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Fixed Regge singularity and small-distance behavior in renormalizable field theories: A model in ϕ_6^3 *

Ivan J. Muzinich† and Hung-Sheng Tsao
The Institute for Advanced Study, Princeton, New Jersey 08540
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We present a detailed analysis of the skeleton ladder graphs for ϕ_6^3 theory in six dimensions. The ladder graphs are dominated by an energy-independent branch point in the angular momentum plane reminiscent of similar results in ϕ^4 theory. This model then serves as a prototype for models in other exactly renormalizable field theories. These branch points are contained explicitly in the anomalous dimensions of the local operators that occur in the Wilson expansion. We contrast our results with some previous results for all internal particles of zero mass. A general renormalization-group-type argument is advanced to determine the conditions for which the complete theory would contain these fixed cuts.

I. INTRODUCTION

For quite some time there has been a great deal of interest in the high-energy behavior of the scattering amplitude deduced from quntum field theory in various asymptotic regions. In the beginning of the 1960's, most studies of this problem consisted of using Feynman-diagram models as a guide to the type of J-plane singularities one might expect for the on-shell physical amplitude.[1]

Further interest in Feynman-diagram-model studies was stimulated in the late 1960's by the suggestion of Bjorken scaling which was approximately confirmed experimentally. The literature in this area is quite rich with the development of the full apparatus of the renormalization-group and Callan-Symanzik approach to the small-distance behavior of field theory.[2] In addition, at large values of the scaling variable both the Regge and Bjorken regions become intertwined, and there has been extensive research both phenomenological and field theoretical[3] on the connection between these limits.

In this paper we investigate this question in a particular model, namely ϕ^3 theory in six dimensions. In particular, the model consists of the iteration of t-channel ladders using the Bethe-Salpeter equation in a skeleton approximation (bare propagators and point vertices). This model can be solved exactly at forward scattering for zero-mass exchanged particles in the t channel. We give in detail the Regge and Bjorken limits. This analysis is a prototype for other models in exactly renormalizable theories, for example, the bubble iterations in ϕ^4 theory. In addition, the ϕ^3 theory in six dimensions is endowed with the property of asymptotic freedom. Therefore, it is quite possible that the results of the skeleton ladder model bear some resemblance to the complete theory since the propagators and vertices are modified by mild logarithmic factors in their asymptotic behaviors.

A general feature of all of these studies in exactly renormalizable theories is that the leading J-plane singularity is a fixed (t-independent) cut. While this fact has been known for some time[4-8] the results on ϕ^3 theory are to our knowledge new. These fixed J-plane cuts are a consequence of the singular small-distance (non-Fredholm) behavior of the Bethe-Salpeter kernel. A consequence of this singular behavior of the Bethe-Salpeter kernel is that the small-distance behavior of the Bethe-Salpeter wave function is controlled by a nontrivial anomalous dimension. This anomalous dimension contains the fixed angular momentum branch points explicitly, as has been pointed out by Kugler and Nussinov,[8] Gatto and Menotti,[8] and Lovelace.[9] In these examples the anomalous dimension controls the asymptotic behavior of the amplitude in the Bjorken limit and its associated fixed cut controls the asymptotic behavior in the Regge limit.

The organization of this paper is as follows: Sec. II deals with an elementary diagonalization of the Bethe-Salpeter equation at forward scattering and a solution of a prototype equation we encounter in both the ϕ^3 and ϕ^4 models. We calculate the moments or partial-wave amplitudes and the asymptotic behavior of the scattering amplitude in both the Regge and Bjorken limits. We also contrast our results with those of some model calculations where all internal particles have zero mass.[5,9] In Sec. III we make some comments and speculation on the general implications of this work for the complete field theory.

II. MODEL CALCULATIONS

A. Diagonalization

The Bethe-Salpeter equation and its subsequent diagonalization are by now well known and we present a very brief outline of the procedure here. A prototype equation is provided by the ϕ_6^3 model and is simply

$$T(p,q)=K(p,q)-i\int\frac{d^6k}{(2\pi)^6}K(p,k)[D(k)]^2T(k,q). \tag{2.1}$$

The equation is explicitly written down for forward scattering; $K(p, k)$ and $D(k)$ are, respectively, the two-particle irreducible kernel and the one-particle propagator. The extension to other field theories and other dimensions of space-time is straightforward.

After Wick rotation of the time component of four-vectors, the momenta are spacelike, which in our convention ($g_{00}=-g_{ii}=-1$) means that $q^2=um^2$, $p^2=vm^2$, $k^2=wm^2$ with $u, v, w>0$, and m is the propagator mass. The equation can then be diagonalized trivially with the expansion (letting $\hat{p}\cdot\hat{q}=z$)

$$T(p,q)\equiv T(u,v,z)=\sum_{n=0}^{\infty}C_n^2(z)T_n(u,v), \tag{2.2}$$

with its corresponding inversion

$$T_n(u,v)=\frac{8}{\pi(n+1)(n+3)}\int_{-1}^{1}C_n^2(z)T(u,v,z)(1-z^2)^{3/2}dz, \tag{2.3}$$

where C_n^2 are Gegenbauer polynomials with the properties

$$\int_{-1}^{1}C_n^2(z)C_{n'}^2(z)(1-z^2)^{3/2}dz=\frac{\pi(n+3)(n+1)}{8}\delta_{nn'} \tag{2.4}$$

and

$$\int d\Omega(\hat{k})C_n^2(\hat{p}\cdot\hat{k})C_{n'}^2(\hat{k}\cdot\hat{q})=\delta_{nn'}\frac{2\pi^3}{n+2}C_n^2(\hat{p}\cdot\hat{q}), \tag{2.5}$$

where $d\Omega(\hat{k})$ is the appropriate solid-angle element.

Use of Eqs. (2.2) and (2.5) in Eq. (2.1) yields at once the diagonalized equation

$$T_n(u,v)=K_n(u,v)+\frac{1}{64\pi^3(n+2)}\int_0^{\infty}w^2dw\,K_n(u,w)D^2(w)T_n(w,v). \tag{2.6}$$

This equation is a one-dimensional integral equation which is much simpler than the full equation, Eq. (2.1), and will be solved in a particular model below. This procedure of diagonalization differs from the usual diagonalization now in use in which the Laplace transform is used to diagonalize the absorptive-part equation.[10] The two procedures give identical results for the absorptive part when the momenta of the propagators in Eq. (2.1) are spacelike. The timelike case, which is applicable to inclusive annihilation, requires a separate procedure.

B. ϕ_6^3 Model

Next we explicitly solve the Bethe-Salpeter equation for the skeleton ladders of Fig. 1. By skeleton we mean bare vertices and propagators. If the mass of the exchanged particles is zero, we can solve the equation exactly. We then have

$$K(p,q)=\frac{g^2}{(p-q)^2}, \tag{2.7}$$

$$D(k)=\frac{1}{k^2+m^2}, \tag{2.8}$$

and from Eq. (2.3)

$$K_n(u,v)=\frac{g^2m^{-2}}{(uv)^{1/2}}\left[\frac{1}{n+1}\left(\frac{u_<}{u_>}\right)^{(n+1)/2}-\frac{1}{n+3}\left(\frac{u_<}{u_>}\right)^{(n+3)/2}\right], \tag{2.9}$$

where $u_<$ ($u_>$) is the smaller (greater) of u,v, and g is the coupling constant in a $(1/3!)g\phi^3$ Lagrangian.

Remark. It will be easily recognized that the kernel, Eq. (2.9), and the diagonalized Bethe-Salpeter equation (2.6), are identical up to a displacement of $n\to n-1$ and a factor of $(p^2q^2)^{-1/2}$ with the ϕ^4 model with bubble iterations solved over ten years ago.[5,6] This problem can be solved exactly. We outline the details here.

Firstly, we recognize that Eq. (2.6) can be reduced to a fourth-order differential equation[11,12] with the use of the identity

$$O_n{}^2K_n(u,v)=g^2(n+2)v^{-2}m^{-2}\delta(u-v), \tag{2.10}$$

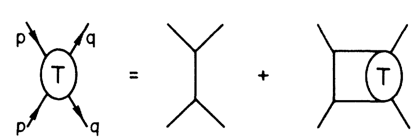

FIG. 1. Bethe-Salpeter equation in the ladder approximation.

where

$$O_n = u\left(\frac{d}{du}\right)^2 + 3\,\frac{d}{du} - \frac{n(n+4)}{4u} \;.$$

Substitution of Eqs. (2.10), (2.9), and (2.8) into Eq. (2.6) yields the resulting fourth-order differential equation

$$\left(O_n{}^2 - \frac{g^2}{64\pi^3(u+1)^2}\right) T_n(u,\,v) = g^2(n+2)v^{-2}m^{-2}\delta(u-v) \;. \tag{2.11}$$

This equation can be factorized by the identity

$$O_{\nu_+}O_{\nu_-} = (1+u)^2 O_n{}^2 - \frac{g^2}{64\pi^3} \;, \tag{2.12}$$

with

$$O_{\nu_\pm} = (1+u)u\,\frac{d^2}{du^2} + (3+2u)\,\frac{d}{du} + \frac{1-\nu_\pm{}^2}{4} - \frac{n(n+4)}{4u} \tag{2.13}$$

and

$$\begin{aligned}\nu_\pm &= 1 + (n+2)^2 \pm [4(n+2)^2 + g^2/4\pi^3]^{1/2}\\ &\simeq n+2 \pm \frac{2g^2}{(4\pi)^3}\,\frac{1}{(n+2\pm 1)(n+2)} \;.\end{aligned} \tag{2.14}$$

The latter expression is valid in the weak-coupling approximation. Equation (2.11) can then be solved explicitly in terms of hypergeometric functions, where we choose the boundary conditions

$$\begin{aligned}&T_n(u,\,v) \to 0 \quad (u \gg v) \;,\\ &T_n(u,\,v) \to u^{n/2} \quad (u \ll v) \;.\end{aligned} \tag{2.15}$$

The solution is

$$\begin{aligned}T_n(u,\,v) = \frac{2g^2\nu}{m^2(4\nu^2+a)^{1/2}}\,&[M_{\nu_-}(u_<)N_{\nu_-}(u_>)\\ &- M_{\nu_+}(u_<)N_{\nu_+}(u_>)] \;,\end{aligned} \tag{2.16}$$

with $\nu = n+2$, $a = g^2/4\pi^3$, and

$$M_{\nu_\pm}(u) = u^{(\nu-2)/2}\,{}_2F_1\big(\tfrac{1}{2}(\nu+\nu_\pm-1),\,\tfrac{1}{2}(\nu-\nu_\pm-1),\,1+\nu,\,-u\big), \tag{2.17}$$

$$\begin{aligned}N_{\nu_\pm}(u) = &\frac{\Gamma\big(\tfrac{1}{2}(3+\nu_\pm+\nu)\big)\Gamma\big(\tfrac{1}{2}(\nu+\nu_\pm-1)\big)}{\Gamma(\nu+1)\Gamma(\nu_\pm+1)}\,u^{-(1+\nu_\pm)/2}\\ &\times {}_2F_1\big(\tfrac{1}{2}(\nu_\pm+\nu-1),\,\tfrac{1}{2}(\nu_\pm-\nu-1),\,1+\nu_\pm,\,-1/u\big),\end{aligned}$$

where the solution has been normalized according to the discontinuity condition dictated by the differential equation, Eq. (2.11),

$$N\frac{dM}{du} - M\frac{dN}{du} = \frac{1+u}{u^3} \;. \tag{2.18}$$

We would like to make the following remarks about the solution.

(1) One will immediately recognize that the quantities $\nu_\pm + 1$ in Eq. (2.14) determine the scale dimension $d(n)$ of the local operators $\phi\partial_{\mu_1}\cdots\partial_{\mu_n}\phi$ of the Wilson expansion of $\phi(x)\phi(0)$ in the following sense. Following Lovelace,[9] we have

$$d(n) = 2d(\phi) + \nu_-(n) + 1 \;, \tag{2.19}$$

where $d(\phi)$ is the dimension of the elementary field. Using the fact that the energy-momentum tensor is conserved and has canonical dimension, we have

$$d(2) = 6 = 2d(\phi) + \nu_-(2) \;. \tag{2.20}$$

It then follows from Eq. (2.19) that

$$\begin{aligned}d(n) &= 6 - \nu_-(2) + \nu_-(n)\\ &\simeq n+4 + \frac{2g^2}{(4\pi)^3}\left[\frac{1}{12} - \frac{1}{(n+1)(n+2)}\right] .\end{aligned} \tag{2.21}$$

This result agrees with the perturbative result of Mack.[13] The apparent poles of the perturbative results, Eq. (2.21), seem to be a general feature.[8,9]

(2) These branch-point singularities are t-independent even though we have considered only the $t=0$ problem in detail. This can be seen from the fact that the high-momentum behavior of the kernel at fixed nonzero t is the same as the $t=0$ kernel. And one can develop an iteration procedure in the difference between the forward and nonforward kernels which is a Fredholm problem. Therefore, this solution contains the same branch points as the forward-scattering problem.

(3) Our solution also reduces immediately to a form that agrees with the solution of the corresponding problem in ϕ^4 theory if all internal particles are of zero mass.[5,9] In the mass-zero limit the solution, Eqs. (2.16) and (2.17), reduces to

$$\begin{aligned}&(q^2p^2)^{1/2}T_n(p^2,\,q^2)\\ &\quad = \frac{2g^2\nu}{(4\nu^2+a)^{1/2}}\left[\left(\frac{q_<{}^2}{q_>{}^2}\right)^{\nu_-/2}\frac{1}{\nu_-} - \left(\frac{q_<{}^2}{q_>{}^2}\right)^{\nu_+/2}\frac{1}{\nu_+}\right] .\end{aligned} \tag{2.22}$$

The poles in $\nu_\pm$ occur because the infrared problem has been enhanced by taking the mass-zero limit for all internal particles.

Next we will construct the Regge and Bjorken limits. Both of these limits are constructed by using the inversion formula Eq. (2.2) and converting to the familiar Sommerfeld-Watson transformation with appropriate analytic continuation in z.

In the Regge limit for on-mass-shell external particles $u = v = -1$ we have from Eqs. (2.16) and (2.17)

$$T(s)=\int_c \frac{d\nu}{2i\sin\pi\nu}\,\frac{16g^2}{(4\nu^2+a^2)^{1/2}}\,\frac{1}{m^2a\pi}\times(e^{i\pi\nu_-}-e^{i\pi\nu_+})C_n^2(-z)\,, \tag{2.23}$$

where for $s\to\infty$ we take

$$-z\sim\frac{s}{2m^2}\,,\qquad C_n^2(-z)\sim(\nu-1)\left(\frac{s}{m^2}\right)^{\nu-2}. \tag{2.24}$$

The physical amplitude is found by taking the crossing-symmetric combination $T(s)+T(u)$. The leading contribution will come from the contour deformed around the branch point at $\nu_c=(1+\sqrt{a}\,)^{1/2}$ (see Fig. 2). Letting $\nu=\nu_c+y$ and noting that

$$\nu_-\sim\left(\frac{2\nu_c(\nu_c^{\,2}-1)y}{\nu_c^{\,2}+1}\right)^{1/2}$$

for small y, we then have

$$T(s)=\int_{-y_0}^{0}\frac{dy}{\sin\pi(\nu_c+y)}\,\frac{g^2(\nu_c+y)}{[4(\nu_c+y)^2+a]^{1/2}}\,\frac{8}{m^2a\pi}\times(\nu_c-1+y)\left(\frac{s}{m^2}\right)^{\nu-2}(-2\sinh\pi\nu_-)\,. \tag{2.25}$$

In Eq. (2.25) a background integral centered at $y=-y_0$ parallel to the imaginary axis has been discarded (Fig. 2). We then obtain for the leading term, which comes from the tip of the cut $y\approx 0$,

$$T(s)=\frac{4\sqrt{\pi}\,(\nu_c-1)}{\sin\pi(\nu_c-1)}\left(\frac{g^2}{s}\right)\left(\frac{s}{m^2}\right)^{\nu_c-1}[b\ln(s/m^2)]^{-3/2}\,, \tag{2.26}$$

with $b=(\nu_c^{\,4}-1)/2\nu_c$.

An alternative procedure valid for weak coupling is obtained by setting $\nu=1+y$. Then

$$T(s)=\int_c\frac{1}{2i\sin\pi(y-1)}\,\frac{g^2(1+y)16}{[4(1+y)^2+a]^{1/2}}\times\frac{1}{m^2a\pi}\,y\left(\frac{s}{m^2}\right)^{-1+y}\exp[i\pi(y^2-\tfrac{1}{4}a)^{1/2}]\,. \tag{2.27}$$

The contribution at $y=0$ when a is small is

$$T(s)\approx-\frac{8g^2}{as}\sum_{l=1}^{l=\infty}\frac{[\ln(s/m^2)]^{2l}}{(2l)!}\binom{\frac{1}{2}}{l+1}\left(-\frac{a}{4}\right)^{l+1}=\frac{g^2}{s}\sum_{l=0}^{l=\infty}\frac{[\frac{1}{4}\sqrt{a}\ln(s/m^2)]^{2l}}{l!(l+1)!}\,. \tag{2.28}$$

This expression is easily recognized as a modified Bessel function of order 1. The result is

$$T(s)\approx\frac{g^2}{s}\,\frac{I_1(\frac{1}{2}\sqrt{a}\ln(s/m^2))}{\frac{1}{4}\sqrt{a}\ln(s/m^2)}\,. \tag{2.29}$$

In the limit of large s we have

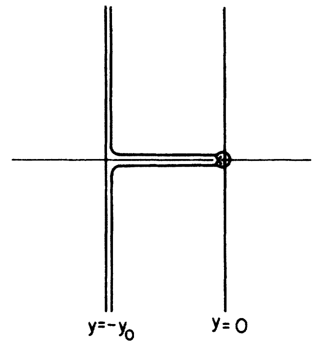

FIG. 2. Integration contour in the y plane.

$$T(s)\sim\frac{4g^2}{\sqrt{\pi}}\,\frac{1}{s}\left(\frac{s}{m^2}\right)^{\sqrt{a}/2}\left(\sqrt{a}\ln\frac{s}{m^2}\right)^{-3/2}, \tag{2.30}$$

which agrees with Eq. (2.26) in the weak-coupling limit. Equations (2.29) and (2.30) are useful if one wants to compare with leading-logarithm calculations in perturbation theory.[14]

In the Bjorken limit $q^2\to\infty$, $s\to\infty$, $\omega=2p\cdot q/q^2$ fixed, and $p^2=-m^2$. After an elementary rearrangement of the inversion formula Eq. (2.2) and the amplitude Eq. (2.17) the result for $q^2/m^2\gg 1$ is

$$\frac{2p\cdot q}{m^2}\,T(q^2,\omega)=\int_{2-i\infty}^{2+i\infty}\frac{d\nu}{2i}\,\frac{E(\nu)(-\omega)^{\nu-1}(\nu-1)}{\sin\pi\nu}\left(\frac{q^2}{m^2}\right)^{(-1-\nu_-+\nu)/2}, \tag{2.31}$$

where

$$E(\nu)=\frac{2g^2\nu}{m^2(4\nu^2+a)^{1/2}}\,\frac{\Gamma(\frac{1}{2}(\nu+\nu_--1))}{\Gamma(\frac{1}{2}(3+\nu-\nu_-))\Gamma(\nu_-+1)}\,.$$

The quantity $1+\nu_--\nu$ is the anomalous part of the dimension of the twist-four operators of the Wilson expansion. Now, to extract the Bjorken limit we see that the factor $(q^2/m^2)^{-(1-\nu_-+\nu)/2}$ oscillates rapidly at large q^2. Therefore, one must resort to the method of steepest descents to estimate the asymptotic behavior for large q^2 and fixed ω. One can rewrite Eq. (2.31) in the more standard way for the imaginary part of $T(q^2,\omega)$ with respect to ω,

$$\omega\,\frac{2p\cdot q}{m^2}\,\mathrm{Im}\,T(q^2,\omega)=\int_{2-i\infty}^{2+i\infty}\frac{d\nu}{2i}\,E(\nu)\exp\left(\nu\ln\omega+\frac{\nu-\nu_--1}{2}\ln\frac{q^2}{m^2}\right). \tag{2.32}$$

It is relatively easy to convince oneself that as $\ln(q^2/m^2)\to\infty$ and fixed ω, there is a saddle point at $\nu=\nu_0$ where

$$\frac{d}{d\nu}(\nu-\nu_-)\bigg|_{\nu=\nu_0}=\frac{\ln\omega}{\frac{1}{2}\ln(q^2/m^2)}\ . \tag{2.33}$$

The asymptotic behavior in the Bjorken limit is given by

$$\frac{2\omega p\cdot q}{m^2}\,\mathrm{Im}\,T(q^2,\omega)$$
$$\approx\frac{\pi E(\nu_0)(\nu_0-1)\omega^{\nu_0}}{[-\pi\ln(q^2/m^2)d^2\nu_-(\nu_0)/d\nu^2]^{1/2}}\left(\frac{q^2}{m^2}\right)^{[\nu_0-\nu_-(\nu_0)-1]/2}. \tag{2.34}$$

In general, it is very tedious to compute the position of the saddle point. However, for large q^2, fixed ω the saddle point will occur at a large value of ν, and one can use the weak-coupling large ν limit of Eq. (2.14)

$$\nu-\nu_--1\approx\frac{a}{8\nu(\nu-1)} \tag{2.35}$$

to estimate ν_0 the position of the saddle point. Using Eq. (2.33) we find

$$\nu_0\approx\left(\frac{a\ln(q^2/m^2)}{8\ln\omega}\right)^{1/3}. \tag{2.36}$$

The function $E(\nu)$ is relatively unimportant at large ν and contributes only a power of ν_0 in Eq. (2.34). In Eq. (2.34)

$$E(\nu_0)\simeq\frac{q^2}{m^2\nu_0}\ , \tag{2.37}$$

and

$$-\frac{d^2\nu_-(\nu_0)}{d\nu^2}\approx\frac{3}{4}\,\frac{a}{\nu_0^{\,4}}\ .$$

For the Regge region in electroproduction $\omega \gg q^2/m^2$, we obtain a result completely similar to Eq. (2.30) in weak coupling. The procedure is the same as before and we merely state the result here:

$$T(q^2,\omega)\underset{\omega\gg q^2/m^2}{\sim}\frac{4g^2}{\sqrt{\pi}}\left(\frac{1}{-\omega q^2}\right)\left(\frac{-\omega q}{m}\right)^{\sqrt{a}/2}\left(a^{1/2}\ln\frac{-\omega q}{m}\right)^{-3/2}. \tag{2.38}$$

If all internal masses are set equal to zero, the Regge limit Eq. (2.26) and Eq. (2.38) is changed owing to the enhancement of infrared logarithms; the modified formula is

$$T(s)=\frac{g^2}{s}\,\frac{\pi(\nu_c-1)}{\sin\pi(\nu_c-1)}\left(\frac{s}{m^2}\right)^{\nu_c-1}\frac{1}{[\pi b\ln(s/m^2)]^{1/2}}\ . \tag{2.39}$$

In contrast the Bjorken limit ($q^2\to\infty$, ω fixed) depends only upon the underlying zero-mass theory. From Eq. (2.35) only large ν is important and both the zero-mass and finite-mass results, Eqs. (2.22) and (2.16), agree in this limit. Also, for the leading term only the perturbative large ν (small g) limit of the anomalous dimension is important. This is to be contrasted, however, with the Regge limit [which depends on the details of the anomalous dimensions $\nu_\pm(\nu)$].

III. DISCUSSION AND SPECULATION

All of the previous considerations and most of the model calculations in the literature have been carried out in the approximation of no insertions, bare propagators, and vertices. At best this can only be an approximation and may have no resemblance to the complete theory. In this section we speculate under what conditions the full theory may have some similarities to the model studies here and elsewhere, at least as far as the existence of the fixed branch points is concerned.

As we remarked previously, the existence of the fixed cuts in the model studied is immediately related to the high off-shell behavior of the kernel, Eq. (2.9). The kernel has a scale-invariant form and a high-momentum behavior that immediately renders it non-Fredholm in the mathematical sense.

Next we remark that if there exists a fixed ultraviolet-stable point in the Gell-Mann–Low sense,[2] the high off-shell behavior of the kernel will again be controlled by a scale-invariant form weighted by a factor of $u^{-4\gamma}$, where γ is the anomalous dimension of the elementary field. Renormalization-group arguments can be used, and the mass can then be scaled to zero in the Bethe-Salpeter kernel with no difficulty, since it lacks two-particle intermediate states in the zero-momentum channel.[15] The propagator contains at high u a factor of $u^{2\gamma}$ which in the integral equation (2.7) compensates that of the kernel.[16] Therefore, at high u and w in Eq. (2.6) the kernel multiplied by propagators has a scale-invariant non-Fredholm form very much like that of the skeleton-model studies in the ϕ_6^3 theory of the previous section. The ϕ_6^3 model serves as a prototype for other renormalizable theories with decent infrared behavior such as ϕ^4 or Yukawa theory if there exist small anomalous dimensions. Therefore, we expect the branch points of the anomalous dimensions in n to be a general property of such theories.[17] Unfortunately, we cannot and will not make any reliable calculations of the anomalous dimensions and more rigorous statements because of lack of knowledge of detailed analytic structure in n of the completely general Bethe-Salpeter kernel.

In the case of ϕ_6^3 which is asymptotically free the extension to the complete theory involves the moving coupling constant of the renormalization group which asymptotically goes to zero, $g^2(u) \sim \ln u^{-1}$. After the anomalous-dimension factors between the kernel and propagators are taken into account, the resulting integral equation is still singular and non-scale-invariant. Needless to say, we have not been able to construct the solution to the equation in general. At large q^2 we have only been able to construct the perturbative large ν result of Sec. II which is consistent in an asymptotically free theory. Although we have not been able to construct the solution in general we expect that non-Regge (nonpole) singularities will exist here also.

Note added in proof. Since this manuscript was prepared, two other papers by Cardy[18] and Lovelace[19] have appeared concerning the question of the Regge singularity when the asymptotic freedom of the full ϕ_6^3 theory is taken into account. Cardy argues that the kernel is of the $\mathcal{L}_2$ class and that there is a point spectrum for the resolvent kernel with moving Regge poles. Lovelace, on the other hand, solves an approximation to the diagonalized Bethe-Salpeter equation and concludes that there exists an accumulation of Regge poles near $n=-1$ and an essential singularity. We believe that more study is needed to resolve this question and further work in this direction is worthwhile. Both Cardy and Lovelace agree with this work on the existence of fixed cuts in the full non-asymptotically-free case with the existence of a renormalization-group fixed point.

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†Permanent address: Brookhaven National Laboratory, Upton, New York 11973.

[1]R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S Matrix* (Cambridge Univ. Press, Cambridge, England, 1966). For recent discussion of Reggeization see M. T. Grisaru, H. J. Schnitzer, and H.-S. Tsao, Phys. Rev. D 8, 4498 (1973); D 9, 2864 (1974).

[2]M. Gell-Mann and F. Low, Phys. Rev. 95, 1300 (1954); K. G. Wilson, Phys. Rev. 179, 1499 (1969); C. G. Callan Jr., Phys. Rev. D 2, 1541 (1970); K. Symanzik, Commun. Math. Phys. 18, 227 (1970); N. Christ, B. Hasslacher, and A. H. Mueller, Phys. Rev. D 6, 3543 (1972).

[3]H. D. I. Abarbanel, M. L. Goldberger, and S. B. Treiman, Phys. Rev. Lett. 22, 500 (1969).

[4]J. D. Bjorken and T. T. Wu, Phys. Rev. 130, 2566 (1963); R. F. Sawyer, Phys. Rev. 131, 1857 (1963); A. Bastai, L. Bertocchi, S. Fubini, G. Furlan, and M. Tonin, Nuovo Cimento 30, 1512 (1963); 30, 1532 (1963).

[5]M. Baker and I. J. Muzinich, Phys. Rev. 132, 2291 (1963).

[6]B. W. Lee and A. Swift, J. Math. Phys. 5, 908 (1964); R. Dashen and D. Gross, Phys. Rev. D 8, 559 (1973).

[7]In electrodynamics there are results by H. Cheng and T. T. Wu, Phys. Rev. Lett. 24, 759 (1970); 24, 1456 (1970); G. V. Frolov and V. N. Gribov, Phys. Lett. 31B, 34 (1973).

[8]M. Kugler and S. Nussinov, Nucl. Phys. B28, 97 (1971); R. Gatto and P. Menotti, Nuovo Cimento 2A, 881 (1971).

[9]C. Lovelace, Rutgers Univ. report (unpublished); C. Callan and M. Goldberger, Phys. Rev. D 11, 1542 (1975); 11, 1553 (1975).

[10]S. Nussinov and J. Rosner, J. Math. Phys. 7, 1670 (1966).

[11]The method employed here follows a method developed by M. Banerjee, L. Kisslinger, and C. A. Levinson (unpublished) rather than the configuration-space procedure of Ref. 6. Of course, the results are identical. We thank Professor L. Kisslinger for a correspondence on the old results.

[12]R. S. Willey, Phys. Rev. 153, 1364 (1967).

[13]G. Mack, *Springer Lecture Notes in Physics*, edited by J. Ehlers *et al*. (Springer, New York, 1973), Vol. 17, p. 300.

[14]R. W. Brown, L. B. Gordon, T. F. Wong, and B.-L. Young, following paper, Phys. Rev. D 11, 2209 (1975).

[15]Basically the kernel behaves as if it were not being evaluated at exceptional-momenta forward scattering in this case. For an account of this and other related questions see A. Mueller, Phys. Rev. D 9, 963 (1974).

[16]T. Appelquist and E. Poggio, Phys. Rev. D 10, 3280 (1974).

[17]The argument relies upon a further diagonalization of the Bethe-Salpeter equation with the use of a Mellin transform. This Mellin transform contains poles which occur in pairs and pinch the contour of the inverse Mellin transform. It is this mechanism that produces the branch points. See M. Banerjee, M. Kugler, C. A. Levinson, and I. J. Muzinich, Phys. Rev. 137, B1280 (1965).

[18] J. Cardy, Phys. Lett. 53B, 355 (1974); Univ. of California Santa Barbara report (unpublished).

[19]C. Lovelace, Rutgers Univ. report (unpublished)