclusion, we have presented diagram techniques to isolate the "geometrical" coefficients associated with the leading singularities in the OPE of current products in asymptotically free theories. Knowledge of these helps us to choose the correct linear combination for the spectral functions and derive sum rules corresponding to their zeroth and first moments, the so-called first and second spectral sum rules. We have illustrated our methods by giving simple examples and also by pointing out that certain sum rules widely used in the literature cannot be valid in asymptotically free theories.

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[6]See S. Borchardt and V. S. Mathur, Phys. Rev. D 9, 2371 (1974), for a discussion of this and other sum rules.

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[8]Note that in the algebra-of-fields model of T. D. Lee, S. Weinberg, and B. Zumino [Phys. Rev. Lett. 18, 1029 (1967)] one can derive this sum rule, since one finds explicitly that Schwinger terms here are c numbers.

[9]Of course, $\bar{\mathcal{P}}'\gamma_\mu\mathcal{P}'$ is a linear combination of currents belonging to the $(15,1)+(1,15)$ and $(1,1)$ representations under $SU(4)_L\times SU(4)_R$. That is why one can close one loop in Fig. 4(c).

[10]The densities U^i and V^i are defined as follows:

$$U^i=\tfrac{1}{2}\bar{q}\lambda_i q,\quad V^i=\tfrac{1}{2}i\bar{q}\lambda_i\gamma_5 q.$$