Modified field theory for quark binding*†

Joe Kiskis
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305
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We have investigated the structure of modified field theories. Higher derivatives are introduced into the Lagrangian in such a way that the bare propagator becomes q^{-4} rather than q^{-2}. A modification of this type to the gluon propagator in a quark-gluon model gives the possibility of permanent quark binding. We find that such a model has difficulties with unitarity and with infrared divergences when treated in naive perturbation theory. If the problems associated with going beyond perturbation theory can be overcome, these difficulties may be eliminated.

I. INTRODUCTION

Although the quark-parton model has had some impressive successes,[1] no particles which can be identified as the hadronic constituents have been observed. One reaction to this frustrating situation has been to speculate that the constituents are permanently bound into the observed hadrons and cannot be separated by even arbitrarily large energies. In this paper, we will study some properties of an unconventional field theory model which is designed to produce this kind of permanent binding. The model involves a q^{-4} bare propagator for the gluon field. The static-limit potential associated with this propagator rises linearly with the distance from a point source. Other approaches to permanent binding have been discussed by Casher, Kogut, and Susskind,[2] Wilson,[3] and Chodos, Jaffe, Johnson, Thorn, and Weisskopf.[4]

Two-dimensional quantum electrodynamics

We can ease into the subject by briefly discussing two-dimensional QED which displays some of the properties in which we are interested. Although this theory is trivial in the sense that it is in the equivalence class of free fields, it has the advantage of being completely soluble.

The theory is formulated by writing down the Lagrangian for a spin-$\frac{1}{2}$ massless fermion interacting with a photon in the usual way. However, the equations are interpreted in a space of one time and one space dimension. This theory has been solved and discussed from several points of view.[5] The most interesting discussion for our purposes has been given by Casher, Kogut, and Susskind.[5] They emphasized the novel permanent binding features of this theory.

The work on this theory has demonstrated that the photon acquires a mass and that the fermions cannot exist separately. At an intuitive level, the impossibility of producing separated fermions can be understood from looking at the Coulomb interaction in two dimensions. (In fact, in two dimensions, the Coulomb interaction is everything since the space does not allow transverse fields.) Gauss's Law is

$$\frac{\partial^2}{\partial x^2}\,\phi(t,x) = -\,e\rho(t,x)\ .$$

A solution for the source

$$-\,e\rho(t,x) = \delta(x)$$

is

$$\phi = \tfrac{1}{2}\,|x|\ .$$

This shows that the potential of a point charge grows linearly with distance from the charge. An infinite energy would be required to produce an asymptotically separated fermion pair.

Linearly rising potential in four dimensions

Returning now to four-dimensional Minkowski space, we wonder whether a linearly growing potential arises in any natural way. In fact, it does. The function ϕ solves

$$\nabla^2\nabla^2\phi = -\,8\pi\,\delta^3(\vec{x})\ .$$

This Green's function equation could be expected to arise from the static limit of the Poincaré-invariant field equation

$$\partial^2\partial^2\phi = j\ . \tag{1.1}$$

In momentum space, the propagator for this differential equation is k^{-4}.

These considerations suggest the construction of a quantum field theory in which the hadron constituents interact through the exchange of a gluon whose field satisfies an equation such as Eq. (1.1). In constructing a realistic theory, it is necessary to make a decision as to whether the gluon field should be scalar, pseudoscalar, Abelian or non-Abelian gauge vector, etc. The bulk of this paper will be involved with elucidating the structure of a quantum field satisfying Eq. (1.1). In order to keep the discussion as simple as possible, we have concentrated on the scalar case. However, in constructing a realistic theory of quark binding, an Abelian vector or non-Abelian vector field will probably be needed. The scalar case has the disadvantage that like charges attract. The Abelian vector case may be thought of as QED with the q^{-2} photon propagator replaced by q^{-4}. Here we have like charges repelling and opposites attracting as needed. By comparing with QED, we can see that the theory will have improved ultraviolet behavior, which will relate favorably to the Bjorken scaling phenomenon. The infrared behavior will be much worse and will result in permanent quark binding if the heuristic arguments concerning the potential can be carried through in a complete theory. A more complicated theory[6] involving colored quarks would employ a non-Abelian vector gluon.

This is not the first appearance of field theories involving higher derivatives. For early work one should refer to the paper of Pais and Uhlenbeck.[7] Much more recently, Kauffmann[6] has independently discussed ideas very similar to those appearing here. His work contains a clear discussion of the advantages of the non-Abelian vector-gluon version. He emphasizes that the self-coupling of the gluons in this theory could prevent the appearance of these unusual particles in the scattering states. Another independent approach has been developed by Blaha.[8] He modified the usual quantization procedure to get a gluon propagator

$$\mathrm{P}\,\frac{1}{k^4}$$

rather than the more conventional

$$1/(k^2+i\epsilon)^2$$

propagator that we will be using.

Binding

We have seen that the classical potential that results from a field equation such as Eq. (1.1) is linearly rising and suggests permanent quark binding. It is an open question whether or not this result will obtain in the quantum field theory. As an indication of what may happen, we can consider the work of Johnson[9] and Wilson.[10]

Johnson considered a field theory with a differential equation for the quark field ψ of the form

$$(\partial^2+m^2)\,\psi(x)=I(x)\,\psi(x)\ .$$

He observes that if the quark-quark matrix elements of $I(x)$ are sufficiently singular, permanent quark binding could result. Without going into the details of his development, we will simply check our theory for the required type of singularity.

In a simple scalar version of the theory the interaction could be

$$\phi(x)\,\psi^2(x)$$

and give

$$I(x)=\phi(x)\ .$$

To get an expression for ϕ in terms of ψ, we can solve the gluon field equation

$$\partial^2\partial^2\phi(x)=\psi^2(x)$$

to get

$$\phi(x)=\phi_{\rm in}(x)+\int d^4y\,D_R(x-y)\,\psi^2(y)\ .$$

The crucial matrix element

$$\langle \text{quark } q\,|\,I(0)\,|\,\text{quark } q'\rangle$$

then contains the term

$$\int d^4y\,D_R(-y)\,\langle q\,|\,\psi^2(0)\,|\,q'\rangle\,e^{i(q-q')\cdot y}\ . \tag{1.2}$$

The retarded propagator satisfies

$$\partial^2\partial^2 D_R(x)=\delta^4(x)$$

and is given by

$$D_R(x)=\frac{1}{(2\pi)^4}\int d^4k\,\frac{e^{-ik\cdot x}}{(k^2+i\epsilon k^0)^2}\ .$$

Expression (1.2) becomes

$$\frac{1}{(Q^2+i\epsilon Q^0)^2}\,\langle q\,|\,\psi^2(0)\,|\,q'\rangle\ ,$$

with

$$Q\equiv q'-q\ .$$

Since the imaginary part of the propagator goes like

$$\epsilon(Q^0)\,\delta'(Q^2)\ ,$$

the matrix element contains a singularity of the type that Johnson argued will result in quark binding. A more detailed analysis of our model from this approach would be worthwhile.

For another indication of whether or not the quantum field theory will result in quark binding, we will consider the ideas of Wilson. Although the specific mechanism he was interested in is not related to our model, his introductory discussion was more general. He discusses the matrix element

$$\langle 0\,|\,T[\,j_\mu(x)\,j_\nu(0)]\,|\,0\rangle \tag{1.3}$$

from the Feynman path approach. After all gluon field configurations are summed over, the contribution of a particular quark path to expression (1.3) depends on

$$\exp\left[-g^2\int ds^\mu\int ds'^\nu\,D_{F\mu\nu}(x-x')\right]\ .$$

D_F is the free propagator for an Abelian gauge field coupled with strength g to the quarks. The integrals are over the quark path being considered. The observation which he makes is that quark binding results from the important distinction between the usual propagator

$$\frac{1}{(x-x')^2}$$

and the

$$\ln(x-x')^2$$

which appears in two-dimensional QED. As we will see later, the coordinate-space counterpart of our q^{-4} propagator is

$$\ln x^2 \ .$$

So, again, we are encouraged in the hope that a complete analysis of the type of model we are interested in will show that permanent quark binding results.

Such a complete analysis is, as in any nontrivial field theory, very difficult. In this paper, we will content ourselves with developing the formal structure of the model. Attempts at results which go beyond the usual perturbation approach to field theory will be left for later work. Section II is concerned with the formal development of the quantum field theory. The usual canonical quantization prescription cannot be applied in a direct and unambiguous way to theories with higher derivatives. The method we will use seems to be equivalent to the old method of Peierls[11] and (at least for the case under consideration) to a method recently proposed by Dürr.[12] An alternative quantization method involving a dummy field rather than higher derivatives is discussed in the Appendix. Although our methods are somewhat different from those of earlier workers, most of the general results of Sec. II have been obtained previously. Pais and Uhlenbeck[7] concentrated on the problem with negative energies and referred to Matthews,[13] who discussed indefinite-metric problems. Nagy[14] has given a nice discussion of indefinite-metric state spaces. General higher-derivative Lagrangians have been treated more formally by Barut and Mullen,[15] and by Cukierda and Lukeirski.[16] Section III develops the interacting field and perturbation theory. It is worth noting that simple interactions involve the introduction of a fundamental scale by way of the coupling constant. Section IV contains our speculations on what may happen if one can transcend the difficulties of going beyond perturbation theory.

II. FORMAL DEVELOPMENT

Euler - Lagrange equations

In this section we will lay out the derivation of the Euler-Lagrange equations from a Lagrangian in which higher-than-first derivatives of the fields appear. The fields which appear will be denoted $\phi_i(x)$. The usual notation

$$\partial_\mu = \frac{\partial}{\partial x^\mu}$$

will be used.[17] The Lagrangian will be a function of ϕ_i, $\partial_\mu \phi_i$, $\partial_\mu \partial_\nu \phi_i$, ... The action is given by

$$A = \int d^4x\, \mathcal{L} \ .$$

The first thing to notice is that

$$\partial_\mu \partial_\nu \phi_i = \partial_\nu \partial_\mu \phi_i \ ,$$

so these quantities should not be varied independently. We now proceed with the variation of A in the usual way. Using the relationships

$$\delta \partial_\mu \phi_i = \partial_\mu \delta \phi_i \ ,$$

$$\delta \partial_\mu \partial_\nu \phi_i = \partial_\mu \partial_\nu \delta \phi_i \ ,$$

$$\ldots ,$$

partially integrating, and taking

$$0 = \delta \phi_i = \partial_\mu \delta \phi_i = \partial_\mu \partial_\nu \delta \phi_i = \cdots$$

on the surface, we get

$$0 = \frac{\delta \mathcal{L}}{\delta \phi_i} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_i} + \sum_\mu \sum_{\nu \leq \mu} \partial_\mu \partial_\nu \frac{\delta \mathcal{L}}{\delta \partial_\mu \partial_\nu \phi_i} - \cdots . \quad (2.1)$$

We have taken care in restricting ourselves to the independent quantities

$$\partial_\mu \partial_\nu \phi_i \text{ with } \nu \leq \mu \ .$$

In this context, when the Lagrangian contains a term such as $\partial_\mu \partial_\nu \phi_i F^{\mu\nu i}$, the variation $\delta/\delta\partial_\mu\partial_\nu\phi_i$ will give $F^{\mu\nu i}$ if $\mu=\nu$ and $F^{\mu\nu i}+F^{\nu\mu i}$ when $\mu \neq \nu$. This bit of inelegance can be eliminated by observing that

$$\sum_\mu \sum_{\nu \leq \mu} \partial_\mu \partial_\nu [F^{\mu\nu i} + (1-\delta^\mu_\nu) F^{\nu\mu i}] = \partial_\mu \partial_\nu F^{\mu\nu i} \ ,$$

with the usual summation convention operating on the right-hand side. If we then interpret

$$\frac{\delta}{\delta \partial_\mu \partial_\nu \phi_i} \partial_\alpha \partial_\beta \phi_j F^{\alpha\beta j} \text{ as } F^{\mu\nu i} \ ,$$

the Euler-Lagrange equation becomes

$$0 = \frac{\delta \mathcal{L}}{\delta \phi_i} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi_i} + \partial_\mu \partial_\nu \frac{\delta \mathcal{L}}{\delta \partial_\mu \partial_\nu \phi_i} - \cdots . \quad (2.2)$$

Conserved currents

In order to keep the indices under control, we will restrict ourselves to Lagrangians which contain, at most, second derivatives of the fields. The stress-energy tensor is found by applying translation invariance in the usual way:

$$T_c^{\mu\nu}(x) = \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\partial^\nu\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\lambda\phi_i}\partial^\nu\partial_\lambda\phi_i - \left(\partial_\lambda\frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\lambda\phi_i}\right)\partial^\nu\phi_i - g^{\mu\nu}\mathcal{L}\,,$$
$$\partial_\mu T_c^{\mu\nu}(x) = 0\,. \tag{2.3}$$

The momentum is

$$P^\mu = \int d^3x\, T_c^{0\mu}(x)\,.$$

The subscript c is for canonical.

To study the effects of Lorentz invariance, consider a variation of the fields

$$\delta^{\alpha\beta}\phi_i = (x^\alpha\partial^\beta - x^\beta\partial^\alpha)\phi_i + \Sigma_{ij}^{\alpha\beta}\phi_j\,.$$

The change in $\mathcal{L}$ is found to be

$$\delta^{\alpha\beta}\mathcal{L} = \partial_\rho(g^{\rho\beta}x^\alpha\mathcal{L} - g^{\rho\alpha}x^\beta\mathcal{L}) + L^{\alpha\beta}\,,$$
$$L^{\alpha\beta} = \frac{\delta\mathcal{L}}{\delta\phi_i}\Sigma_{ij}^{\alpha\beta}\phi_j + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi_i}\Sigma_{ij}^{\alpha\beta}\partial_\mu\phi_j + \frac{\delta\mathcal{L}}{\delta\partial_\mu\partial_\nu\phi_i}\Sigma_{ij}^{\alpha\beta}\partial_\mu\partial_\nu\phi_j + \frac{\delta\mathcal{L}}{\delta\partial_\alpha\phi_i}\partial^\beta\phi_i - \frac{\delta\mathcal{L}}{\delta\partial_\beta\phi_i}\partial^\alpha\phi_i + 2\frac{\delta\mathcal{L}}{\delta\partial_\alpha\partial_\mu\phi_i}\partial^\beta\partial_\mu\phi_i - 2\frac{\delta\mathcal{L}}{\delta\partial_\beta\partial_\mu\phi_i}\partial^\alpha\partial_\mu\phi_i\,. \tag{2.4}$$

Since Lorentz invariance is assumed, we infer

$$L^{\alpha\beta} = 0\,.$$

Translation invariance

$$\partial_\mu\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi_i}\partial_\mu\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\nu\phi_i}\partial_\mu\partial_\nu\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\nu\partial_\sigma\phi_i}\partial_\mu\partial_\nu\partial_\sigma\phi_i \tag{2.5}$$

was used in deriving this expression for $\delta\mathcal{L}$. The equations of motion were not used.

When the equations of motion are used to compute $\delta\mathcal{L}$, we get

$$\delta^{\alpha\beta}\mathcal{L} = \partial_\rho\left[\frac{\delta\mathcal{L}}{\delta\partial_\rho\phi_i}\delta^{\alpha\beta}\phi_i - \left(\partial_\nu\frac{\delta\mathcal{L}}{\delta\partial_\rho\partial_\nu\phi_i}\right)\delta^{\alpha\beta}\phi_i + \frac{\delta\mathcal{L}}{\delta\partial_\rho\partial_\nu\phi_i}\partial_\nu\delta^{\alpha\beta}\phi_i\right]\,.$$

The difference of these two expressions for $\delta\mathcal{L}$ is then

$$\mathfrak{M}_c^{\rho\alpha\beta}(x) = x^\alpha T_c^{\rho\beta} - x^\beta T_c^{\rho\alpha} + \left(\frac{\delta\mathcal{L}}{\delta\partial_\rho\phi_i} - \partial_\nu\frac{\delta\mathcal{L}}{\delta\partial_\rho\partial_\nu\phi_i}\right)\Sigma_{ij}^{\alpha\beta}\phi_j + \frac{\delta\mathcal{L}}{\delta\partial_\rho\partial_\nu\phi_i}\left[\Sigma_{ij}^{\alpha\beta}\partial_\nu\phi_j + (g_\nu^\alpha\partial^\beta - g_\nu^\beta\partial^\alpha)\phi_i\right]\,. \tag{2.6}$$

$\mathfrak{M}_c^{\rho\alpha\beta}(x)$ is conserved,

$$\partial_\rho\mathfrak{M}^{\rho\alpha\beta}(x) = 0\,,$$

and the angular momentum is

$$M^{\alpha\beta} = \int d^3x\,\mathfrak{M}_c^{0\alpha\beta}(x)\,.$$

Symmetrization of $T^{\mu\nu}$

We have seen that

$$\mathfrak{M}_c^{\rho\alpha\beta} = x^\alpha T_c^{\rho\beta} - x^\beta T_c^{\rho\alpha} + K^{\rho\alpha\beta}\,,$$

with

$$K^{\rho\alpha\beta} = -K^{\rho\beta\alpha}\,.$$

Set

$$T_s^{\mu\nu} = T_c^{\mu\nu} + \tfrac{1}{2}\partial_\rho(K^{\rho\mu\nu} - K^{\mu\rho\nu} - K^{\nu\rho\mu})$$

and

$$\mathfrak{M}_s^{\rho\alpha\beta} = x^\alpha T_s^{\rho\beta} - x^\beta T_s^{\rho\alpha}\,. \tag{2.7}$$

Since the difference between T_s and T_c is the divergence of a term which is antisymmetric in ρ and μ, T_s and T_c will give the same momentum. Similarly, a little work will show that the difference between $\mathfrak{M}_s$ and $\mathfrak{M}_c$ is a term which does not contribute to the angular momentum. Thus, T_s and $\mathfrak{M}_s$ can be used to calculate P and M as well as T_c and $\mathfrak{M}_c$. It should be noted, however, that the local commutators such as

$$[T^{\mu\nu}(x'), \phi_i(x)]$$

are not the same for T_s and T_c.

Now

$$T_s^{\mu\nu} - T_s^{\nu\mu} = T_c^{\mu\nu} - T_c^{\nu\mu} + \partial_\rho K^{\rho\mu\nu}\,.$$

When the explicit form of K is used to calculate the right-hand side of this equation, we find an expression which is equal to $L^{\mu\nu}$. Since Lorentz invariance required L to be zero, we conclude that T_s is the symmetric stress-energy tensor.

Commutation relations

In order to avoid as many complications as possible in understanding the quantum mechanics of a field which has higher derivatives in the Lagrangian, we begin with a simple case. Consider a scalar field ϕ with a Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial^2\phi)(\partial^2\phi) \ .$$

The equation of motion is

$$\partial^2\partial^2\phi = 0 \ .$$

In order to quantize this theory, we will propose a simple expression for

$$[\phi(x), \phi(y)]$$

which has the general properties usually found in this commutator. The crucial test will be confronted when the commutators of P and M with ϕ which follow from this are evaluated.

Since we are dealing with a free field, we expect the commutator to be a c number which is Lorentz invariant and translation invariant. In addition, it should be a solution to the field equation which is odd in $(x-y)$ and which vanishes for

$$(x-y)^2 < 0 \ .$$

We can satisfy these requirements by taking

$$[\phi(x), \phi(y)] = \frac{-1}{(2\pi)^3}\int d^4k\, \epsilon(k^0)\, \delta'(k^2)\, e^{-ik\cdot(x-y)} \ . \tag{2.8}$$

To check that the commutator vanishes for $(x-y)^2 < 0$, we can use its manifest Poincaré invariance to choose a frame where

$$y = 0 \text{ and } x^0 = 0 \ .$$

Then

$$[\phi(0,\vec{x}), \phi(0)] = \frac{-1}{(2\pi)^3}\int d^4k\, \epsilon(k^0)\, \delta'(k^2)\, e^{+i\vec{k}\cdot\vec{x}} = 0 \ .$$

Thus,

$$[\phi(x), \phi(y)] = 0 \text{ for } (x-y)^2 \leqslant 0 \ .$$

Since

$$k^2\delta'(k^2) = -\delta(k^2) \ ,$$

it is easy to verify

$$\partial^2\partial^2[\phi(x), \phi(y)] = 0 \ .$$

We should note at this point that we could have added to the commutator a piece proportional to $\Delta(x-y)$.[17] However, since the Lagrangian tells us that ϕ is dimensionless, this would require introducing a dimensional constant. This seems unnatural.

For the well-known singular functions of field theory, we will use the Δ notation of Bjorken and Drell.[17] For the singular functions associated with the $\partial^2\partial^2$ operator, we will use a D notation. Thus,

$$[\phi(x), \phi(y)] = i\,D(x-y) \ ,$$

and

$$D(x) = +\frac{\partial}{\partial m^2}\,\Delta(x, m^2)\Big|_{m^2=0} = \frac{i}{(2\pi)^3}\int d^4k\, \epsilon(k^0)\, \delta'(k^2)\, e^{-ik\cdot x} \ .$$

The important equal-time commutators can now be calculated. From the expression for $D(x-y)$, we can prove a general relationship which is useful:

$$[\partial_0{}^n\phi(x), \partial_0{}^m\phi(y)] = -[\partial_0{}^{n-1}\phi(x), \partial_0{}^{m+1}\phi(y)] \ . \tag{2.9}$$

To calculate an equal-time commutator we proceed as follows:

$$[\phi(x), \phi(y)] = \frac{\partial}{\partial m^2}\frac{1}{(2\pi)^3}\int d^4k\, \epsilon(k^0)\, \delta(k^2-m^2)\, e^{-ik\cdot(x-y)}\Big|_{m^2=0}$$
$$= \frac{-1}{(2\pi)^3}\int \frac{d^3k}{4|\vec{k}|^3}\{[1 + i|\vec{k}|(x^0-y^0)]\,e^{-ik\cdot(x-y)} - \text{c.c.}\} \ , \tag{2.10}$$
$$[\phi(t,\vec{x}), \phi(t,\vec{y})] = \frac{-1}{(2\pi)^3}\int\frac{d^3k}{4|\vec{k}|^3}\,(e^{+i\vec{k}\cdot(\vec{x}-\vec{y})} - \text{c.c.}) = 0 \ .$$

By the same method, we can get

$$[\dot\phi(t,\vec{x}), \phi(t,\vec{y})] = [\ddot\phi(t,\vec{x}), \phi(t,\vec{y})] = [\phi^{(\text{iv})}(t,\vec{x}), \phi(t,\vec{y})] = 0 \ , \tag{2.11}$$
$$[\dddot\phi(t,\vec{x}), \phi(t,\vec{y})] = i\,\delta^3(\vec{x}-\vec{y}) \ ,$$
$$[\phi^{(\text{v})}(t,\vec{x}), \phi(t,\vec{y})] = 2i\nabla^2\delta^3(\vec{x}-\vec{y}) \ .$$

The stress-energy tensor which results from the Lagrangian is[18]

$$T_c^{\mu\nu} = (\partial^2\phi)(\partial^\mu\partial^\nu\phi) - (\partial^\mu\partial^2\phi)(\partial^\nu\phi) - \tfrac{1}{2}g^{\mu\nu}(\partial^2\phi)(\partial^2\phi) \ . \tag{2.12}$$

The commutation relations give

$$[T_c^{\mu\nu}(t,\vec{x}\,'), \phi(t,\vec{x})] = -i\,g^{0\mu}\delta^3(\vec{x}\,'-\vec{x})\,\partial^\nu\phi(t,\vec{x}\,') \ ,$$

so that

$$i[P^\mu, \phi(x)] = \partial^\mu\phi(x) \ .$$

A little work gives

$$i[M^{\alpha\beta}, \phi(x)] = x^\alpha\partial^\beta\phi(x) - x^\beta\partial^\alpha\phi(x) \ ,$$
$$i[M^{\alpha\beta}, \partial^\mu\phi(x)] = (x^\alpha\partial^\beta - x^\beta\partial^\alpha)\,\partial^\mu\phi(x) + (g^{\alpha\mu}g^\beta_\nu - g^{\beta\mu}g^\alpha_\nu)\,\partial^\nu\phi(x) \ ,$$

etc. The conclusion is that the operators P and M are a representation of the Poincaré group which is carried by the field ϕ. Thus, our quantization of the field has passed a crucial test.

Particle content

We now have a quantum field theory for the field ϕ. In order to find out what physical interpretation can be attached to the field, we will go over to momentum space. As is usually done, we will look for the particle content of the theory. The procedure will be to expand ϕ in momentum space and demand that the commutation relations of the Fourier components be arranged to give back the correct commutation relations in coordinate space.

The expansion of ϕ will be

$$\phi(x) = \int d^4k\, \theta(k^0)\, \delta'(k^2)\,[\phi(k)\, e^{-ik\cdot x} + \text{H.c.}]\,. \qquad (2.13)$$

When the relation

$$k^2\delta'(k^2) = -\delta(k^2)$$

is used, it is easy to see that this expression for ϕ satisfies the field equation. The k^0 integration is done by pulling out the mass derivative:

$$\phi(x) = -\frac{\partial}{\partial m^2}\int d^4k\, \theta(k^0)\,\delta(k^2-m^2)\,[\phi(k)\,e^{-ik\cdot x} + \text{H.c.}]\Big|_{m^2=0}$$
$$= \int \frac{d^3k}{4|\vec{k}|^3}\,\{[\chi(k) + i|\vec{k}|x^0\phi(k)]\,e^{-ik\cdot x} + \text{H.c.}\}\ ,$$

with $k^0 = |\vec{k}|$. We have defined

$$\chi(k) \equiv \phi(k) - \psi(k)\ ,$$
$$\psi(k) \equiv k^0\frac{\partial}{\partial k^0}\,\phi(k)\Big|_{k^0=|\vec{k}|}\ . \qquad (2.14)$$

Since $\phi(k)$ appears only with $k^0 = |\vec{k}|$, $\psi(k)$ is independent of $\phi(k)$.

If we assume that

$$[\phi(k), \phi(k')] = [\phi(k), \psi(k')]$$
$$= [\phi^\dagger(k), \phi^\dagger(k')]$$
$$= [\phi^\dagger(k), \psi^\dagger(k')]$$
$$= 0\ , \qquad (2.15)$$

we find that

$$[\phi(x), \phi(y)] = \int\frac{d^3k}{4|\vec{k}|^3}\int\frac{d^3k'}{4|\vec{k}'|^3}\{[\chi(k) + i|\vec{k}|x^0\phi(k), \chi^\dagger(k') - i|\vec{k}'|y^0\phi^\dagger(k')]\,e^{-ik\cdot x}e^{-ik'\cdot y} - \text{H.c.}\}\ .$$

The result we want is

$$[\phi(x), \phi(y)] = \int\frac{d^3k}{4|\vec{k}|^3}\,\frac{-1}{(2\pi)^3}\,\{[1 + i|\vec{k}|(x^0-y^0)]\,e^{-ik\cdot(x-y)} - \text{c.c.}\}\ .$$

It is effected by taking

$$[\phi(k), \phi^\dagger(k')] = 0\ ,$$
$$[\phi(k), \psi^\dagger(k')] = \frac{4|\vec{k}|^3}{(2\pi)^3}\,\delta^3(\vec{k}-\vec{k}')\ , \qquad (2.16)$$
$$[\psi(k), \psi^\dagger(k')] = \frac{4|\vec{k}|^3}{(2\pi)^3}\,\delta^3(\vec{k}-\vec{k}')\ .$$

For later convenience, note that

$$[\chi(k), \phi^\dagger(k')] = -\frac{4|\vec{k}|^3}{(2\pi)^3}\,\delta^3(\vec{k}-\vec{k}')\ ,$$
$$[\chi(k), \psi^\dagger(k')] = 0\ , \qquad (2.17)$$
$$[\chi(k), \chi^\dagger(k')] = -\frac{4|\vec{k}|^3}{(2\pi)^3}\,\delta^3(\vec{k}-\vec{k}')\ .$$

At this point, the correct interpretation for the momentum-space operators is not evident. It will be helpful to write P^μ in terms of these operators. The calculation is completely straightforward but rather tedious. The result is

$$P^\mu = (2\pi)^3\int\frac{d^3k}{4|\vec{k}|^3}\,k^\mu\,[\phi(k)\,\psi^\dagger(k) + \phi^\dagger(k)\,\psi(k) - (1-g^{0\mu})\,\phi^\dagger(k)\,\phi(k)]\ . \qquad (2.18)$$

The momentum-space commutators are used in a simple way to verify that P^μ is self-adjoint. It is also easy to show that

$$i[P^\mu, \phi(k)] = -ik^\mu\phi(k)\ ,$$

and that

$$i[P^\mu, \psi(k)] = -ik^\mu[\psi(k) + g^{0\mu}\phi(k)]\ .$$

These are (as they must be) the correct expressions to give

$$i[P^\mu, \phi(x)] = \partial^\mu\phi(x)\ .$$

The next step is to form linear combinations of $\phi(k)$ and $\psi(k)$ which have commutation relations close to those with which we are familiar from conventional theories. For this purpose, set

$$a = x\phi + y\psi \text{ and } a' = x'\phi + y'\psi$$

and calculate

$$[a, a'^\dagger] = (xy' + yx' + yy')[\psi, \psi^\dagger] ,$$
$$[a, a^\dagger] = (2xy + y^2)[\psi, \psi^\dagger] ,$$
$$[a', a'^\dagger] = (2x'y' + y'^2)[\psi, \psi^\dagger] .$$

It will be nice to have

$$[a, a'^\dagger] = 0 .$$

If y were zero, that would require x or y' to be zero. If x were zero, a would be zero. If y' were zero, we would have

$$a \propto a' .$$

Thus, we assume that y is not zero. We can then solve for x':

$$x' = -y'(1+r) ,$$

with

$$r = x/y .$$

We now have

$$a = y[r\phi + \psi] ,$$
$$a' = y'[-(1+r)\phi + \psi] ,$$
$$[a, a'^\dagger] = 0 ,$$
$$[a, a^\dagger] = y^2(1+2r)[\psi, \psi^\dagger] ,$$
$$[a', a'^\dagger] = -y'^2(1+2r)[\psi, \psi^\dagger] .$$

Taking

$$(1+2r) = 0$$

is ruled out because that would give

$$a' \propto a .$$

The choices

$$1 + 2r > 0$$

and

$$1 + 2r < 0$$

are equivalent. We take

$$1 + 2r > 0 .$$

That gives

$$[a, a^\dagger] > 0$$

and

$$[a', a'^\dagger] < 0 .$$

Choosing a convenient normalization and renaming a and a' to a_+ and a_-, we get

$$a_+(k) = N[\psi(k) + r\phi(k)] ,$$
$$a_-(k) = N[\psi(k) - (1+r)\phi(k)] , \tag{2.19}$$

with

$$N = \left(\frac{(2\pi)^3}{4|\vec{k}|^3(1+2r)}\right)^{1/2} . \tag{2.20}$$

These operators satisfy

$$[a_+(k), a_+(k')] = [a_-(k), a_-(k')]$$
$$= [a_+(k), a_-^\dagger(k')]$$
$$= 0 \tag{2.21}$$

and

$$[a_+(k), a_+^\dagger(k')] = \delta^3(\vec{k} - \vec{k}') ,$$
$$[a_-(k), a_-^\dagger(k')] = -\delta^3(\vec{k} - \vec{k}') . \tag{2.22}$$

Other expressions which will be useful later are

$$a_+ - a_- = N(1+2r)\phi ,$$
$$a_+ + a_- = N(2\psi - \phi) ,$$
$$\psi = \frac{1}{2N}\left[a_+ + a_- + \frac{1}{1+2r}(a_+ - a_-)\right] , \tag{2.23}$$
$$\phi = \frac{1}{N}\frac{1}{1+2r}(a_+ - a_-) .$$

With a bit of work the momentum can be expressed in terms of a_+ and a_-.

$$P^\mu = \int d^3k\, k^\mu \left\{ a_+^\dagger(k) a_+(k) - a_-^\dagger(k) a_-(k) + \frac{g^{0\mu}}{1+2r}[a_+^\dagger(k) - a_-^\dagger(k)][a_+(k) - a_-(k)] \right\} . \tag{2.24}$$

As usual, we have dropped an infinite c number. After studying this expression for P^μ for a moment, one can see that it will be useful to know that

$$[a_+(k) - a_-(k), a_+^\dagger(k') - a_-^\dagger(k')] = 0 ,$$
$$[a_+(k) + a_-(k), a_+^\dagger(k') + a_-^\dagger(k')] = 0 , \tag{2.25}$$
$$[a_+(k) + a_-(k), a_+^\dagger(k') - a_-^\dagger(k')] = 2\delta^3(\vec{k} - \vec{k}') ,$$

and most importantly,

$$[a_+^\dagger(k) a_+(k) - a_-^\dagger(k) a_-(k), \{a_+^\dagger(k') - a_-^\dagger(k')\}\{a_+(k') - a_-(k')\}] = 0 . \tag{2.26}$$

The $a_+(k)$ operators have the same commutation relations as and appear in the momentum in the way that is usual for destruction and creation operators. We will choose a vacuum such that

$$a_+(k)|0\rangle = 0 . \tag{2.27}$$

The operators $a_+^\dagger(k)$ and $a_+(k)$ will then create and destroy particles of momentum k in the usual way.

For the a_- operators, there are two choices:

(1) We interchange the roles of creation and destruction by defining

$$b = a_-^\dagger \text{ and } b^\dagger = a_- .$$

The vacuum is given the property

$$b(k)|0\rangle = 0 .$$

The b operators satisfy

$$[b(k), b^\dagger(k')] = \delta^3(\vec{k} - \vec{k}') .$$

The $b^\dagger$ and b operators will then create and destroy particles in the usual way. However, they appear in the momentum as

$$-b^\dagger(k)\, b(k) .$$

The theory then contains states with negative energy. Worse than that is the appearance of $a_+^\dagger b^\dagger$ in the Hamiltonian. This shows that eigenstates of H will have an infinite number of particles.

Rather than struggle with such difficulties, we prefer to deal with those which arise from the second option.

(2) Here, we assume for the vacuum

$$a_-(k)|0\rangle = 0 . \tag{2.28}$$

The basis states of the theory are then

$$\frac{a_+^\dagger(k_1)^{n_1}}{(n_1!)^{1/2}} \frac{a_-^\dagger(k_1)^{m_1}}{(m_1!)^{1/2}} \frac{a_+^\dagger(k_2)^{n_2}}{(n_2!)^{1/2}} \frac{a_-^\dagger(k_2)^{m_2}}{(m_2!)^{1/2}} \cdots |0\rangle .$$

It is important to note that states with an odd number of "−" particles have a negative norm. For instance,

$$\begin{aligned}\langle a_-^\dagger(k')0|a_-^\dagger(k)|0\rangle &= \langle 0|a_-(k')\, a_-^\dagger(k)|0\rangle \\ &= \langle 0|[a_-(k'), a_-^\dagger(k)]|0\rangle \\ &= -\delta^3(\vec{k} - \vec{k}') .\end{aligned}$$

Another important observation which can easily be verified is that

$$-a_-^\dagger(k')\, a_-(k')[a_-^\dagger(k)]^m|0\rangle = m[a_-^\dagger(k)]^m|0\rangle\, \delta^3(\vec{k} - \vec{k}') .$$

The next step in our program will be to find the eigenstates of energy and momentum. Since the energy and momentum operators are just (continuous) sums of operators for each momentum, we can simplify matters by working in the subspace corresponding to just one momentum k. We have

$$P^\mu = \int d^3k\, p^\mu(k) ,$$

with

$$\vec{p}(k) = \vec{k}[a_+^\dagger(k)\, a_+(k) - a_-^\dagger(k)\, a_-(k)] ,$$

$$p^0(k) = h(k) = h_0(k) + h'(k) ,$$

$$h_0(k) = |\vec{k}|[a_+^\dagger(k)\, a_+(k) - a_-^\dagger(k)\, a_-(k)] ,$$

$$h'(k) = \frac{|\vec{k}|}{1+2r}[a_+^\dagger(k) - a_-^\dagger(k)][a_+(k) - a_-(k)] .$$

We have already calculated commutators which show that

$$[\vec{p}, h_0] = [\vec{p}, h'] = [h_0, h'] = 0 .$$

Now, since h_0 and h' commute and h_0 and $\vec{p}$ are essentially the same, our problem reduces to finding simultaneous eigenstates of h_0 and h'. Since

$$(a_+^\dagger a_+ - a_-^\dagger a_-)(a_+^\dagger)^n (a_-^\dagger)^m|0\rangle = (n+m)(a_+^\dagger)^n(a_-^\dagger)^m|0\rangle ,$$

all states

$$(a_+^\dagger)^n (a_-^\dagger)^m|0\rangle$$

are eigenstates of h_0 with non-negative energies. Linear combinations of these which mix states with the same $n+m$ are also eigenstates of h_0.

$$h_0 \sum_{i=0}^{N} c_i (a_+^\dagger)^{N-i}(a_-^\dagger)^i|0\rangle = N|\vec{k}| \sum_{i=0}^{N} c_i (a_+^\dagger)^{N-i}(a_-^\dagger)^i|0\rangle .$$

So much for the eigenstates of h_0.

The next important observation is that h' leaves the subspace spanned by the

$$(a_+^\dagger)^{N-i}(a_-^\dagger)^i|0\rangle \quad i = 0, 1, 2, \ldots, N, \quad N \text{ fixed}$$

invariant. Attention is then restricted to this subspace. Because $a_+^\dagger - a_-^\dagger$ and $a_+ - a_-$ commute,

$$(a_+^\dagger - a_-^\dagger)^N|0\rangle \tag{2.29}$$

is easily seen to be an eigenstate of h' with eigenvalue zero. Indeed,

$$\begin{aligned}h'(a_+^\dagger - a_-^\dagger)^N|0\rangle &= \frac{|\vec{k}|}{1+2r}(a_+^\dagger - a_-^\dagger)(a_+ - a_-)(a_+^\dagger - a_-^\dagger)^N|0\rangle \\ &= \frac{|\vec{k}|}{1+2r}(a_+^\dagger - a_-^\dagger)(a_+^\dagger - a_-^\dagger)^N(a_+ - a_-)|0\rangle \\ &= 0 .\end{aligned}$$

We will now show that there are no other eigenstates of h'. First, observe that the states

$$(a_+^\dagger - a_-^\dagger)^{N-i}(a_+^\dagger + a_-^\dagger)^i|0\rangle$$

also span the subspace characterized by a particular value of N. The commutation relations are used to verify that these $N+1$ states are indeed linearly independent. Since the subspace is $N+1$ dimensional they must span it. Next, we verify by direct calculation that

$$\begin{aligned}h'(a_+^\dagger - a_-^\dagger)^{N-i}&(a_+^\dagger + a_-^\dagger)^i|0\rangle \\ &= \frac{|\vec{k}|}{1+2r}\, 2i\,(a_+^\dagger - a_-^\dagger)^{N-(i-1)}(a_+^\dagger + a_-^\dagger)^{i-1}|0\rangle , \\ &\qquad i \neq 0 .\end{aligned}$$

For $i=0$, we have already shown that the right-hand side is zero. Now consider the action of h' on an arbitrary state. Letting

$$|i\rangle=(a_+^\dagger-a_-^\dagger)^{N-i}(a_+^\dagger+a_-^\dagger)^i|0\rangle \ ,$$

we calculate

$$h'|\psi\rangle=h'\sum_{i=0}^{N}c_i|i\rangle=\frac{2|\vec{k}|}{1+2r}\sum_{i=1}^{N}ic_i|i-1\rangle \ .$$

If this is to be equal to $\lambda|\psi\rangle$, we must have

$$\lambda c_N=0 \ ,$$

$$\lambda c_{N-1}=\frac{2|\vec{k}|}{1+2r}Nc_N \ ,$$

$$\ldots \ ,$$

$$\lambda c_1=\frac{2|\vec{k}|}{1+2r}2c_2 \ ,$$

$$\lambda c_0=\frac{2|\vec{k}|}{1+2r}c_1 \ .$$

If λ is not zero, all the c's must be zero. If λ is zero, only c_0 is allowed to be nonzero. This completes the demonstration.

Let us pause to review what we have discovered about the structure of the state space. The theory contains positive- and negative-norm particles. Because of the negative-norm particles the state space is not a Hilbert space. And because of this, although the Hamiltonian is self-adjoint, it cannot be diagonalized. There is a subspace corresponding to each momentum. These subspaces are composed of subspaces characterized by specifying the total number of particles in a state. Within a subspace where all the states have N particles, there is only one direction which is an eigenstate of the Hamiltonian. For instance, corresponding to the momentum

$$\vec{k}=0$$

there is a one-dimensional subspace which is the constant multiple of the vacuum. The vacuum is an eigenstate of H. For

$$N=1$$

there is a two-dimensional space spanned by

$$a_+^\dagger(0)|0\rangle \text{ and } a_-^\dagger(0)|0\rangle \ .$$

The direction

$$[a_+^\dagger(0)-a_-^\dagger(0)]|0\rangle$$

is an eigenstate of H. This is continued for higher N in an obvious way. Corresponding to some

$$\vec{k}\neq 0$$

there is no one-dimensional subspace. The

$$N=1$$

subspace is spanned by

$$a_+^\dagger(k)|0\rangle \text{ and } a_-^\dagger(k)|0\rangle \ .$$

The state

$$[a_+^\dagger(k)-a_-^\dagger(k)]|0\rangle$$

is an eigenstate of H with energy $|\vec{k}|$. The two-particle subspace is spanned by

$$a_+^\dagger(k)a_+^\dagger(k)|0\rangle, \quad a_+^\dagger(k)a_-^\dagger(k)|0\rangle, \text{ and } a_-^\dagger(k)a_-^\dagger(k)|0\rangle \ .$$

The state

$$[a_+^\dagger(k)-a_-^\dagger(k)]^2|0\rangle$$

has energy $2|\vec{k}|$. It is easy to continue this for higher numbers of particles. It should be clear that the eigenstates of H all have positive energy.

Our final observation will be that, except for the vacuum, all these eigenstates have zero norm:

$$\begin{aligned}\langle(a_+^\dagger-a_-^\dagger)^N0|(a_+^\dagger-a_-^\dagger)^N|0\rangle&=\langle 0|(a_+-a_-)^N(a_+^\dagger-a_-^\dagger)^N|0\rangle\\&=\langle 0|(a_+^\dagger-a_-^\dagger)^N(a_+-a_-)^N|0\rangle\\&=0\end{aligned} \qquad (2.30)$$

for $N\neq 0$.

Quantum mechanics of a one-particle subspace

Studying the quantum mechanics of a one-particle subspace gives some insight into the peculiarities of this theory. We have

$$\begin{aligned}&|1,0\rangle\equiv a_+^\dagger|0\rangle \ ,\\&|0,1\rangle\equiv a_-^\dagger|0\rangle \ ,\\&|1\rangle\equiv\frac{1}{\sqrt{2}}(|1,0\rangle+|0,1\rangle) \ ,\\&|2\rangle\equiv\frac{1}{\sqrt{2}}(|1,0\rangle-|0,1\rangle) \ ,\\&H=H_0+H' \ ,\\&H_0=E(a_+^\dagger a_+-a_-^\dagger a_-) \ ,\\&H'=E\frac{1}{1+2r}(a_+^\dagger-a_-^\dagger)(a_+-a_-) \ ,\\&H_0|1\rangle=E|1\rangle \ ,\\&H_0|2\rangle=E|2\rangle \ ,\\&H'|1\rangle=\frac{2E}{1+2r}|2\rangle \ ,\\&H'|2\rangle=0 \ ,\\&\langle 1,0|1,0\rangle=1 \ ,\\&\langle 0,1|0,1\rangle=-1 \ ,\\&\langle 1|1\rangle=\langle 2|2\rangle=0 \ ,\\&\langle 1|2\rangle=1 \ .\end{aligned} \qquad (2.31)$$

Before solving the Schrödinger equation, it is important to note the form of the completeness sum in a theory which contains negative-metric particles. The more general form also applies to the usual theories:

$$1 = \sum_n |n\rangle\langle n|n\rangle\langle n| \ . \tag{2.32}$$

The sum runs over some complete set of states which satisfy

$$\langle n'|n\rangle = \pm\delta_{nn'} \ .$$

For the special subspace in which we are now working completeness becomes

$$\begin{aligned} 1 &= |1,0\rangle\langle 1,0| - |0,1\rangle\langle 0,1| \\ &= |1\rangle\langle 2| + |2\rangle\langle 1| \ . \end{aligned}$$

To solve the Schrödinger equation we write

$$|\psi(t)\rangle = a_1(t)|1\rangle + a_2(t)|2\rangle \ .$$

The Schrödinger equation

$$\frac{\partial}{\partial t}|\psi(t)\rangle = -iH|\psi(t)\rangle$$

gives

$$\begin{aligned} \dot{a}_1 &= -iEa_1 \ , \\ \dot{a}_2 &= -iE\left(a_2 + \frac{2a_1}{1+2\gamma}\right) \ . \end{aligned}$$

The solution is

$$\begin{aligned} a_1(t) &= a_1(0)\,e^{-iEt} \ , \\ a_2(t) &= \left(a_2(0) - iEt\,\frac{2}{1+2\gamma}\,a_1(0)\right)e^{-iEt} \ . \end{aligned} \tag{2.33}$$

Here we begin to see that there are going to be difficulties associated with giving the theory a physical interpretation. To begin with, there is an ambiguity in identifying $P_2(t)$, the probability that the system will be in the state $|2\rangle$ at time t. Should we use

$$|a_2(t)|^2$$

or

$$|\langle 2|\psi(t)\rangle|^2 = |a_1(t)|^2 \ ?$$

When

$$a_1(0) = 0$$

the system has simple behavior which suggests that we should take

$$P_2(t) = |a_2(t)|^2 \ .$$

However, when $a_1(0)$ is not zero, P_2 grows like t^2 and certainly cannot be a probability. Faced with this, we may wonder just exactly what states or combinations of states are actually physically relevant. This cannot be answered in the noninteracting theory that we are using. At this point, we are content to note that completeness guarantees that there is one combination of amplitudes which is well behaved:

$$\begin{aligned} a_1^*(t)\,a_2(t) + a_2^*(t)\,a_1(t) &= |a_{1,0}(t)|^2 - |a_{0,1}(t)|^2 \\ &= a_1^*(0)\,a_2(0) + a_2^*(0)\,a_1(0) \\ &= |a_{1,0}(0)|^2 - |a_{0,1}(0)|^2 \ . \end{aligned} \tag{2.34}$$

The discussion of interactions will determine whether or not this can be given a physical interpretation.

Remarks

We will close this section with a couple of general remarks about theories with negative-metric particles. We would like to emphasize that there is nothing inherently wrong with a theory which contains negative-metric particles. The norm of a state is not an observable quantity. Only transition probabilities are observable. The only thing which is required is that the transition probabilities be non-negative and that probability be conserved. Thus, it is the form of the interactions which is important. For instance, there will be no problems with a theory in which the S matrix does not connect positive- and negative-metric states. Gupta-Bleuler QED is a more complicated example.

Finally, we will bring up a point which, although it is very simple, was not mentioned in the discussions of negative-metric theories that we encountered. The point is that the diagonal matrix element of an observable loses its meaning as an expectation value when negative-metric states are present. The correct generalization is easily derived. Suppose there exists a complete basis in which the self-adjoint observable A is diagonal:

$$A|n\rangle = A_n|n\rangle \text{ with } \langle n|n'\rangle = \pm\delta_{nn'} \ .$$

The probability to observe the value A_n in the state $|\psi\rangle$ is

$$P(A_n) = |\langle n|\psi\rangle|^2 \ .$$

The expected value of A is then

$$\begin{aligned} \bar{A} &= \sum_n A_n|\langle n|\psi\rangle|^2 \\ &= \sum_n \langle\psi|n\rangle A_n\langle n|\psi\rangle \\ &= \sum_n \langle\psi|A|n\rangle\langle n|\psi\rangle \\ &= \left\langle\psi\left|A\left(\sum_n |n\rangle\langle n|\right)\right|\psi\right\rangle \ . \end{aligned}$$

If we define an operator η by

$$\eta|n\rangle=\langle n|n\rangle|n\rangle ,$$

we get

$$\bar{A}=\langle\psi|A\sum_n\langle n|n\rangle|n\rangle\langle n|n\rangle\langle n|\psi\rangle$$
$$=\langle\psi|A\eta|\psi\rangle$$
$$\neq\langle\psi|A|\psi\rangle . \tag{2.35}$$

III. INTERACTIONS

In this section, we will investigate the properties of an interacting field theory with higher derivatives in the Lagrangian. Our purpose will be to continue the formal development of the theory and to resolve the questions which appeared in the discussion of the free field.

Scattering from a classical current

As a means of easing into our subject, we will consider the scattering of the field ϕ from a c-number source. If an interaction term

$$\mathcal{L}_I=-e\phi(x)J(x) \tag{3.1}$$

is added to the free Lagrangian that we have already studied, the Euler-Lagrange equation becomes

$$\partial^2\partial^2\phi(x)=eJ(x) . \tag{3.2}$$

The source $J(x)$ is a given c-number function. We will assume that its support is restricted to a bounded region of space-time.

The field equation can be solved by introducing the Green's functions for the differential operator $\partial^2\partial^2$:

$$\partial^2\partial^2 D_R(x)=\delta^4(x) ,$$
$$D_R(x)=0 \text{ for } x^2<0 ,$$
$$D_R(x)=0 \text{ for } 0<x^0 ,$$
$$\partial^2\partial^2 D_A(x)=\delta^4(x) ,$$
$$D_A(x)=0 \text{ for } x^2<0 ,$$
$$D_A(x)=0 \text{ for } x^0<0 .$$

These are related to the usual Green's functions by

$$D_{R(A)}(x)=-\frac{\partial}{\partial m^2}\Delta_{R(A)}(x,m^2)\bigg|_{m^2=0} .$$

It is then easy to see that

$$D_R(x)-D_A(x)=\frac{\partial}{\partial m^2}[\Delta_A(x)-\Delta_R(x)]\bigg|_{m^2=0}$$
$$=\frac{\partial}{\partial m^2}\Delta(x,m^2)\bigg|_{m^2=0}$$
$$=D(x) .$$

When the in and out fields are introduced in the usual way

$$\phi_{\text{in}}(x)=\phi(x)-e\int d^4y\,D_R(x-y)J(y) ,$$

$$\phi_{\text{out}}(x)=\phi(x)-e\int d^4y\,D_A(x-y)J(y) ,$$

we find that

$$\phi_{\text{out}}(x)=\phi_{\text{in}}(x)+e\int d^4y\,D(x-y)J(y) . \tag{3.3}$$

As usual, ϕ is quantized by assuming it has the same equal-time commutation relations as the corresponding free field. By their definitions, we can see that the in and out fields are free and that they have the same equal-time commutators as ϕ. The in and out fields are then two copies of the free field that we have studied in detail. In particular, they will have the momentum-space structure of the free field. The relationship between in and out fields in momentum space is then

$$\begin{aligned}\phi_{\text{out}}(k)&=\phi_{\text{in}}(k)+2\pi ieJ(k) ,\\ \psi_{\text{out}}(k)&=\psi_{\text{in}}(k)+2\pi ieK(k) ,\end{aligned} \tag{3.4}$$

where

$$k^0=|\vec{k}|$$

and

$$K(k)\equiv k^0\frac{\partial}{\partial k^0}J(k)\bigg|_{k^0=|\vec{k}|} .$$

We will now use this solution to continue the discussion of the one-particle subspace that we began in Sec. II. Suppose that the initial state is the in vacuum $|0\,\text{in}\rangle$. By using the solution to the field equation, we can expand the in vacuum in terms of out states. The result is

$$\begin{aligned}|0\,\text{in}\rangle=&|0\,\text{out}\rangle\langle 0\,\text{out}|0\,\text{in}\rangle+|1k\,\text{out}\rangle\frac{N(1+2r)}{\sqrt{2}}2\pi ieJ(k)\langle 0\,\text{out}|0\,\text{in}\rangle\\ &+|2k\,\text{out}\rangle\frac{N}{\sqrt{2}}2\pi ie[2K(k)-J(k)]\langle 0\,\text{out}|0\,\text{in}\rangle\\ &+\cdots .\end{aligned} \tag{3.5}$$

The notation is

$$|1k\,\text{out}\rangle = \frac{1}{\sqrt{2}}\,[a^\dagger_{+\text{out}}(k) + a^\dagger_{-\text{out}}(k)]|0\,\text{out}\rangle\ ,$$

$$|2k\,\text{out}\rangle = \frac{1}{\sqrt{2}}\,[a^\dagger_{+\text{out}}(k) - a^\dagger_{-\text{out}}(k)]|0\,\text{out}\rangle\ ,$$

$$N = \left[\frac{(2\pi)^3}{4|\vec{k}|^3(1+2r)}\right]^{1/2}\ .$$

The dots represent states of other momenta and more particles. This result shows that the source can create both $|1\rangle$ and $|2\rangle$ type states. Therefore, the problems that we touched on in Sec. II will not be avoided in any simple way.

In that discussion, we also brought up the possibility of attaching physical significance only to the well-behaved combination of amplitudes

$$\langle\psi|1\rangle\langle 2|\psi\rangle + \langle\psi|2\rangle\langle 1|\psi\rangle\ .$$

Let us explore this idea in more detail. The motivation for this suggestion is the need to find an interpretation of the theory which is consistent with the conservation of probability. In a Hilbert space of states, the conservation of probability follows from the completeness relation for the in and out states. Consider

$$P_{A\to B} = |\langle B\,\text{out}|A\,\text{in}\rangle|^2\ ,$$

$$\sum_B P_{A\to B} = \sum_B \langle A\,\text{in}|B\,\text{out}\rangle\langle B\,\text{out}|A\,\text{in}\rangle$$

$$= \langle A\,\text{in}|A\,\text{in}\rangle$$

$$= 1\ .$$

In the theory we are dealing with, it is

$$|1\rangle\langle 2| + |2\rangle\langle 1| = |1,0\rangle\langle 1,0| - |0,1\rangle\langle 0,1|$$

which appears in the completeness relation. If we could find a consistent interpretation of this combination as a transition probability, the conservation of probability would follow. This, of course, cannot be done by fiat. One must supply detailed physical arguments to show why

$$\langle\psi|1,0\rangle\langle 1,0|\psi\rangle \text{ and } \langle\psi|0,1\rangle\langle 0,1|\psi\rangle$$

are not separately observable. Even before doing that, though, it will be necessary to show that

$$\langle\psi|1\rangle\langle 2|\psi\rangle + \langle\psi|2\rangle\langle 1|\psi\rangle$$

is positive. Let us check this for the case of scattering from a c-number source. We find that

$$\langle 0\,\text{in}|1\,\text{out}\rangle\langle 2\,\text{out}|0\,\text{in}\rangle + \langle 0\,\text{in}|2\,\text{out}\rangle\langle 1\,\text{out}|0\,\text{in}\rangle = (2\pi)^2 e^2 N^2 (1+2r)|\langle 0\,\text{out}|0\,\text{in}\rangle|^2 X\ ,$$

with

$$X(k) = J^*(k)K(k) + K^*(k)J(k) - J^*(k)J(k)\ .$$

Since the coefficient of X is positive, we are interested in whether or not X is positive. Using the definition of K, we find that

$$X = \left[k^0 \frac{\partial}{\partial k^0}\{J^*(k^0,\vec{k})J(k^0,\vec{k})\} - \{J^*(k^0,\vec{k})J(k^0,\vec{k})\}\right]_{k^0=|\vec{k}|}\ .$$

To show that this need not be positive, we will construct a simple counterexample. Suppose $J(t,\vec{x})$ is a product of a function of t and a function of $\vec{x}$. Then $J(k^0,\vec{k})$ will also be a product

$$J(k,\vec{k}) = T(k^0)S(\vec{k})\ .$$

This gives

$$X = |S(\vec{k})|^2\left(k^0\frac{\partial}{\partial k^0}A(k^0) - A(k^0)\right)_{k^0=|\vec{k}|}\ ,$$

with

$$A(k^0) \equiv |T(k^0)|^2\ .$$

Now assume that the function of time is a good function. This implies that $A(k^0)$ is a good function. In particular,

$$\lim_{k^0\to\infty} A(k^0) = 0\ .$$

A few manipulations show that

$$X \geq 0$$

implies

$$A(k^0) \to \infty \text{ as } k^0 \to \infty\ .$$

The conclusion is that for scattering from a c-number source, we cannot guarantee that

$$\langle\psi|1\rangle\langle 2|\psi\rangle + \langle\psi|2\rangle\langle 1|\psi\rangle$$

is positive.

This does not bode well for our theory. Our next task will be to determine whether or not similar unfortunate results are obtained in a fully interacting theory.

Perturbation theory in the Heisenberg picture

In this section, we will develop perturbation theory in the Heisenberg picture for a sample interacting theory. The goal of the discussion will be to answer (within the context of perturbation theory) the questions about physical interpretation which have come up in earlier sections. We have chosen to work in the Heisenberg picture so that our results cannot be questioned on the basis of certain technical difficulties which appear in the derivation of the Feynman diagrams. They will be discussed later.

Let ψ be an ordinary scalar field, and consider the Lagrangian

$$\mathcal{L}=\tfrac{1}{2}\partial^2\phi\partial^2\phi+\tfrac{1}{2}(\partial_\mu\psi\partial^\mu\psi-M^2\psi)-e\phi\psi^3 . \qquad (3.6)$$

The field equations are

$$\partial^2\partial^2\phi=e\psi^3 ,$$
$$(\partial^2+M^2)\psi=e'\phi\psi^2 ,$$

with

$$e'\equiv-3e .$$

Since our considerations will be restricted to the lowest order of perturbation theory, we will not need to know the wave-function renormalization or the mass shift beyond zeroth order. It will simplify matters then to set

$$Z_\phi=Z_\psi=1 \text{ and } \delta M_\phi=\delta M_\psi=0$$

from the beginning. The in and out fields are then introduced in the usual way:

$$\phi_{\text{in(out)}}(x)=\phi(x)-e\int d^4y\, D_{R(A)}(x-y)\psi^3(y) ,$$
$$\psi_{\text{in(out)}}(x)=\psi(x)-e'\int d^4y\, \Delta_{R(A)}(x-y)\phi(y)\psi^2(y) .$$

By applying the "full" energy-momentum operator to these expressions we can verify that the in and out fields are Heisenberg operators:

$$i[P^\mu,\phi_{\text{in(out)}}(x)]=\partial^\mu\phi_{\text{in(out)}}(x) ,$$
$$i[P^\mu,\psi_{\text{in(out)}}(x)]=\partial^\mu\psi_{\text{in(out)}}(x) .$$

From the properties of the Green's functions it follows that

$$\partial^2\partial^2\phi_{\text{in(out)}}(x)=0 ,$$
$$(\partial^2+M^2)\psi_{\text{in(out)}}(x)=0 .$$

These results are, of course, nothing novel. As is usually done, we will give the in and out fields the structure of the free fields that were studied in detail. The states which are created and destroyed by these fields are identified with the asymptotic scattering states.

The first part of our program will be to calculate the nature of the ϕ particles that can be created in the collision of two ψ particles with momenta p_1 and p_2. The state of the system is $|p_1p_2\text{in}\rangle$, which we will expand in terms of the out states to order e. This state is

$$|p_1p_2\text{in}\rangle=a_{\text{in}}^\dagger(p_1)a_{\text{in}}^\dagger(p_2)|0\rangle .$$

The notation for the ψ field is

$$a_{\text{in}}(p)=\frac{1}{\sqrt{2\omega}}A_{\text{in}}(p) ,$$
$$\omega=(\vec{p}^{\,2}+M^2)^{1/2} ,$$
$$\psi_{\text{in}}(x)=\frac{1}{(2\pi)^{3/2}}\int d^4k\,\delta(k^2-M^2)\times\theta(k^0)[A_{\text{in}}(k)e^{-ik\cdot x}+\text{H.c.}] ,$$
$$A_{\text{in}}(k)=\frac{i}{(2\pi)^{3/2}}\int d^3x\, e^{ik\cdot x}\overleftrightarrow{\partial}_0\psi_{\text{in}}(x) .$$

Similar relationships hold for the out field. We then get

$$|p_1p_2\text{in}\rangle=\frac{1}{2(\omega_1\omega_2)^{1/2}}\frac{-i}{(2\pi)^{3/2}}\frac{-i}{(2\pi)^{3/2}}\int d^3x_1\int d^3x_2 e^{-ip_1\cdot x_1}e^{-ip_2\cdot x_2}\overleftrightarrow{\partial}_{x_1^0}\overleftrightarrow{\partial}_{x_2^0}\psi_{\text{in}}(x_1)\psi_{\text{in}}(x_2)|0\rangle .$$

To order e the field equations give

$$\psi_{\text{in}}(x)\cong\psi_{\text{out}}(x)+e'\int d^4y\,\Delta(x-y)\phi_{\text{out}}(y)\psi_{\text{out}}^2(y) .$$

When this approximation for ψ_{in} is used and the integrations are carried out, we find that

$$\begin{aligned}|p_1p_2\text{in}\rangle\cong|p_1p_2\text{out}\rangle+\frac{e'2\pi i}{(2\pi)^{3/2}(\omega_1\omega_2)^{1/2}}\int\frac{d^3k}{4|\vec{k}|^3}\frac{A_{\text{out}}^\dagger(\vec{p}-\vec{k})}{2[(\vec{P}-\vec{k})^2+M^2]^{1/2}}\\ \times\{\phi_{\text{out}}^\dagger(k)\delta(|\vec{k}|+[(\vec{P}-\vec{k})^2+M^2]^{1/2}-E)\\ -\psi_{\text{out}}^\dagger(k)\delta(|\vec{k}|+[(\vec{P}-\vec{k})^2+M^2]^{1/2}-E)\\ -\phi_{\text{out}}^\dagger(k)|\vec{k}|\delta'(|\vec{k}|+[(\vec{P}-\vec{k})^2+M^2]^{1/2}-E)\}|0\rangle ,\end{aligned} \qquad (3.7)$$

with

$$\vec{P}=\vec{p}_1+\vec{p}_2 \text{ and } E=\omega_1+\omega_2 .$$

It is important to notice that even though $|p_1p_2\text{in}\rangle$ contains $|1\rangle$ type out states it is nevertheless an eigenstate of H. This can be verified by direct calculation with

$$H(\phi,\psi)=H_{\text{out}}(\phi_{\text{out}},\psi_{\text{out}})\ .$$

By H_{out} (or H_{in}), we mean the Hamiltonian that results from the Lagrangian with the interaction term dropped. It is a sum of the free Hamiltonians for each field.

Now consider the final states

$$|1pk\,\text{out}\rangle=\frac{N(k)}{\sqrt{2}}a^\dagger_{\text{out}}(p)[2\psi^\dagger_{\text{out}}(k)-\phi^\dagger_{\text{out}}(k)]\,|0\rangle\ ,$$

$$|2pk\,\text{out}\rangle=\frac{N(k)}{\sqrt{2}}a^\dagger_{\text{out}}(p)\phi^\dagger_{\text{out}}(p)\phi^\dagger_{\text{out}}(k)\,|0\rangle\ .$$

For simplicity the parameter r has been set to zero. The matrix elements are

$$\begin{aligned}\langle 1pk\,\text{out}\,|p_1p_2\text{in}\rangle&=\frac{ie'}{2(2\pi)^2}\frac{1}{(\omega_1\omega_2)^{1/2}}\frac{1}{(\vec{p}^{\,2}+M^2)^{1/4}}\frac{1}{(4|\vec{k}|^3)^{1/2}}\delta^3(\vec{p}+\vec{k}-\vec{P})\\&\quad\times[\delta(|\vec{k}|+(\vec{p}+M^2)^{1/2}-E)-2|\vec{k}|\delta'(|\vec{k}|+(\vec{p}^{\,2}+M^2)^{1/2}-E)]\ ,\\ \langle 2pk\,\text{out}\,|p_1p_2\text{in}\rangle&=-\frac{ie'}{2(2\pi)^2}\frac{1}{(\omega_1\omega_2)^{1/2}}\frac{1}{(\vec{p}^{\,2}+M^2)^{1/4}}\frac{1}{(4|\vec{k}|^3)^{1/2}}\delta^3(\vec{p}+\vec{k}-\vec{P})\delta(|\vec{k}|+(\vec{p}^{\,2}+M^2)^{1/2}-E)\ .\end{aligned}\tag{3.8}$$

The two combinations which may have a physical interpretation are

$$|\langle 2pk\,\text{out}\,|p_1p_2\text{in}\rangle|^2$$

and

$$\langle p_1p_2\text{in}\,|2pk\,\text{out}\rangle\langle 1pk\,\text{out}\,|p_1p_2\text{in}\rangle+\langle p_1p_2\text{in}\,|1pk\,\text{out}\rangle\langle 2pk\,\text{out}\,|p_1p_2\text{in}\rangle\ .$$

The quantity

$$|\langle 1pk\,\text{out}\,|p_1p_2\text{in}\rangle|^2$$

is ruled out since we have not been able to define

$$\delta'(x)\delta'(x)\ .$$

The expressions for the interesting combinations are

$$|\langle 2pk\,\text{out}\,|p_1p_2\text{in}\rangle|^2=\frac{e'^2}{4(2\pi)^4}\frac{1}{\omega_1\omega_2}\frac{1}{(\vec{p}^{\,2}+M^2)^{1/2}}\frac{1}{4|\vec{k}|^3}\delta^4(p+k-P)\delta^4(0)\tag{3.9}$$

and

$$\begin{aligned}&\langle p_1p_2\text{in}\,|2pk\,\text{out}\rangle\langle 1pk\,\text{out}\,|p_1p_2\text{in}\rangle+\langle p_1p_2\text{in}\,|1pk\,\text{out}\rangle\langle 2pk\,\text{out}\,|p_1p_2\text{in}\rangle\\&\qquad=-\frac{e'^2}{2(2\pi)^4}\frac{1}{\omega_1\omega_2}\frac{1}{(\vec{p}^{\,2}+M^2)^{1/2}}\frac{1}{4|\vec{k}|^3}\delta^3(0)\delta^3(\vec{p}+\vec{k}-\vec{P})\delta(|\vec{k}|+(\vec{p}^{\,2}+M^2)^{1/2}-E)\\&\qquad\quad\times[\delta(|\vec{k}|+(\vec{p}^{\,2}+M^2)^{1/2}-E)-2|\vec{k}|\delta'(|\vec{k}|+(\vec{p}^{\,2}+M^2)^{1/2}-E)]\ .\end{aligned}\tag{3.10}$$

Although this last expression is unusual, it leads to a finite cross section when integrated over the acceptance of the counters. We will show how this is done when we discuss unitarity.

The next part of the program is to calculate the e^2 contribution to the scattering of two ψ particles. In fact, we will only be interested in the part of the amplitude which has an imaginary part. To order e^2, we have

$$\psi_{in}(x)=\psi_{out}(x)+e'\int d^4y\,\Delta(x-y)\phi_{out}(y)\psi_{out}^2(y)+e'e\int d^4y_1\Delta(x-y_1)\int d^4y_2D_A(y_1-y_2)\psi_{out}^3(y_2)\psi_{out}^2(y_1)$$
$$+e'^2\int d^4y_1\Delta(x-y_1)\phi_{out}(y_1)\int d^4y_2\Delta_A(y_1-y_2)\phi_{out}(y_2)\psi_{out}^2(y_2)\psi_{out}(y_2)$$
$$+e'^2\int d^4y_1\Delta(x-y_1)\phi_{out}(y_1)\psi_{out}(y_1)\int d^4y_2\Delta_A(y_1-y_2)\phi_{out}(y_2)\psi_{out}^2(y_2)\ .$$

This is substituted into the expression for the initial state $|p'p_2\text{in}\rangle$ in terms of ψ_{in} that we used before. The computations which follow are complicated and boring. The result is

$$\langle P_1P_2\text{out}|p_1p_2\text{in}\rangle_I=\frac{2i}{4(\omega_1\omega_2\Omega_1\Omega_2)^{1/2}}\frac{1}{(2\pi)^3}\int d^4y_1\int d^4y_2e^{-i(p_1+p_2)\cdot y_1}e^{i(P_1+P_2)\cdot y_2}$$
$$\times[6ee'D_A(y_1-y_2)\Delta_-(y_1-y_2)+4e'^2D_+(y_1-y_2)\Delta_A(y_1-y_2)]\ .$$

In this expression, the momenta of the outgoing ψ particles are P_1 and P_2 with zeroth components Ω_1 and Ω_2. The subscript I on the matrix element indicates that we have retained only the terms which contribute to the imaginary part in the kinematic region for the scattering process. The new singular function is

$$D_+(x)=-\frac{\partial}{\partial m^2}\ \Delta_+(x,m^2)\Big|_{m^2=0}\ .$$

In momentum space this becomes

$$\langle P_1P_2\text{out}|p_1p_2\text{in}\rangle_I=\frac{ie'^2}{(2\pi)^5}\frac{1}{(\omega_1\omega_2\Omega_1\Omega_2)^{1/2}}\delta^4(p_1+p_2-P_1-P_2)$$
$$\times\frac{\partial}{\partial m^2}\int d^4k\left(\frac{2\theta(-p_1^0-p_2^0-k^0)\delta((p_1+p_2+k)^2-m^2)}{k^2-M^2-ik^0\epsilon}-\frac{\theta(p_1^0+p_2^0+k^0)\delta((p_1+p_2+k)^2-M^2)}{k^2-m^2-ik^2\epsilon}\right)\Bigg|_{m^2=0}\ .\tag{3.11}$$

To prepare for the discussion of the conservation of probability, the zeroth-order contribution should be added on:

$$\langle P_1P_2\text{out}|p_1p_2\text{in}\rangle_U=\delta^3(\vec p_1-\vec P_1)\delta^3(\vec p_2-\vec P_2)+\delta^3(\vec p_1-\vec P_2)\delta^3(\vec p_2-\vec P_1)+\langle P_1P_2\text{out}|p_1p_2\text{in}\rangle_I\ .$$

When this is used to calculate the transition probability to order e^2, we find

$$|\langle P_1P_2\text{out}|p_1p_2\text{in}\rangle_U|^2=2\delta^3(0)\delta^3(0)\delta^3(\vec p_1-\vec P_1)\delta^3(\vec p_2-\vec P_2)+2\delta^3(0)\delta^3(0)\delta^3(\vec p_1-\vec P_2)\delta^3(\vec p_2-\vec P_1)$$
$$+[\delta^3(\vec p_1-\vec P_1)\delta^3(\vec p_2-\vec P_2)+\delta^3(\vec p_1-\vec P_2)\delta^3(\vec p_2-\vec P_1)]\delta^4(0)\frac{ie'^2}{(2\pi)^5}\frac{1}{(\omega_1\omega_2\Omega_1\Omega_2)^{1/2}}M\ ,\tag{3.12}$$

with

$$M=\frac{\partial}{\partial m^2}\int d^4k\,2\pi i[2\theta(-p_1^0-p_2^0-k^0)\delta((p_1+p_2+k)^2-m^2)\epsilon(k^0)\delta(k^2-M^2)$$
$$-\epsilon(-p_1^0-p_2^0-k^0)\delta((p_1+p_2+k)^2-m^2)\theta(-k^0)\delta(k^2-M^2)]\ .$$

This expression for M results from observing that

$$\frac{1}{k^2-M^2-ik^0\epsilon}-\frac{1}{k^2-M^2+ik^0\epsilon}=2\pi i\epsilon(k^0)\delta(k^2-M^2)\ .$$

A change of variables

$$k\to-k-p_1-p_2$$

was also made in the second term. In the frame

$$\vec P=\vec p_1+\vec p_2=0$$

the k integration is easy. After differentiating with respect to m^2, setting m^2 to zero and substituting back in to the formula for the probability we find that

$$|\langle P_1P_2\text{out}|p_1p_2\text{in}\rangle_U|^2=2\delta^3(0)\delta^3(0)\delta^3(\vec p_1-\vec P_1)\delta^3(\vec p_2-\vec P_2)+2\delta^3(0)\delta^3(0)\delta^3(\vec p_1-\vec P_2)\delta^3(\vec p_2-\vec P_1)$$
$$+[\delta^3(\vec p_1-\vec P_1)\delta^3(\vec p_2-\vec P_2)+\delta^3(\vec p_1-\vec P_2)\delta^3(\vec p_2-\vec P_1)]\delta^4(0)\frac{e'^2}{4(2\pi)^3}\frac{1}{\omega_1\omega_2}\frac{1}{P^2}\frac{P^2+M^2}{P^2-M^2}\tag{3.13}$$

with

$$P^2=(p_1+p_2)^2\ .$$

Now that the calculations are out of the way, we can think about the physical interpretation of the results. To simplify the notation, let $P(P,p,k)$ stand for whichever expression

$$A(P,p,k)\equiv|\langle 2pk\text{out}|p_1p_2\text{in}\rangle|^2$$

or

$$S(P,p,k)\equiv\langle p_1p_2\text{in}|2pk\text{ out}\rangle\langle 1pk\text{ out}|p_1p_2\text{in}\rangle + \langle p_1p_2\text{in}|1pk\text{ out}\rangle\langle 2pk\text{ out}|p_1p_2\text{in}\rangle$$

might be used to represent the probability for production of ϕ particles. In addition, let

$$E(P,P_1P_2)\equiv|\langle P_1P_1\text{out}|p_1p_2\text{in}\rangle_U|^2\ .$$

The conservation of probability requires that

$$1=\sum_n P(P\to n)\ .$$

Assuming that this holds order by order in perturbation theory, a condition on the quantities that we have calculated results:

$$\delta^3(0)\delta^3(0)+\delta^3(0)\delta^3(0)=\frac{1}{2}\int d^3P_1\int d^3P_2E(P,P_1,P_2) + \int d^3p\int d^3k\,P(P,p,k)\ .$$

The δ functions on the left-hand side arise from the normalization conventions:

$$\begin{aligned}``1"&=\langle p_1p_2\text{in}|p_1p_2\text{in}\rangle\\ &=\lim_{p_1'\to p_1}\lim_{p_2'\to p_2}\langle p_1'p_2'\text{in}|p_1p_2\text{in}\rangle\\ &=\lim_{p_1'\to p_1}\lim_{p_2'\to p_2}[\delta^3(\vec{p}_1-\vec{p}_1')\delta^3(\vec{p}_2-\vec{p}_2') + \delta^3(\vec{p}_1-\vec{p}_2')\delta^3(\vec{p}_2-\vec{p}_1')]\\ &=\delta^3(0)\delta^3(0)+\delta^3(0)\delta^3(0)\ .\end{aligned}$$

The factor of $\frac{1}{2}$ appears to avoid the double counting of states when calculating a total cross section that has identical particles in the final state. After carrying out the p_1 and p_2 integrations the conservation condition becomes

$$0=\delta^4(0)\frac{e'^2}{4(2\pi)^3}\frac{1}{\omega_1\omega_2}\frac{1}{P^2}\frac{P^2+M^2}{P^2-M^2} + \int d^3p\int d^3k\,P(P,p,k)\ . \tag{3.14}$$

This equation shows that we are in trouble already. The first term is positive, which, if the condition is satisfied, requires the second term to be negative. However, the second term is supposed to be a transition probability, which should be positive. Whether the sign of the first term can be changed by higher orders of perturbation theory or by other forms for the interaction is not known. Since $A(P,p,k)$ is positive, it cannot possibly satisfy the condition, and there is no point in integrating it.

The integration of $S(P,p,k)$ is a bit tricky so we will indicate how it is done. The vector δ function is used to do the p integration and the k integration is switched to polar coordinates:

$$\int d^3p\int d^3k\,S(P,p,k)=-\delta^3(0)\frac{e'^2}{8(2\pi)^4}\frac{1}{\omega_1\omega_2}\int\frac{dk\,d\Omega}{k}\frac{1}{[(\vec{P}-\vec{k})^2+M^2]^{1/2}} \times\delta(k+[(\vec{P}-\vec{k})^2+M^2]^{1/2}-E)\{\delta(k+[(\vec{P}-\vec{k})^2+M^2]^{1/2}-E) - 2k\delta'(k+[(\vec{P}-\vec{k})^2+M^2]^{1/2}-E)\}\ .$$

The first term is of the usual type and produces a $\delta(0)$ to go with $\delta^3(0)$. The second term presents us with the problem of defining an integral of the form

$$I=\int dx f(x)\delta(g(x))\delta'(g(x))\ .$$

We use the following method:

$$\begin{aligned}I&=\int dx f(x)\frac{d}{dg}[\tfrac{1}{2}\delta(g(x))\delta(g(x))]\\ &=\tfrac{1}{2}\int dx f(x)\frac{1}{g'(x)}\frac{d}{dx}[\delta(g(x))\delta(g(x))]\\ &=-\tfrac{1}{2}\int dx\,\delta(g(x))\delta(g(x))\frac{d}{dx}\frac{f(x)}{g'(x)}\\ &=-\tfrac{1}{2}\delta(0)\int dx\,\delta(g(x))\frac{d}{dx}\frac{f(x)}{g'(x)}\\ &=-\tfrac{1}{2}\delta(0)\int dx\frac{\delta(x-x_0)}{|g'(x_0)|}\frac{d}{dx}\frac{f(x)}{g'(x)}\ .\end{aligned}$$

With this the k integrations can be done (most easily in the $\vec{P}=0$ frame). The result is

$$\int d^3k\int d^3p\,S(P,p,k) = -\delta^4(0)\frac{e'^2}{4(2\pi)^3}\frac{1}{\omega_1\omega_2}\frac{1}{P^2}\frac{P^2+M^2}{P^2-M^2}\ .$$

As expected, this satisfies the condition but has the wrong sign for a probability.

The conclusion of this section is, then, that the theory we are using does not allow the usual sort of probability interpretation. To show that this unfortunate result holds for all orders of perturbation theory and in all theories with higher derivatives in the Lagrangian will require much theoretical labor.

Diagrammatic approach

In this section, we will discuss the derivation of the Feynman rules for the sample theory we have been working with. We do not intend to present a complete detailed derivation. Rather, we will follow through the treatment in Bjorken and Drell[19] and supply the changes that are necessary for the theory we have.

The first step is the derivation of the reduction formulas. Recall that the one-particle states are created by operating on the vacuum:

$$|1k\,\text{in}\rangle = (N/\sqrt{2})[2\psi_{\text{in}}^\dagger(k) + \phi_{\text{in}}^\dagger(k)]\,|0\rangle \;,$$

$$|2k\,\text{in}\rangle = (N/\sqrt{2})\phi_{\text{in}}^\dagger(k)\,|0\rangle \;.$$

Again we have set r to zero. Recall that the expansion of the field is

$$\phi_{\text{in}}(x) = \int d^4k\, \delta'(k^2)\theta(k^0)[\phi_{\text{in}}(k)e^{-ik\cdot x} + \text{H.c.}] \;.$$

Operating with ∂^2, we find a simple result,

$$\partial^2\phi_{\text{in}}(x) = \int d^4k\, \delta(k^2)\theta(k^0)[\phi_{\text{in}}(k)e^{-ik\cdot x} + \text{H.c.}] \;,$$

which can be inverted in the usual way to get a formula for $\phi_{\text{in}}(k)$:

$$\phi_{\text{in}}(k) = \frac{i}{(2\pi)^3}\int d^3x\, e^{ik\cdot x}\, \overleftrightarrow{\partial}_0 \partial^2\phi_{\text{in}}(x) \;. \tag{3.15}$$

To get the formula for $2\psi(k) - \phi(k)$, we combine and operate until we get

$$(4\partial_0{}^2 - 2t\partial_0\partial^2 - 3\partial^2)\phi_{\text{in}}(x) = \int \frac{d^3k}{2|\vec{k}|}\,\{[2\psi(k) - \phi(k)]\,e^{-ik\cdot x} + \text{H.c.}\} \;,$$

which inverts to

$$2\psi(k) - \phi(k) = \frac{i}{(2\pi)^3}\int d^3x\, e^{ik\cdot x}\, \overleftrightarrow{\partial}_0[4\partial_0{}^2 - 2t\partial_0\partial^2 - 3\partial^2]\phi_{\text{in}}(x) \;. \tag{3.16}$$

In reducing an in particle of type $|1\rangle$ from the initial state, we find a formula like

$$\frac{-i}{(2\pi)^3}\int d^3x\, e^{-ik\cdot x}\, \overleftrightarrow{\partial}_0(4\partial_0{}^2 - 2t\partial_0\partial^2 - 3\partial^2)\langle A\,|[X\phi_{\text{in}}(x) - \phi_{\text{out}}(x)X]\,|B\rangle \;,$$

which we can write as

$$\frac{1}{\sqrt{Z}}(\lim_{t\to\infty} - \lim_{t\to-\infty})\,\frac{i}{(2\pi)^3}\int d^3x\, e^{-ik\cdot x}\, \overleftrightarrow{\partial}_0(4\partial_0{}^2 - 2t\partial_0\partial^2 - 3\partial^2)\langle A\,|\,T[\phi(x)X]\,|B\rangle \;.$$

This expression reveals the first difficulty with the derivation. The limits may not exist and the asymptotic condition may not be satisfied due to bad behavior at large t. We will simply hope for the best and proceed. The standard manipulations then reduce this to

$$\frac{1}{\sqrt{Z}}\,\frac{i}{(2\pi)^3}\int d^4x\, e^{-ik\cdot x}\,\partial^2(4\partial_0{}^2 - 2t\partial_0\partial^2 - 3\partial^2)\langle A|\,T[\phi(x)X]\,|B\rangle$$
$$= \frac{1}{\sqrt{Z}}\,\frac{-i}{(2\pi)^3}\int d^4x\, e^{-ik\cdot x}(3 + 2t\partial_0)\,\partial^2\partial^2\langle A|\,T[\phi(x)X]\,|B\rangle. \tag{3.17}$$

For the $|2\rangle$ particle the corresponding result is

$$\frac{1}{\sqrt{Z}}\,\frac{i}{(2\pi)^3}\int d^4x\, e^{-ik\cdot x}\,\partial^2\partial^2\langle A|\,T[\phi(x)X]\,|B\rangle. \tag{3.18}$$

When all of the in and out particles have been reduced into the time-ordered product, our attention is shifted to the τ functions

$$\tau(x_1, x_2, \ldots) = \langle 0|\,T[\phi(x_1)\psi(x_2)\cdots]\,|0\rangle.$$

The perturbation expansion for the τ functions is developed by introducing the U matrix which has the property

$$\phi(x) = U^{-1}(t)\phi_{\text{in}}(x)U(t),$$

with similar expressions for $\dot{\phi}$, $\ddot{\phi}$, $\dddot{\phi}$, ψ, and $\dot{\psi}$. The derivation of the properties of U proceeds smoothly. It is convenient to introduce the operator

$$U(t,t')=U(t)U^{-1}(t').$$

No problems are encountered in obtaining the usual expression for $U(t,t')$ in terms of the time-ordered products of the interaction Hamiltonian. The next step is to use the U matrix to write the τ functions in terms of the in fields:

$$\tau(x_1x_2\cdots x_n)=\langle 0|T[U^{-1}(t_1)\phi_{\rm in}(x_1)U(t_1,t_2)\psi_{\rm in}(x_2)\cdots U(t_n)]|0\rangle$$

$$=\lim_{t\to+\infty}\langle 0|U^{-1}(t)T[U(t,t_1)\phi_{\rm in}(x_1)U(t_1,t_2)\psi_{\rm in}(x_2)\cdots U(t_n,-t)]U(-t)|0\rangle .$$

The U operators outside of the time-ordered product are handled by showing that

$$\lim_{t\to\pm\infty}U(t)|0\rangle=\lambda_\pm|0\rangle .$$

The second technical difficulty is encountered when we attempt to verify this condition. Following Bjorken and Drell, we consider an in state which contains a ϕ particle of momentum p plus anything else α. Observe that

$$\langle \alpha p\ {\rm in}|U(t)|0\rangle=\frac{N}{\sqrt{2}}\frac{i}{(2\pi)^3}\int d^3x\, e^{ip\cdot x}\overleftrightarrow{\partial}_0\langle\alpha|\Phi_{\rm in}(x)U(t)|0\rangle . \qquad (3.19)$$

The operator Φ is

$$(4\partial_0{}^2-2x^0\partial_0\partial^2-3\partial^2)\phi_{\rm in}(x)\quad\text{or}\quad \partial^2\phi_{\rm in}(x),$$

depending upon whether the state contains a $|1\rangle$ or $|2\rangle$ type particle. Using the properties of U, calculate

$$\langle\alpha p|U(t)|0\rangle=\frac{N}{\sqrt{2}}\frac{i}{(2\pi)^3}\int d^3x\, e^{ip\cdot x}\overleftrightarrow{\partial}_0\langle\alpha|U(x^0)\Phi(x)U^{-1}(x^0)U(t)|0\rangle$$

$$=\frac{N}{\sqrt{2}}\frac{i}{(2\pi)^3}\left\langle\alpha\middle|U(x^0)\left[\int d^3x\, e^{ip\cdot x}\overleftrightarrow{\partial}_0\Phi(x)\right]U^{-1}(x^0)U(t)\middle|0\right\rangle$$

$$+\frac{N}{\sqrt{2}}\frac{i}{(2\pi)^3}\int d^3x\, e^{ip\cdot x}\langle\alpha|[\dot{U}(x^0)\Phi(x)U^{-1}(x^0)U(t)+U(x^0)\Phi(x)\dot{U}^{-1}(x^0)U(t)]|0\rangle .$$

Set $x^0=t$ and let $t\to-\infty$. The first term will contain the quantities

$$\langle\alpha|U(-\infty)[2\psi_{\rm in}(p)-\phi_{\rm in}(p)]|0\rangle$$

or

$$\langle\alpha|U(-\infty)\phi_{\rm in}(p)|0\rangle ,$$

which are zero. A little manipulation shows that the matrix element in the second term is

$$\lim_{t\to-\infty}\langle\alpha|[-iH_I(t),\Phi_{\rm in}(t,\vec{x})]U(t)|0\rangle ,$$

with

$$H_I(t)=-\mathcal{L}_I(\phi_{\rm in},\psi_{\rm in})$$
$$=e\phi_{\rm in}\psi_{\rm in}{}^3 .$$

If the original ϕ particle was a $|1\rangle$ type, $\Phi_{\rm in}$ contains $\dddot{\phi}_{\rm in}$ and the commutator is not zero. The only way around this that we can see is to say that $H_I(t)$ should be interpreted to contain an adiabatic switching factor

$$e^{-\epsilon|t|}.$$

To understand this better, we can use lowest-order perturbation theory to calculate the matrix element in Eq. (3.19) directly:

$$\lim_{t\to-\infty}\langle \alpha 1p \text{ in}|U(t)|0\rangle = \lim_{t\to-\infty}\lim_{t'\to-\infty}\langle \alpha 1p \text{ in}|U(t,t')|0\rangle$$
$$\cong -i\lim_{t\to-\infty}\lim_{t'\to-\infty}\left\langle \alpha 1p \text{ in}\left|\int_{t'}^{t}dt_1\, H_I(t_1)\right|0\right\rangle$$
$$=-ie\lim_{t\to-\infty}\lim_{t'\to-\infty}\int_{t'}^{t}dt_1\int d^3x\,\frac{1}{\sqrt{2}\,(2\pi)^3}\left[\frac{(2\pi)^3}{4|\vec{p}|^3}\right]^{1/2}(1-2i|\vec{p}|t_1)\,e^{ip\cdot x}\langle\alpha|\psi^3(x)|0\rangle.$$

If

$$\langle\alpha|\psi^3(x)|0\rangle\sim e^{iq\cdot x},$$

the resulting integral will be of the form

$$\int_{t'}^{t}dt_1\int d^3x\,(1-2i|\vec{p}|t_1)e^{i(p^0+q^0)t}\,e^{-i(\vec{p}+\vec{q})\cdot\vec{x}}=(2\pi)^3\delta^3(\vec{p}+\vec{q})\left(1-2p^0\frac{\partial}{\partial p^0}\right)\int_{t'}^{t}dt_1 e^{i(p^0+q^0)t_1}\bigg|_{p^0=|\vec{p}|}. \tag{3.20}$$

Without a convergence factor, this integral is not defined if we intend to take the limit $t'\to-\infty$ followed by $t\to-\infty$. This shows again the need for the adiabatic switching factor in H_I. Including this factor will modify the integral in Eq. (3.20) to

$$\left(1-2p^0\frac{\partial}{\partial p^0}\right)\int_{t'}^{t}dt_1\, e^{[\epsilon+i(p^0+q^0)]t_1}\bigg|_{p^0=|\vec{p}|};$$

with t' and t going to minus infinity this is certainly zero. The conclusion is that this part of the derivation of the Feynman rules requires that H_I be interpreted with an adiabatic switching factor.

No further problems are encountered in completing the derivation of the Feynman rules. The only work which is left for us to do is to examine the properties of the propagator

$$\langle 0|T[\phi_{\text{in}}(x)\phi_{\text{in}}(x')]|0\rangle.$$

For this discussion the designation "in" on the fields will not be indicated.

Using the momentum-space commutation relations, the time-ordered product is easily calculated. Since it is formed from

$$\langle 0|\phi(x)\phi(x')|0\rangle \text{ and } \langle 0|\phi(x')\phi(x)|0\rangle,$$

we begin by looking at

$$\langle 0|\phi(x)\phi(x')|0\rangle=\int\frac{d^3k}{4|\vec{k}|^3}\int\frac{d^3k'}{4|\vec{k}|^3}\langle 0|\{[\chi(k)+i|\vec{k}|t\phi(k)]e^{-ik\cdot x}+\text{H.c.}\}\{[\chi(k')+i|\vec{k}'|t'\phi(k')]e^{-ik'\cdot x'}+\text{H.c.}\}|0\rangle$$
$$=\int\frac{d^3k}{4|\vec{k}|^3}\int\frac{d^3k}{4|\vec{k}|^3}[\chi(k)+i|\vec{k}|t\phi(k),\chi^\dagger(k')-i|\vec{k}'|t'\phi^\dagger(k')]e^{-ik\cdot x}e^{+ik'\cdot x'}.$$

Working out the commutators and using the resulting δ function to do the k' integration gives

$$\langle 0|\phi(x)\phi(x')|0\rangle=-\frac{1}{(2\pi)^3}\int\frac{d^3k}{4|\vec{k}|^3}[1+i|\vec{k}|(t-t')]e^{-ik\cdot(x-x')}.$$

To get $\langle 0|\phi(x')\phi(x)|0\rangle$ simply interchange x and x'.
After observing that

$$\langle 0|\phi(x)\phi(x')|0\rangle=\frac{\partial}{\partial m^2}\Delta_+(x-x',m^2)\bigg|_{m^2=0}.$$

It is no problem to show that

$$\langle 0|T[\phi(x)\phi(x')]|0\rangle=\frac{\partial}{\partial m^2}i\Delta_F(x-x',m^2)\bigg|_{m^2=0}$$
$$=\frac{i}{(2\pi)^4}\int d^4k\,e^{-ik\cdot(x-x')}\frac{1}{(k^2+i\epsilon)^2}. \tag{3.21}$$

This shows that the time-ordered product solves the differential equation

$$\partial^2\partial^2\langle 0|T[\phi(x)\phi(x')]|0\rangle = i\delta^4(x-x').$$

By direct calculation one can verify that

$$\partial^2\partial^2\frac{1}{4(2\pi)^2}\ln(-x^2+i\epsilon) = i\delta^4(x).$$

Thus, the propagator in coordinate space is

$$\langle 0|T[\phi(x)\,\phi(x')]|0\rangle = \frac{1}{4(2\pi)^2}\ln[-(x-x')^2+i\epsilon]+C\ , \tag{3.22}$$

where C is a (perhaps infinite) constant.

Equation (3.22) should be interpreted with some care. Considered as an ordinary integral the k integration diverges at $k=0$. (This is the same behavior that is found for the photon propagator two-dimensional QED.) Considered as a distribution the result depends on the method used to regulate the integral. It is also interesting to note that although the left-hand side of Eq. (3.22) is formally scale invariant the right-hand side is not.

As an alternate way to calculate the propagator we can do a (trivial) spectral sum

$$\langle 0|\phi(x)\,\phi(x')|0\rangle = \int d^3k\,\langle 0|\phi(x)\,(|1k\rangle\langle 2k| + |2k\rangle\langle 1k|)\,\phi(x')|0\rangle\ .$$

A typical matrix element is

$$\langle 0|\phi(x)|1k\rangle = \langle 0|\phi(0)\,e^{-iP\cdot x}|1k\rangle = \langle 0|\phi(0)\,e^{-iHt}|1k\rangle\,e^{i\vec{k}\cdot\vec{x}}\ .$$

Recall that

$$H = H_0 + H'$$

and

$$[H_0, H'] = 0\ .$$

Using

$$H_0|1\vec{k}\rangle = |\vec{k}|\ |1k\rangle$$

and

$$H'|1\vec{k}\rangle = \frac{2|\vec{k}|}{1+2r}\ |2k\rangle$$

and

$$H'|2k\rangle = 0$$

we find

$$\langle 0|\phi(x)|1k\rangle = \langle 0|\phi(0)\left(|1k\rangle - it\,\frac{2|\vec{k}|}{1+2r}\,|2k\rangle\right)e^{-ik\cdot x}\ .$$

This and similar results for the other matrix elements combine to give

$$\langle 0|\phi(x)\,\phi(x')|0\rangle = \int d^3k\,e^{-ik\cdot(x-x')}\Big[\langle 0|\phi(0)\,(|1k\rangle\langle 2k| + |2k\rangle\langle 1k|)\,\phi(0)|0\rangle - \frac{2}{1+2r}\,i|\vec{k}|(t-t')\langle 0|\phi(0)|2k\rangle\langle 2k|\phi(0)|0\rangle\Big]\ . \tag{3.23}$$

When the definitions of the states and the momentum-space commutation relations are used to calculate

$$\langle 0|\phi(0)|1k\rangle = \frac{1}{(2\pi)^3}\frac{N}{\sqrt{2}},\quad \langle 0|\phi(0)|2k\rangle = -\frac{1}{(2\pi)^3}\frac{N}{\sqrt{2}}(1+2r)\quad ,$$

Eq. (3.23) becomes

$$\langle 0|\phi(x)\,\phi(x')|0\rangle = -\frac{1}{(2\pi)^3}\int\frac{d^3k}{4|\vec{k}|^3}\,e^{-ik\cdot(x-x')}[1+i|\vec{k}|(t-t')]$$

as before.

We now turn to the relationship of the time-ordered product to the normal-ordered product. As usual, ϕ can be decomposed according to

$$\phi = \phi^{(+)} + \phi^{(-)}\quad ,$$

with

$$\phi^{(+)}(x)|0\rangle = 0,\quad \langle 0|\phi^{(-)}(x) = 0\ .$$

Using the definition of the normal-ordered product we find that

$$\phi(x)\,\phi(x') = :\phi(x)\,\phi(x'): - [\phi^{(+)}(x), \phi^{(-)}(x')] = :\phi(x)\,\phi(x'): - \langle 0|\phi(x)\,\phi(x')|0\rangle\ ,$$

since the commutator is a c number. With the corresponding expression for $\phi(x')\phi(x)$ and the observation that

$$:\phi(x)\,\phi(x'):=:\phi(x')\,\phi(x): \;,$$

we obtain

$$T[\phi(x)\,\phi(x')]=:\phi(x)\,\phi(x'):+\langle 0|\,T[\phi(x)\,\phi(x')]\,|0\rangle \;. \tag{3.24}$$

With the properties of time-ordered product known, we can return to the derivation of the Feynman rules. Equation (3.24) allows us to prove Wick's theorem. The rest of the derivation presents no difficulties. The usual rules are modified by replacing the momentum-space scalar propagator $1/k^2$ by $1/k^4$. The other modification results from the form of the reduction formulas. The differential operator which serves to remove the external line propagators becomes $\partial^2\partial^2$ for $|2\rangle$ type states and $-(3+2t\partial_0)\partial^2\partial^2$ for $|1\rangle$ type states. As an example, we will recalculate the matrix element

$$\langle pk1\,|\,p_1p_2\rangle \;.$$

Use of the reduction formulas gives

$$\begin{aligned}\langle pk1 \text{ out}\,|\,p_1p_2 \text{ in}\rangle &= -\left(\frac{i}{(2\pi)^{3/2}}\right)^3 \frac{i}{(2\pi)}\left(\frac{1}{2\omega_1 2\omega_2 2\omega_p}\right)^{1/2}\left(\frac{(2\pi)^3}{8|\vec{k}|^3}\right)^{1/2} \\ &\quad\times\int d^4x_1\int d^4x_2\int d^4x_3\int d^4x_4\; e^{-ip_1\cdot x_1}e^{-ip_2\cdot x_2}e^{ip\cdot x_3}e^{ik\cdot x_4}(3+2t\partial_0)\,\partial^2\partial_4{}^2(\partial^2+M^2)_1(\partial^2+M^2)_2(\partial^2+M^2)_3 \\ &\quad\times\langle 0|\,T[\psi(x_1)\,\psi(x_2)\,\psi(x_3)\,\phi(x_4)]\,|0\rangle \;.\end{aligned} \tag{3.25}$$

In lowest order the τ function is

$$\begin{aligned}\tau(x_1x_2x_3x_4) &= -ie\int d^4y\,\langle 0|\,T[\psi_{\rm in}(x_1)\,\psi_{\rm in}(x_2)\,\psi_{\rm in}(x_3)\,\phi_{\rm in}(x_4):\phi(y)\,\psi^3(y):]\,|0\rangle \\ &= -ie3!\int d^4y\, i\Delta_F(x_1-y)\,i\Delta_F(x_2-y)\,i\Delta_F(x_3-y)\,\langle 0|\,T[\phi_{\rm in}(x_4)\,\phi_{\rm in}(y)]\,|0\rangle \;.\end{aligned}$$

Inserting this in Eq. (3.25), letting the differential operators act, and carrying out the x_1, x_2, and x_3 integrations we find that

$$\begin{aligned}\langle pk1 \text{ out}\,|\,p_1p_2 \text{ in}\rangle &= -ie3!\,(2\pi)^{-9/2}\left(\frac{1}{2\omega_1 2\omega_2 2\omega_p}\right)^{1/2}\frac{1}{(2\pi)^3}\left(\frac{(2\pi)^3}{8|\vec{k}|^3}\right) \\ &\quad\times\int d^4y\, e^{-i(p_1+p_2-p)\cdot y}\int d^4x\, e^{ik\cdot x}(3+2x^0\partial_0)\,\delta^4(x_4-y) \;.\end{aligned} \tag{3.26}$$

After the x integration is done, the y integral we are left with is

$$\int d^4y\, e^{-i(p_1+p_2-p)\cdot y}(1-2i|\vec{k}|\,y^0)=(2\pi)^4\,\delta^3(\vec{p}+\vec{k}-\vec{P})\,[\delta(\,|\vec{k}|+(\vec{p}^{\,2}+M^2)^{1/2}-E)-2|\vec{k}|\,\delta'(\,|\vec{k}|-(\vec{p}^{\,2}+M^2)^{1/2}-E)] \;.$$

When this is substituted in Eq. (3.26) the same expression that we obtained in the Heisenberg picture is reproduced.

Infrared problems

As we have already seen, the $1/k^4$ propagator presents some infrared difficulties. From a general point of view, this is a good feature which may lead to quark binding. However, in terms of perturbation theory calculations, it is a disaster. The radiative corrections to scattering processes cannot be rendered infrared finite as can be done in QED. To give an idea of what can happen we will calculate the order-e^2 contribution to the fermion self-energy in the Abelian vector-meson version of the theory. By this we mean a theory like QED except that the $1/k^2$ photon propagator is replaced by $1/k^4$. We choose this model so that comparison with the familiar results of QED will be facilitated. The coupling constant e then has units of mass. The gauge properties are not affected. It is interesting to note that because of this the Ward identity is also not affected.

The self-energy insertion is

$$\Sigma(p) = \frac{-i\alpha}{4\pi^3}\int d^4k \frac{1}{(k^2+i\epsilon)^2}\gamma_\mu \frac{\not p - \not k + m}{(p-k)^2 - m^2}\gamma^\mu .$$

We will regulate this in the infrared region by using the propagator

$$\frac{1}{(k^2-\lambda^2+i\epsilon)^2} .$$

A little γ algebra gives

$$\delta m = \Sigma(p)|_{\not p = m} = \frac{-i\alpha}{4\pi^3}\int d^4k \frac{1}{(k^2-\lambda^2+i\epsilon)^2}\frac{\not k + m}{k^2 - 2k\cdot p} . \tag{3.27}$$

After parametrizing the denominators, shifting the origin in k space, and doing the k integration, Eq. (3.27) becomes

$$\delta m = -\frac{\alpha}{4\pi} m \int_0^1 dx_1 \int_0^1 dx_2 \int_0^1 dx_3\, \delta(1-x_1-x_2-x_3)\frac{x_3+1}{[x_3{}^2 m^2 + (1-x_3)\lambda^2 - i(1-x_3)\epsilon]} .$$

These integrals are elementary. The result is

$$\delta m = \frac{m}{4\pi}\frac{\alpha}{m^2}\left[1 - 2\frac{1+k-\frac{1}{2}k^2}{(4k-k^2)^{1/2}}\left(\arctan\frac{2-k}{(4k-k^2)^{1/2}} - \arctan\frac{-k}{(4k-k^2)^{1/2}}\right)\right] ,$$

with

$$k \equiv \lambda^2/m^2 .$$

If this expression is carefully expanded in the $k \to 0$ limit we find

$$\delta m = \frac{m}{4\pi}\frac{\alpha}{m^2}\left(\frac{3}{2} - \frac{\pi}{2}\frac{m}{\lambda}\right) . \tag{3.28}$$

In normal QED, δm depends on the ultraviolet cutoff but not on the photon mass. A mass shift which depends strongly on λ suggests that the observed fermion mass will depend on the size of the room in which it is measured. This should serve to illustrate the infrared problems in perturbation theory. If, in the full theory, the fermions were permanently bound into charge-zero particles this infrared divergence would presumably be mitigated.

IV. SPECULATIONS

The presentation of our results is now complete. However, the task of elucidating the structure of our model is by no means complete. We will conclude the paper by speculating on the sort of interesting possibilities that might be established in a more ambitious analysis.

The development that we have carried out so far will be referred to as naive perturbation theory. We have seen that it has some problems. The conservation of probability is violated by the production of negative-metric states. In a finite order, the quark pole will not be eliminated, and the quarks will not be permanently bound. The radiative corrections are infrared divergent at the one-loop level. (Presumably, this is related to the idea that the quarks actually cannot be asymptotically separated.) These problems are serious. They indicate a need to go beyond the unimaginative confines of naive perturbation theory.

We will begin the discussion with some general remarks[20] about the charge operator in the Abelian vector-gluon version of the model that was introduced at the end of the last section. The "electromagnetic field" now satisfies a higher-order field equation

$$\partial^2\partial_\nu F^{\mu\nu} = ej^\mu .$$

$F^{\mu\nu}$ is constructed from the vector potential in the usual way:

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu .$$

In the Lorentz gauge

$$\partial_\mu A^\mu = 0 ,$$

the field equation becomes

$$\partial^2\partial^2 A^\mu = ej^\mu .$$

This reveals a field equation of the type (although now for the vector case) that we have studied in detail.

If Gauss's Law

$$\partial^2\vec{\nabla}\cdot\vec{E} = e\rho$$

is integrated to get the charge in a volume V with surface S, we find

$$Q_V = \int_S d\vec{\sigma}\cdot\partial^2\vec{E} . \tag{4.1}$$

This is to be compared with the usual result

$$Q_V = \int_S d\vec{\sigma}\cdot\vec{E} . \tag{4.2}$$

In QED, vacuum polarization effects do not alter the q^{-2} behavior of the photon propagator at small q^2. The exchange of these massless quanta gives an electric field which falls as r^{-2}. A finite contribution to Eq. (4.2) results even as the surface is moved to infinity.

In the same way, if the vector propagator in the model maintains its q^{-4} behavior, we will have

$$\vec{E} \sim \vec{r}/r, \quad \nabla^2 \vec{E} \sim \vec{r}/r^3 .$$

A finite contribution to Eq. (4.1) will result. On the other hand, if the electric field goes to zero as r goes to infinity, there will be no contribution to Eq. (4.1) as the surface is expanded to infinity. We then have $Q = 0$, and we must conclude that only neutral states are allowed. However, the bare quarks are charged. As a consequence the physical states must be in the

$$\psi^\dagger \psi |0\rangle$$

channel and not in the

$$\psi^\dagger |0\rangle$$

channel. (ψ is the field for the spin-$\frac{1}{2}$ quark.) This is just what we mean by permanent quark binding. As a reflection of this, the single-particle pole in the quark propagator should be eliminated.

Now, the long-range structure of the "electric field" comes from the q^{-4} singularity in the gluon propagator. It arises from the exchange of the unusual massless gluons we studied in Secs. II and III. If the vacuum polarization modifies the $q^2 \to 0$ behavior of the propagator to q^{-2} or some even weaker singularity, $\vec{E}$ will go to zero at infinity and the quarks will be bound.

Now that we have focused on the importance of the vacuum polarization insertion, we will discuss various possibilities for its behavior. The first important point is that the Abelian vector version of our model has the same structure of coupled integral equations for the Green's functions as QED except that the expression for the vector propagator is

$$D'_{F\mu\nu}(q) = \frac{-g_{\mu\nu}}{q^2[q^2 + e_0^2 \Pi(q^2)]} + \text{gauge terms.} \quad (4.3)$$

When we restrict our attention to perturbation theory, there are two initially reasonable looking possibilities. In naive perturbation theory, we assume that we can begin the perturbation expansion with the initial estimates

$$S'_F \to \frac{Z_2}{\not{p} - m} \text{ as } \not{p} \to m ,$$

$$D'_{F\mu\nu} = \frac{-g_{\mu\nu} Z_3}{q^4} + \text{gauge terms as } q^2 \to 0 .$$

Problems arise immediately when the one-loop contribution to $\Pi(q^2)$ is calculated. This is the same as the usual QED calculation. $\Pi(0)$ is found to be an infinite constant. The basic structure of our D'_F is then modified to q^{-2} and we see that the initial q^{-4} guess was a poor one. Naive perturbation theory will be useful only if an infinite class of graphs can be summed.

Another perturbation theory possibility is to renormalize the integral equations in the usual way and to assume the structure

$$\tilde{S}'_F \to \frac{1}{\not{p} - m} \text{ as } \not{p} \to m , \qquad \tilde{D}'_{F\mu\nu} \to \frac{-g_{\mu\nu}}{q^2} \text{ as } q^2 \to 0 . \quad (4.4)$$

The renormalized expression for the gluon propagator is

$$\tilde{D}'_{F\mu\nu} = \frac{-g_{\mu\nu}}{q^2[Z_3 q^2 + e^2 \Pi(q^2)]} + \text{gauge terms} .$$

The structure of Eq. (4.4) requires of Z_3

$$Z_3 e_0^2 \Pi(0) = e^2 \Pi(0) = 1 . \quad (4.5)$$

However, when we use this structure to calculate the one-loop contribution to $\Pi(0)$ we find again that it is an infinite constant. Equation (4.5) then demands that

$$e^2 = 0 .$$

Thus, if the renormalized theory is to be finite, it must be free. This possibility does not appear to be productive either.

A need to go beyond this sort of unimaginative perturbation theory is indicated. All indications are that the vector propagator will not maintain the q^{-4} behavior. This means that the quarks will be permanently bound. This should be reflected in our estimate for the quark propagator. In particular, it should not have the $(\not{p} - m)^{-1}$ single-particle pole. We do not know what structure would be more appropriate as an initial estimate in a more sophisticated perturbation theory. For the vacuum polarization which determines the vector propagator structure, we have already commented that $\Pi(q^2) \sim q^2$ as $q^2 \to 0$ seems unlikely. When the quark propagator is modified, $\Pi(q^2) \to$ constant might be consistent. This would give a long-range "electric" field falling like r^{-2} and quark binding as discussed. Another interesting possibility would be that $\Pi(q^2)$ would develop a pole as it does in two-dimensional QED. The result of that would be a gluon propagator with no singularity at $q^2 \to 0$ and only a short-range interaction remaining.

It is worth noting that these changes could serve to cure the unitarity problems of naive perturbation theory. If, for instance, the vacuum polarization modified the gluon propagator to q^{-2}, a standard positive-metric particle content would result.

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APPENDIX

As we have seen, the field equation

$$\partial^2\partial^2\phi = J$$

follows from a Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial^2\phi)(\partial^2\phi) - J\phi \ . \qquad \text{(A1)}$$

This theory has been quantized in a natural way. The free case

$$J = 0$$

was completely solved in Sec. II.

It is interesting to note that the same results can be obtained by introducing an auxiliary field rather than higher derivatives into the Lagrangian.[21] Consider

$$\mathcal{L} = +\partial_\mu\phi_1\partial^\mu\phi_2 - \tfrac{1}{2}m^2\phi_2{}^2 - (1/m)J\phi_1 \ . \qquad \text{(A2)}$$

If the canonical quantization method is used, we find

$$\pi_1 \equiv \frac{\delta\mathcal{L}}{\delta\dot{\phi}_1} = \dot{\phi}_2$$

and

$$\pi_2 \equiv \frac{\delta\mathcal{L}}{\delta\dot{\phi}_2} = \dot{\phi}_1 \ .$$

We then require that

$$[\pi_1(t,\vec{x}), \phi_1(t,\vec{y})] = [\dot{\phi}_2(t,\vec{x}), \phi_1(t,\vec{y})] = -i\delta^3(\vec{x}-\vec{y}) \ ,$$
$$[\pi_2(t,\vec{x}), \phi_2(t,\vec{y})] = [\dot{\phi}_1(t,\vec{x}), \phi_2(t,\vec{y})] = -i\delta^3(\vec{x}-\vec{y}) \ , \qquad \text{(A3)}$$

with all other commutators zero.

The Euler-Lagrange equations are

$$\partial^2\phi_2 = -(1/m)J \ ,$$
$$\partial^2\phi_1 = -m^2\phi_2 \ .$$

They imply that

$$\partial^2\partial^2\phi_1 = mJ \ .$$

The Hamiltonian density is

$$\mathcal{H} = \pi_1\dot{\phi}_1 + \pi_2\dot{\phi}_2 - \mathcal{L} \ ,$$

which becomes

$$\mathcal{H} = \dot{\phi}_1\dot{\phi}_2 + \vec{\nabla}\phi_1\cdot\vec{\nabla}\phi_2 + \tfrac{1}{2}m^2\phi_2{}^2 + (1/m)J\phi_1 \ .$$

Commuting with ϕ_1 and ϕ_2 we find

$$i[\mathcal{H}(t,\vec{x}), \phi_{1,2}(t,\vec{y})] = \dot{\phi}_{1,2}(t,\vec{x})\,\delta^3(\vec{x}-\vec{y})$$

as required. This method is, thus, an alternate formalism for obtaining a field equation of the type in which we are interested.

It is easy to show that this theory is equivalent to the one which begins with the Lagrangian in Eq. (A1) and which we discussed in Sec. II. Introduce

$$\phi \equiv (1/m)\,\phi_1 \ . \qquad \text{(A4)}$$

We then have

$$\phi_1 = m\phi \ ,$$

$$\phi_2 = -(1/m)\,\partial^2\phi \ ,$$

and

$$\partial^2\partial^2\phi = J \ .$$

The Lagrangian in Eq. (A2) becomes

$$\mathcal{L} = \tfrac{1}{2}(\partial^2\phi)(\partial^2\phi) - J\phi \ .$$

Substituting in the commutation relations Eq. (A3), we find that ϕ as given by Eq. (A4) satisfies the same commutation relations as the ϕ of Eq. (A1) which was discussed in Sec. II. The fact that the methods which start from Eq. (A1) and Eq. (A2) give the same results suggests that we are doing things correctly.

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