High-energy scattering of spinning particles by external fields and "exponentiation"

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A semiclassical description allows a simple geometrical interpretation of the condition ($g = 2$) found by Weinberg, generalizing earlier results of Cheng and Wu, and Meng, for "exponentiation" of the scattering amplitude for high-energy particles by external electromagnetic fields. The same argument shows that the presence of any nonzero electric dipole moment is inimical to exponentiation. Similar discussion of scattering of isospin-bearing particles by a given Yang-Mills field leads to the conclusion that exponentiation cannot be expected to occur for that case.

I. INTRODUCTION

Scattering of high-energy particles in external electromagnetic fields has recently received considerable attention[1-3] as a calculable model in which one may test the validity of the eikonal approximation. Following earlier work by Pauli[4] and Moliére,[5] it was noted by Cheng and Wu[1] that the scattering of an electron (obeying the Dirac equation) by a given slowly varying electromagnetic field may be well described by the eikonal approximation in the high-energy limit, in which case the scattering assumes a simple exponential form. It was subsequently remarked[1] that this property is lost if the "electron" is allowed to possess an anomalous (Pauli) magnetic moment while for a spin-1 particle, exponentiation does not occur for the "normal" magnetic moment,[2] corresponding to $g=1$, but only if the particle possesses an additional "anomalous" moment[3] of equal magnitude. Weinberg[6] combined and extended these results by showing that, for particles possessing no other internal degrees of freedom than the spin, exponentiation always occurs (for any spin value) if the gyromagnetic ratio assumes the particular value $g=2$. In this paper we argue that decoupling of various spin channels, and hence "exponentiation," can be simply seen from a semiclassical treatment of spin motion in an external field, for which the relevant formulas have been derived a long time ago.[7,8]

This paper is organized as follows. In Sec. II, a semiclassical criterion for "exponentiation" is developed employing the approach of Refs. 7 and 8. This criterion is applied to the case of homogeneous external electromagnetic fields, and the condition ($g=2$) for decoupling of spin channels is obtained for particles with arbitrary spin. It is also argued that when the particle possesses (besides the magnetic) an electric dipole moment, such a decoupling ("exponentiation") is impossible. Section III treats the case of scattering by an external non-Abelian (Yang-Mills) gauge field. Both the spin and isospin precessions are discussed. Our conclusions are summarized in Sec. IV.

II. EXTERNAL ELECTROMAGNETIC FIELD

First, let us recall some of the results of Ref. 7 using the same notation. These authors show that one can derive an "equation of motion" for the spin polarization vector when the particle moves in a given manner along a classical trajectory. Since one gets the classical trajectory in the limit $\hbar=0$ and the spin effects appear in the next approximation with respect to $\hbar$, we can see that in such a quasiclassical description the spin motion does not feed back into the trajectory equation. It therefore makes sense to ask the question of how the spin behaves when the particle follows a given quasiclassical trajectory in an external field. Our further discussion is based on the conjecture that such a quasiclassical description may be used to furnish criteria for the occurrence of "exponentiation" *in the high-energy limit* when the wavelengths associated with the particle motion are very short and the conditions of applicability for such a quasiclassical description are satisfied.

Following Refs. 7 and 8, the axial four-vector $s=(s_0,\vec{s})$ representing the polarization of a particle with charge e, mass m, and magnetic and electric dipole moments $ge/2m$ and $g'e/2m$ (with units $\hbar=c=1$), respectively, moving in a homogeneous external field $F=-(\vec{E},\vec{H})$ satisfies the equation

$$\frac{ds}{d\tau}=\frac{e}{m}\{[\tfrac{1}{2}gF\cdot s+\tfrac{1}{2}(g-2)(s\cdot F\cdot u)u] -\tfrac{1}{2}g'[F^{*}\cdot s+(s\cdot F^{*}\cdot u)u]\}\ , \tag{1}$$

where τ is the proper time and $u=(\gamma,\gamma\vec{v})$ is the four-velocity [with $\gamma=(1-v^2)^{-1/2}$]. F^* is the dual of F: $F^*=(-\vec{H},\vec{E})$. In obtaining (1), we made use[7] of the classical equation of motion for the orbit

$$\frac{du}{d\tau}=\frac{e}{m}F\cdot u\,, \tag{2}$$

which, together with Eq. (1), implies that $s\cdot u$ and $s\cdot s$ are constants. In the particle's instantaneous rest frame (R), Eq. (1) reduces to

$$\frac{d\vec{s}}{d\tau}=\frac{e}{2m}\{g[\vec{s}\times\vec{H}]+g'[\vec{s}\times\vec{E}]\} \quad (R) \tag{3}$$

and

$$\frac{ds_0}{d\tau}=\vec{s}\cdot\frac{d\vec{v}}{d\tau} \quad (R)\,. \tag{4}$$

Equation (3) is the standard nonrelativistic equation of motion for the spin, while Eq. (4) expresses the condition assumed in all classical treatments of spin (see, e.g., Ref. 9, Sec. 5) that the four-vector s has components $(0,\vec{s})$ in the particle's rest frame so that in an arbitrary Lorentz frame, it satisfies the relations $s\cdot u=0$ or

$$\vec{s}\cdot\vec{v}=s_0\,, \tag{5}$$

which in turn yields Eq. (4) in the particle's rest system R ($\vec{v}=0$).

It is convenient to refer the spin motion to a (rotating) coordinate system in R. The spin can be covariantly expressed through

$$s=\zeta_1e_1+\zeta_2e_2+\zeta_3e_3\,, \tag{6}$$

where we have defined

$$e_1=\gamma(v,\hat{v}) \quad (L)\,, \tag{7}$$

and $\hat{v}$ is a spatial unit vector along the direction of motion in the laboratory frame (L); e_2 and e_3 are a pair of mutually orthogonal spacelike unit vectors, each of which is orthogonal to both e_1 and u. The three spacelike unit vectors e_1, e_2, e_3 form an orthogonal basis in the rest frame R of the particle, and the spacelike components ζ_i ($i=1,2,3$) of s in this basis form a vector $\vec{\zeta}$ which specifies the spin in this coordinate system. The rate of transformation of longitudinal polarization into transverse (and vice versa) as the particle moves through a homogeneous electromagnetic field can be found by substituting Eq. (6) into Eq. (1) and making some straightforward manipulations.[7,8] Defining $a=\frac{1}{2}(g-2)$, we find

$$\frac{m}{e}\frac{d\zeta_\perp}{dt}=a\vec{\zeta}\cdot[\vec{H}\times\hat{v}]+\tfrac{1}{2}g'\{\vec{\zeta}\cdot[\vec{E}\times\hat{v}]+v\vec{\zeta}_\perp\cdot\vec{H}\} + \left(\frac{1}{v\gamma^2}-av\right)\vec{\zeta}_\perp\cdot\vec{E} \quad (L)\,, \tag{8}$$

where $\vec{\zeta}_\perp$ is the component of $\vec{\zeta}$ orthogonal to $\hat{v}$. Thus we see that in the high-energy limit ($\gamma\to\infty$), the spin polarization preserves its inclination to the direction of motion if $a=g'=0$, i.e., $g=2$ and $g'=0$, for an arbitrary homogeneous electromagnetic field. In a quantum description this corresponds to a decoupling of different spin states in the helicity basis, which is precisely the condition we require for exponentiation to occur. Equation (8) therefore provides a semiclassical criterion for exponentiation, thereby confirming and extending the result of Weinberg. Exponentiation is expected in the high-energy limit only for $g=2$, independent of the magnitude of the spin, while the possession of a nonvanishing electric dipole moment, $g'\neq 0$, destroys it immediately. The last property can be checked rather easily for the case of spin $\frac{1}{2}$ by following the procedure of Cheng and Wu.

III. YANG-MILLS FIELD

In view of its intrinsic interest and possible future importance, we extend our criteria for the decoupling of spin channels in electrodynamics to the case of spin- and isospin-bearing particles interacting with an external c-number Yang-Mills field.[10,11] Since they now possess both ordinary spin and isospin, our particles will now exhibit two kinds of precession. The spin precession is quite similar to that which takes place in an electromagnetic field, which we considered in the last section. The precession of the isospin is, however, rather different and we shall discuss it later.

To write the analog of Eq. (1), one requires the equation of motion for the spin in the (R) system and the equation of motion for the classical trajectory of the particle moving through the given Yang-Mills field. The second equation has already been given by Wong[11]:

$$\frac{du}{d\tau}=\frac{G}{m}(\underline{f}\cdot\underline{I})u\,, \tag{9}$$

where G is the coupling constant (analogous to e), u is the four-velocity as before, and $\underline{I}$ is the isospin vector, while

$$\underline{f}_{\mu\nu}=\partial_\mu\underline{b}_\nu-\partial_\nu\underline{b}_\mu+G\underline{b}_\mu\times\underline{b}_\nu\,, \tag{10}$$

where $\underline{b}_\mu$ is the Yang-Mills field which is a four-vector in space-time and a three-vector in isospin space. Note that Eq. (9) is identical in form to Eq. (2) if one identifies

$$F=\underline{f}\cdot\underline{I}=-(\vec{\underline{e}}\cdot\underline{I},\vec{\underline{h}}\cdot\underline{I})\,. \tag{11}$$

It remains to derive the equation of motion for the spin. For a consistent interpretation, the time component of s must satisfy Eq. (4) as in the previous case, while the three-vector $\vec{s}$ presumably satisfies an equation similar to (3). However, in contrast to the well-known case of electromagnetic interactions, we have no way of knowing it except from a specific theory. We shall adopt the procedure, which gives the correct answer in the electromagnetic case, of finding the equation for the spin precession in (R) by taking the Dirac equation for a spin-$\frac{1}{2}$ particle and proceeding to the (spin-dependent) nonrelativistic limit. By analogy with the electromagnetic case, the resultant equation will be assumed to be valid for any spin.

We start with the Dirac equation with an external Yang-Mills field (see, e.g., Ref. 11)

$$i\frac{\partial\psi}{\partial t}=[\vec{\alpha}\cdot(\vec{\mathrm{p}}-G\vec{\underline{\mathrm{b}}}\cdot\underline{I})+\beta m+G\underline{b}_0\cdot\underline{I}]\,\psi\ , \tag{12}$$

which, in the nonrelativistic limit (see, e.g., Ref. 8), reduces to the two-component spinor Schrödinger equation

$$i\frac{\partial\varphi}{\partial t}=\frac{1}{2m}(\vec{\mathrm{p}}-G\vec{\underline{\mathrm{b}}}\cdot\underline{I})^2\varphi+G\underline{b}_0\cdot\underline{I}\,\varphi-\vec{\sigma}\cdot\vec{\kappa}\,\varphi\ , \tag{13}$$

where the vector

$$\begin{aligned}\vec{\kappa}&=-\frac{i}{2m}(\vec{\mathrm{p}}-G\vec{\underline{\mathrm{b}}}\cdot\underline{I})\times(\vec{\mathrm{p}}-G\vec{\underline{\mathrm{b}}}\cdot\underline{I})\\&=\frac{G}{2m}\underline{I}\cdot(\vec{\nabla}\times\vec{\underline{\mathrm{b}}}+G\vec{\underline{\mathrm{b}}}\times\vec{\underline{\mathrm{b}}})\\&=\frac{G}{2m}\vec{\underline{\mathrm{h}}}\cdot\underline{I}\ .\end{aligned} \tag{14}$$

Here $\vec{\underline{\mathrm{h}}}\cdot\underline{I}$ is the analog of the magnetic field in electrodynamics. It is an axial vector given by the spatial components of the antisymmetric tensor: $\underline{h}\equiv\{f_{ik}\}$. From the nonrelativistic Hamiltonian (13) and Eq. (14) we obtain the spin equation of motion in the (R) system;

$$\frac{d\vec{\mathrm{s}}}{d\tau}=i[H,\vec{\mathrm{s}}]=\frac{G}{m}(\vec{\mathrm{s}}\times\vec{\underline{\mathrm{h}}}\cdot\underline{I})\quad(R)\ , \tag{15}$$

which is the sought-for analog of Eq. (3) with the "gyromagnetic ratio" equal to 2 (as in the case of a Dirac particle moving in a given electromagnetic field).

Since the fourth component of s satisfies Eq. (4) as before, Eqs. (9) and (15) show that the problem of spin precession in an external Yang-Mills field becomes isomorphic with the case of electromagnetic external fields with $g=2$ and $g'=0$. Equation (15) does not contain any free parameter (which would correspond in electrodynamics to an arbitrary magnetic moment or an arbitrary g factor). Thus, as regards the spin motion, the conditions for exponentiation would appear to be exactly satisfied.

On the other hand, the existence of another kind of precession in the Yang-Mills case makes the occurrence of exponentiation virtually impossible. As a result of its interaction with the given Yang-Mills field, the isospin coordinate of the particle can vary as it passes through the field. The equation of motion corresponding to this precession of the isospin has been derived by Wong[11] in the semiclassical limit

$$\frac{d\underline{I}}{d\tau}=G(\underline{I}\,u_\mu\times\underline{b}_\mu)\ . \tag{16}$$

By analogy with the case of ordinary spin, we should expect exponentiation when different isospin channels are decoupled, corresponding to the vanishing of charge-exchange scattering. Equation (16), considered together with Eq. (9) for the trajectory, shows that in general there will not be any such decoupling between isospin channels for an arbitrary external Yang-Mills field. This is in contrast to the case of electromagnetic fields where exponentiation takes place in the high-energy limit when the conditions $g=2$ and $g'=0$ are satisfied, for an *arbitrary* homogeneous external field. The difference between the two cases arises from the fact that while the precession of the mechanical or ordinary spin takes place in the same four-dimensional space as the motion of the particle as a whole, so that spin precession may be compensated by orbit deflection, the isospin precession takes place in a different space. The system of Eqs. (9) and (16) is quite different from that for the electromagnetic case where the trajectory equation (2) does not depend at all on the spin coordinate. Our analysis therefore indicates that exponentiation should not be expected in the case of scattering by a Yang-Mills field. Evidence in this direction has been obtained in field-theoretic calculations by Nieh and Yao.[12] The arguments given above can, of course, be extended to any other semisimple Lie group.

IV. CONCLUSION

The general condition $g=2$ given by Weinberg for exponentiation in the scattering of spinning particles by a given external electromagnetic field is nothing but the classical condition that the spin preserves its orientation relative to the direction of motion. Quantum mechanically, this corresponds to decoupling of different spin channels, in the helicity basis, and hence to exponentiation. This interpretation shows that exponentiation is lost not only when the particle possesses a noncanonical magnetic moment (viz., $g\neq 2$) but also whenever it has a nonzero electric dipole moment. Analogous discussion of scattering by an external Yang-Mills field shows that while the spin channels decouple as in the case of electromagnetic scattering (with a canonical magnetic moment $g=2$), the isospin channels do not in general decouple.

Thus, in both examples, exponentiation of the scattering amplitude occurs only under rather special conditions, and is not at all a general feature of high-energy scattering. It would therefore appear that in high-energy scattering of hadrons, where multiparticle production and the availability of many channels is the general rule, it is too much to expect that exponentiation of scattering amplitudes should be a general circumstance.

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