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Exact results for effective Lagrangians

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A simple method is presented for the evaluation in quantum field theory of the effective Lagrangian induced by one-loop quantum effects. Exact solutions may be obtained in the quasilocal situation where the resulting Lagrangian is allowed to depend on the fields and their first derivatives (and, in some cases, their second derivatives as well). The method is a general one and may be applied to any given field theory. For example, Schwinger's result for the effective Maxwell Lagrangian with constant external field and the Coleman-Weinberg results for effective potentials each emerge as special cases of the general method. By isolating the divergent part of the induced Lagrangian in the general case, moreover, one may recover the 't Hooft-Veltman expression for the one-loop counterterms of an arbitrary field theory. At no stage need Feynman diagrams be evaluated.

I. INTRODUCTION

It was recognized long ago that the effects of closed loops in a perturbation expansion of the S matrix could be summarized by adding to the original classical action functional, S, an "effective" quantum action functional, W.[1,2] All calculations are then reduced to a study of the c-number theory based on $\Gamma = S + W$.

Although it has now become popular to employ the functional integral representation for Γ as a useful device for generating the full irreducible vertex functions of the theory, there have been relatively few attempts to simply calculate Γ explicitly, even though this effective action, if one could only get a handle on it, contains all the information we need ever want to extract from the theory.

Of course Γ is an exceedingly complicated quantity. Even in the one-loop approximation it is a nonlocal functional of the fields depending, as it does, on the field variables and all their derivatives. For arbitrarily varying fields, therefore, one must, it seems, resort to perturbative methods of calculation. In certain situations, however, exact results can be obtained. For example, Schwinger[1] has computed exactly the effective Maxwell Lagrangian induced by closed loops of fermions or bosons, in the case of a constant external field. Schwinger's coordinate-space method is an elegant one, relying on his proper-time formalism, which is accompanied by the introduction of abstract vectors in a nonphysical Hilbert space with an associated "Hamiltonian" and "transition amplitudes" satisfying a "Schrödinger equation."

There is, however, another way to compute effective Lagrangians which relies on straightforward momentum-space methods, which, we feel, are more familiar to most field theorists. Working in momentum space also facilitates the introduction of dimensional regularization whereby the resulting divergences may be rapidly and elegantly removed. It is this route which we advocated in a previous paper[3] for the computation of effective potentials, and which we wish to extend in this paper to effective Lagrangians.

Rather than plunging straight into the evaluation of Γ in the general case, we prefer to begin in Sec. II with the example of a $\lambda\phi^4$ theory so as not to obscure the essential simplicity of the method. Many of the results of this section will be seen to carry over, with one or two modifications, to the general case treated in Sec. III. Using the background-field method of DeWitt,[2] it turns out that the one-loop effective Lagrangian, $\mathcal{L}^{(1)}$, may be computed exactly provided one can obtain an exact solution to the equation of the appropriate Green's function in the presence of the background field, $\phi(x)$. This we are able to do in the quasilocal situation in which $\mathcal{L}^{(1)}$ is allowed to depend on the variables ϕ^2, $\partial_\mu\phi^2$, and $\partial_\mu\partial_\nu\phi^2$.

The essence of the calculation is to write an integral representation for the momentum-space Green's function, $G(p)$, in terms of three unknown functions which are then determined by three elementary first-order differential equations obtained by substituting $G(p)$ back into its defining equation. By working throughout in n dimensions and only taking the limit $n \to 4$ after renormalization, the resulting finite Lagrangian is arrived at without reference to cutoffs or counterterms, and we describe how to avoid the infrared divergence in the massless theory. Our Lagrangian goes smoothly over to the Coleman-Weinberg effective potential in the local approximation obtained by setting the derivatives of the field equal to zero.

In Sec. III, we set up the relevant Green's-function equation for an arbitrary field theory. The major difference from Sec. II is that this is now a matrix equation, depending on parameters which do not, in general, commute with each other. (If one is interested solely in the divergent part of the Lagrangian, then this presents no problems. Indeed, the 't Hooft-Veltman expression for one-loop counterterms in an arbitrary theory follows almost immediately.) The determination of the effective Lagrangian is consequently more difficult, since the number of invariants formed by such commutators is now without limit, unless one specifies which particular group one is interested in. This is an interesting problem in its own right which we postpone to a future publication. In this paper we show instead how, with only slight loss of generality in the form of the interaction, exact results can be obtained in the quasilocal situation, even for an arbitrary group. For an Abelian gauge group, of course, everything is much simpler, and we see how Schwinger's effective Maxwell Lagrangian induced by closed loops of charged scalar mesons emerges as a special case.

Finally, Sec. IV treats the feasibility of extending the method to two or more loops, of introducing a greater degree of arbitrariness to the background field, of including fictitious quanta, and of taking the gravitational interaction into account. We finish with some general remarks on the usefulness of effective Lagrangians in field theory and some problems we hope our method will be able to solve.

Appendix A deals with the more tedious aspects of the matrix algebra, and since Secs. II and III deal only with bosons, we show in Appendix B how fermions may be included as well.

II. A SIMPLE EXAMPLE: $\lambda\phi^4$ THEORY

A. The effective Lagrangian

We begin with the classical action

$$S=\int dx\,\mathcal{L}^{(0)}\,,\quad \mathcal{L}^0=\tfrac{1}{2}(\partial\phi)^2-\tfrac{1}{2}m^2\phi^2-\frac{\lambda}{4!}\phi^4\,. \tag{2.1}$$

To compute the effective Lagrangian, we employ the background-field method of DeWitt[2] and accordingly make the replacement

$$\phi(x)\rightarrow\phi(x)+h(x)\,, \tag{2.2}$$

where $h(x)$ is the quantum field variable and $\phi(x)$ is now to be regarded as the classical background relative to which the quantum fluctuations take place. If $S(\phi+h)$ is now expanded about the background field, then the one-loop effects will be governed by those terms which are bilinear in the quantum field. We denote the corresponding Lagrangian by L,

$$L=-\tfrac{1}{2}h(\partial^2+m^2+\tfrac{1}{2}\lambda\phi^2)h\,. \tag{2.3}$$

The Lagrangian induced by these one-loop effects, $\mathcal{L}^{(1)}$, is then given by the functional integral over the quantum fields,

$$\exp\left(\frac{i}{\hbar}\int dx\,\mathcal{L}^{(1)}\right)=N\int(dh)\exp\left(\frac{i}{\hbar}\int dx\,L\right), \tag{2.4}$$

where N is a normalization constant determined by the requirement that $\mathcal{L}^{(1)}=0$ when $\phi=0$.

As we shall see later, one may solve for $\mathcal{L}^{(1)}$ explicitly provided one can obtain exact solutions to the Green's-function equation

$$[\partial_x{}^2+m^2+\tfrac{1}{2}\lambda\phi^2(x)]\,G(x,x')=\delta(x,x')\,, \tag{2.5}$$

where

$$\frac{\hbar}{i}G(x,x')=\langle h(x)h(x')\rangle \tag{2.6}$$

$$\equiv\frac{\int(dh)h(x)h(x')\exp[(i/\hbar)\int dx\,L]}{\int(dh)\exp[(i/\hbar)\int dx\,L]}\,. \tag{2.7}$$

Since, for the time being, we are ignoring the possibility of real pair production, it is not necessary at this stage to specify what particular boundary conditions $G(x,x')$ obeys.

For an arbitrarily varying background field, $\phi(x)$, Eq. (2.5) is a very difficult equation to solve. The resulting Lagrangian will depend on the field and all its derivatives,

$$\mathcal{L}^{(1)}=\mathcal{L}^{(1)}(\phi,\partial_\mu\phi,\partial_\mu\partial_\nu\phi,\partial_\mu\partial_\nu\partial_\rho\phi,\ldots)\,, \tag{2.8}$$

and one is forced to resort to perturbative methods of solution. Instead, however, we wish to solve Eq. (2.5) *exactly* in the quasilocal situation where $\mathcal{L}^{(1)}$ depends only on ϕ and $\partial_\mu\phi$. To this end, we consider a background field obeying the restriction

$$\partial_\mu\partial_\nu\partial_\rho\phi^2(x)=0 \tag{2.9}$$

and the quantity $\phi^2(x)$ is now expanded about the reference point x',

$$\phi^2(x)=\phi^2(x')+\phi^2{}_{,\mu}(x')(x-x')^\mu+\tfrac{1}{2}\phi^2{}_{,\mu\nu}(x')(x-x')^\mu(x-x')^\nu\,, \tag{2.10}$$

where commas denote differentiation. [Note that to ensure the correct dependence of $\mathcal{L}^{(1)}$ on the first derivative of ϕ, we must keep the second derivative of ϕ^2, and that by discarding third derivatives of ϕ^2 by Eq. (2.6), we are ignoring second derivatives of ϕ.] Substituting (2.10) back into (2.5), we see that we are led to seek a solution of the equation

$$[\partial_x{}^2+\alpha(x')+\beta_\mu(x')(x-x')^\mu + \tfrac{1}{4}\gamma^2{}_{\mu\nu}(x')(x-x')^\mu(x-x')^\nu]\,G(x,x')=\delta(x,x')\,, \tag{2.11}$$

where

$$\alpha = m^2+\tfrac{1}{2}\lambda\phi^2\,,$$
$$\beta_\mu=\tfrac{1}{2}\lambda\phi^2{}_{,\mu}\,, \tag{2.12}$$
$$\gamma^2{}_{\mu\nu}=\lambda\phi^2{}_{,\mu\nu}\,.$$

(We write γ^2 rather than γ simply for convenience.)

Before proceeding to solve Eq. (2.11), we first wish to show how knowledge of its solution enables us to obtain the effective Lagrangian. Substituting (2.10) and (2.12) into the expression (2.3) for L, and then differentiating both sides of Eq. (2.4) with respect to α, we have

$$\frac{\partial \mathcal{L}^{(1)}}{\partial\alpha} = -\tfrac{1}{2}\langle h(x)h(x)\rangle = -\frac{\hbar}{2i}G(x,x)\,. \tag{2.13}$$

Thus the whole procedure devolves upon a determination of

$$G(x,x)=\int\frac{d^n p}{(2\pi)^n}G(p)\,, \tag{2.14}$$

where, in anticipation of the use of dimensional regularization, the momentum-space integral is allowed to be n-dimensional. The Fourier-transformed Green's function,

$$G(p)=\int dx\, e^{ip(x-x')}G(x,x')\,, \tag{2.15}$$

obeys, from (2.11), the p-space equation

$$\left(-p^2+\alpha-i\beta_\mu\frac{\partial}{\partial p_\mu}-\frac{1}{4}\gamma^2{}_{\mu\nu}\frac{\partial^2}{\partial p_\mu\partial p_\nu}\right)G(p)=1\,, \tag{2.16}$$

and were it not for the derivatives in the above equation, it could be solved immediately to yield

$$G(p)=\frac{-1}{p^2-\alpha} = \int_0^\infty ds\, e^{-\alpha s}e^{p^2 s}\,. \tag{2.17}$$

Indeed, if we were interested only in the effective potential, where $\mathcal{L}^{(1)}$ depends on ϕ alone, our task would now be complete, Eqs. (2.13), (2.14), and (2.17) providing the desired result. When the derivatives are present, however, we look for a solution of the form

$$G(p)=\int_0^\infty ds\, e^{-\alpha s}\exp[\,p^\mu A_{\mu\nu}(s)p^\nu+B_\mu(s)p^\mu+C(s)]\,, \tag{2.18}$$

where the unknowns A, B, and C are to be determined in terms of β and γ, and where

$$A_{\mu\nu}(s)\to\delta_{\mu\nu}s\,,\quad B_\mu(s)\to 0\,,\quad C(s)\to 0 \tag{2.19}$$

as we switch off the background field.

From Eqs. (2.14) and (2.18), we now have

$$G(x,x)=\frac{1}{(2\pi)^n}\int_0^\infty ds\, e^{-\alpha s+C}\int d^n p\, e^{p\cdot A\cdot p+B\cdot p}\,, \tag{2.20}$$

noting that, for suitably chosen n, we may freely interchange the orders of integration. At this stage everything is perfectly finite, only at the end do we take the limit $n\to 4$. Under the change of variable

$$q=p+\tfrac{1}{2}A^{-1}\cdot B \tag{2.21}$$

the q integral becomes an elementary Gaussian,

$$G(x,x)=\frac{1}{(2\pi)^n}\int_0^\infty ds\,\exp(-\alpha s+C-\tfrac{1}{4}B\cdot A^{-1}\cdot B)\times\int d^n q\, e^{q\cdot A\cdot q}\,, \tag{2.22}$$

and we obtain

$$G(x,x)=\frac{i}{(4\pi)^{n/2}}\int_0^\infty\frac{ds}{s^{n/2}}\exp[-\alpha s+C-\tfrac{1}{4}B\cdot A^{-1}\cdot B-\tfrac{1}{2}\mathrm{tr}\ln(As^{-1})]\,. \tag{2.23}$$

Finally, the effective Lagrangian $\mathcal{L}^{(1)}$ is obtained from Eq. (2.13) by integrating with respect to α subject to the boundary condition $\mathcal{L}^{(1)}=0$ when $\phi=0$:

$$\mathcal{L}^{(1)}=\frac{\hbar}{2(4\pi)^{n/2}}\int_0^\infty\frac{ds}{s^{1+n/2}}\{e^{-\alpha(\phi)s}\exp[C-\tfrac{1}{4}B\cdot A^{-1}\cdot B-\tfrac{1}{2}\mathrm{tr}\ln(As^{-1})]-e^{-\alpha(0)s}\}\,. \tag{2.24}$$

It now remains to show that the expression (2.18) for $G(p)$ is indeed a solution of Eq. (2.16), and to determine the unknowns A, B, and C. This we do by inserting (2.18) into (2.16), yielding

$$\int_0^\infty ds[\,\alpha-p\cdot(1+A\gamma^2A)\cdot p-(2i\beta\cdot A+\beta\cdot\gamma^2A)\cdot p-(\tfrac{1}{2}\mathrm{tr}\gamma^2A+i\beta\cdot B+\tfrac{1}{4}B\cdot\gamma^2\cdot B)]\,e^{-\alpha s}e^{p\cdot A\cdot p+B\cdot p+C}=1\,. \tag{2.25}$$

By setting

$$1+A\gamma^2 A=\frac{\partial A}{\partial s}, \tag{2.26}$$

$$2i\beta\cdot A+B\cdot\gamma^2 A=\frac{\partial B}{\partial s} \tag{2.27}$$

in the above equation and integrating by parts, we see that (2.18) solves (2.16) provided the function $C(s)$ obeys the equation

$$\tfrac{1}{2}\operatorname{tr}(\gamma^2 A)+i\beta\cdot B+\tfrac{1}{4}B\cdot\gamma^2\cdot B=\frac{\partial C}{\partial s} \tag{2.28}$$

and the boundary condition

$$e^{-\alpha s}e^{p\cdot A\cdot p+B\cdot p+C}\big|_0^\infty=-1\,. \tag{2.29}$$

These equations for A, B, and C, do indeed admit of elementary solutions in terms of trigonometrical functions,

$$\begin{aligned} A&=\gamma^{-1}\tan\gamma s\,,\\ B&=-2i\gamma^{-2}(1-\sec\gamma s)\beta\,,\\ C&=-\tfrac{1}{2}\operatorname{tr}\ln\cos\gamma s-\beta\cdot\gamma^{-3}(\tan\gamma s-\gamma s)\cdot\beta\,.\end{aligned} \tag{2.30}$$

The resulting expression for the effective Lagrangian of Eq. (2.24) is

$$\mathcal{L}^{(1)}=\frac{\hbar}{2(4\pi)^{n/2}}\int_0^\infty\frac{ds}{s^{1+n/2}}\left[e^{-\alpha(\phi)s}e^{-f(s)}-e^{-\alpha(0)s}\right], \tag{2.31}$$

where

$$\begin{aligned} f(s)&=-C+\tfrac{1}{4}B\cdot A^{-1}\cdot B+\tfrac{1}{2}\operatorname{tr}\ln(As^{-1})\\ &=\tfrac{1}{2}\operatorname{tr}\ln[(\gamma s)^{-1}\sin(\gamma s)]\\ &\quad+\beta\cdot\gamma^{-3}[2\tan(\tfrac{1}{2}\gamma s)-\gamma s]\cdot\beta\,,\end{aligned} \tag{2.32}$$

with α, β, and γ given by Eq. (2.12).

This completes the derivation of the effective Lagrangian for $\lambda\phi^4$. Note that had we chosen a $\lambda\phi^3$ theory, the Lagrangian would be the same, save that α, β, and γ would be replaced by

$$\begin{aligned}\alpha&=m^2+\lambda\phi\,,\\ \beta\mu&=\lambda\phi_{,\mu}\,,\\ \gamma^2{}_{\mu\nu}&=2\lambda\phi_{,\mu\nu}\,,\end{aligned}$$

and we would then obtain the correct dependence on the field, its first derivative, *and* its second derivative as well.

B. Renormalization

In the case of massive fields ($m\neq 0$), the total Lagrangian is now

$$\begin{aligned}\mathcal{L}&=\mathcal{L}^{(0)}+\mathcal{L}^{(1)}\\ &=\tfrac{1}{2}(\partial\phi)^2-\tfrac{1}{2}m^2\phi^2-\frac{\lambda}{4!}\phi^4\\ &\quad+\frac{\hbar}{2(4\pi)^{n/2}}\int_0^\infty\frac{ds}{s^{1+n/2}}e^{-m^2 s}\left[e^{-\lambda\phi^2 s/2}e^{-f(s)}-1\right]\end{aligned} \tag{2.33}$$

To obtain a finito result in the limit $n\to 4$, we must first rewrite $\mathcal{L}$ in terms of the renormalized mass m_R and renormalized coupling constant λ_R defined by

$$\begin{aligned} m_R{}^2&=-\frac{\partial^2\mathcal{L}}{\partial\phi^2}\bigg|_{\phi=0},\\ \lambda_R&=-\frac{\partial^4\mathcal{L}}{\partial\phi^4}\bigg|_{\phi=0}.\end{aligned} \tag{2.34}$$

There is no wave-function renormalization. [There is an infinite term in (2.33) proportional to $\operatorname{tr}\gamma^2$, but this is of no consequence since $\operatorname{tr}\gamma^2=\lambda\partial^2\phi^2$, i.e., a total divergence.] In terms of m_R and λ_R, $\mathcal{L}$ becomes (and we now drop the R subscript)

$$\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2-\tfrac{1}{2}m^2\phi^2-\frac{\lambda}{4!}\phi^4+\frac{\hbar}{2(4\pi)^{n/2}}\int_0^\infty\frac{ds}{s^{1+n/2}}e^{-m^2 s}\left[e^{-\lambda\phi^2 s/2}e^{-f(s)}-1+\tfrac{1}{2}\lambda\phi^2 s-\tfrac{1}{8}\lambda^2\phi^4 s^2\right] \tag{2.35}$$

and this is now perfectly finite in the limit $n=4$ (though we recommend that n be kept arbitrary for the purpose of actually evaluating the integral). This closed expression for $\mathcal{L}$ takes into account all orders of the coupling constant λ, and is valid to first order in $\hbar$. This corresponds to summing up the contributions from all one-loop Feynman diagrams possessing arbitrarily many external legs. The local approximation obtained by setting the derivatives of ϕ equal to zero in (2.35) corresponds to the situation where these external legs carry zero momenta. In this limit, then, we obtain the effective potential which, upon performing the s integration, becomes

$$\begin{aligned}-\mathcal{L}(\phi)\equiv V(\phi)=\tfrac{1}{2}m^2\phi^2+\frac{\lambda}{4!}\phi^4-\frac{\hbar}{2(4\pi)^{n/2}}\Big\{&\Gamma(-\tfrac{1}{2}n)[(m^2+\tfrac{1}{2}\lambda\phi^2)^{n/2}-(m^2)^{n/2}]+\tfrac{1}{2}\lambda\phi^2\Gamma(1-\tfrac{1}{2}n)(m^2)^{n/2-1}\\ &-\tfrac{1}{8}\lambda^2\phi^4\Gamma(2-\tfrac{1}{2}n)(m^2)^{n/2-2}\Big\},\end{aligned} \tag{2.36}$$

which, for $n=4$, is

$$V(\phi)=\tfrac{1}{2}m^2\phi^2+\frac{\lambda}{4!}\phi^4+\frac{\hbar}{64\pi^2}\left[(m^2+\tfrac{1}{2}\lambda\phi^2)^2\ln\left(1+\frac{\lambda\phi^2}{2m^2}\right)-\tfrac{1}{2}\lambda m^2\phi^2-\tfrac{3}{8}\lambda^2\phi^4\right],$$

in agreement[4] with that obtained by Coleman and Weinberg.[5,6] If, on the other hand, we keep ϕ and $\partial_\mu\phi$, the first few terms in the expansion of (2.34) are

$$\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2-\frac{m^2}{2}\phi^2-\frac{\lambda}{4!}\phi^4+\frac{\hbar\lambda^2}{32\pi^2}\left(-\frac{5}{12m^2}\phi^2(\partial\phi)^2+\frac{1}{40m^4}(\partial\phi)^4\right)$$
$$+\frac{\hbar\lambda^3}{32\pi^2}\left(-\frac{1}{48m^2}\phi^6+\frac{1}{16m^4}\phi^4(\partial\phi)^2-\frac{97}{360m^6}\phi^2(\partial\phi)^4+\frac{61}{2520m^8}(\partial\phi)^6\right)+\cdots. \quad (2.37)$$

The increasing complexity of each term is a good illustration of why the evaluation of individual Feynman diagrams is *not* the best way of obtaining our effective Lagrangian.

Next we turn to the case $m=0$. Massless fields must be handled with more care owing to the infrared problem. The Lagrangian is

$$\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2-\frac{\lambda}{4!}\phi^4+\frac{\hbar}{2(4\pi)^{n/2}}\int_0^\infty\frac{ds}{s^{1+n/2}}\left[e^{-\lambda\phi^2 s/2}e^{-f(s)}-1\right] \quad (2.38)$$

and the renormalization condition must be modified so as to avoid the origin in ϕ space. Therefore we define

$$\lambda_R=-\frac{\partial^4\mathcal{L}}{\partial\phi^4}\bigg|_{\phi=M}, \quad (2.39)$$

where M is an arbitrary mass. (Note that dimensional regularization avoids the need of a spurious mass renormalization.) $\mathcal{L}$ now becomes (and again we drop the subscript R)

$$\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2+\frac{\hbar}{2(4\pi)^{n/2}}\int_0^\infty\frac{ds}{s^{1+n/2}}\left[e^{-\lambda\phi^2 s/2}(e^{-f(s)}-1)\right]-V(\phi), \quad (2.40)$$

where we have separated out the effective potential

$$V(\phi)=\frac{\lambda\phi^4}{4!}-\frac{\hbar}{2(4\pi)^{n/2}}\int_0^\infty\frac{ds}{s^{1+n/2}}\left[e^{-\lambda\phi^2 s/2}-1-\frac{\phi^4}{4!}(3\lambda^2s^2-6\lambda^3M^2s^3+\lambda^4M^4s^4)e^{-\lambda M^2 s/2}\right]. \quad (2.41)$$

The dependence of $\mathcal{L}$ on the renormalized coupling constant is rather curious in the massless case and requires a separate discussion. Let us look first at the effective potential. Under the rescaling

$$s'=\tfrac{1}{2}\lambda\phi^2 s \quad (2.42)$$

Eq. (2.41) becomes of the form

$$V(\phi)=\frac{\lambda}{4!}\phi^4+\hbar(\lambda\phi^2)^{n/2}F\left(\frac{\phi^2}{M^2}\right), \quad (2.43)$$

where F is independent of λ. Evaluating the s' integral, in fact, gives

$$V(\phi)=\frac{\lambda}{4!}\phi^4-\frac{\hbar\Gamma(-\frac{1}{2}n)}{2}\left(\frac{\lambda\phi^2}{8\pi}\right)^{n/2}\times\left[1-\left(\frac{\phi^2}{M^2}\right)^{2-n/2}\frac{n(n-1)(n-2)(n-3)}{4!}\right], \quad (2.44)$$

yielding the Coleman-Weinberg result for $n=4$:

$$V(\phi)=\frac{\lambda}{4!}\phi^4+\frac{\hbar\lambda^2\phi^4}{256\pi^2}\left[\ln\left(\frac{\phi^2}{M^2}\right)-\frac{25}{6}\right]. \quad (2.45)$$

Thus the sum of all one-loop graphs for the effective potential is proportional to λ^2.

The situation changes, however, (as Coleman and Weinberg anticipated) when the external legs carry nonzero momenta, i.e., when we keep the derivatives of ϕ as in the complete Lagrangian (2.40). Under the rescaling (2.42), the Lagrangian is seen to be of the form

$$\mathcal{L}=\tfrac{1}{2}(\partial\phi)^2-\frac{\lambda}{4!}\phi^4+\hbar\lambda^2\phi^4\left[F\left(\frac{\phi^2}{M^2}\right)+G\left(\frac{(\partial\phi)^2}{\lambda\phi^4}\right)\right] \quad (2.46)$$

and the dependence on the renormalized coupling constant is much more complicated. Clearly, expanding out the factor $e^{-f(s)}$ in Eq. (2.35) and then performing the s integration is a procedure which must be avoided since this would lead to a series in inverse powers of λ. A series in positive powers of λ can be obtained only at the expense of performing some rather unpleasant integrations.

III. THE GENERAL THEORY

A. The formalism

Let us consider a set of real boson fields $h^i(x)$, where the index represents not only an internal-symmetry label but possibly space-time labels as well. (Fermion fields are treated in Appendix B.) The one-loop effects are again governed by that part of the Lagrangian which is bilinear in the quantum field variable. Following 't Hooft and Veltman,[7] we write this Lagrangian in the form

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu h^i W^{\mu\nu}_{ij}\partial_\nu h^j - h^i N^\mu_{ij}\partial_\mu h^j - \tfrac{1}{2}h^i M_{ij}h^j\,, \tag{3.1}$$

with

$$W^{\mu\nu}_{ij}=W^{\mu\nu}_{ji}=W^{\nu\mu}_{ij}\,,\qquad N^\mu_{ij}=-N^\mu_{ji}\,,\quad M_{ij}=M_{ji} \tag{3.2}$$

where W, M, and N are regarded as external, space-time dependent source functions. In all theories save quantum gravity and certain chiral models $W^{\mu\nu}_{ij}$ is simply $\delta^{\mu\nu}\delta_{ij}$. Here we shall deal only with the situation

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu h^i\partial^\mu h_i - h^i N^\mu_{ij}\partial_\mu h_j - \tfrac{1}{2}h^i M_{ij}h^j\,. \tag{3.3}$$

The Green's function G^{jk} is defined by

$$(\partial^2\delta_{ij}+2N^\mu_{ij}\partial_\mu+\partial_\mu N^\mu_{ij}+M_{ij})G^{jk}(x,x')=\delta^k_i\,\delta(x,x')\,. \tag{3.4}$$

The Lagrangian given by (3.3) covers both of the following situations:

(a) A theory with self-interaction. For example, in the $\lambda\phi^4$ theory of Sec. II the h's are the quantum field variables with a background ϕ field; thus

$$N\equiv 0\,,\quad M\equiv m^2+\tfrac{1}{2}\lambda\phi^2\,. \tag{3.5}$$

(b) A quantum field interacting with a (different) classical external field. For example, for charged scalar bosons (π,π^*) of mass m in an external electromagnetic field A_μ we have

$$h_1\equiv\frac{1}{\sqrt{2}}(\pi+\pi^*)\,,\quad h_2\equiv\frac{1}{i\sqrt{2}}(\pi-\pi^*)\,,$$

$$N^\mu_{ij}\equiv eA^\mu\epsilon_{ij}\,,\quad \epsilon_{ij}=\begin{pmatrix}0&1\\-1&0\end{pmatrix}, \tag{3.6}$$

$$M_{ij}\equiv(m^2-e^2A^2)\delta_{ij}\,.$$

Alternatively h^i might represent a multiplet of scalar fields interacting with an external Yang-Mills field, $B^\alpha{}_\mu$, where

$$N^\mu\equiv G_\alpha B^{\alpha\mu}\,,\quad M=m^2+G_\alpha B^{\alpha\mu}G_\beta B^\beta{}_\mu\,. \tag{3.7}$$

The G_α are the generators of the group satisfying

$$[G_\alpha,G_\beta]=G_\gamma c^\gamma{}_{\alpha\beta}\,.$$

If the quantum variable is itself a gauge field, then, in general, one must take into account the fictitious particles. We shall reserve discussion of how this is achieved for a future publication.

Before proceeding to solve Eq. (3.4) it will prove useful to summarize the gauge structure of $\mathcal{L}$ as defined by (3.3).

The requirement that $\mathcal{L}$ is invariant under $h\to\bar{h}=e^{\Lambda(x)}h$, $\Lambda_{ij}=-\Lambda_{ji}$, yields the following transformation properties for M and N:

$$N_\mu\to\bar{N}_\mu=e^\Lambda N_\mu e^{-\Lambda}-(\partial_\mu e^\Lambda)e^{-\Lambda}\,, \tag{3.8}$$

$$M\to\bar{M}=e^\Lambda Me^{-\Lambda}+e^\Lambda N^\mu\partial_\mu e^{-\Lambda}-(\partial_\mu e^\Lambda)N^\mu e^{-\Lambda}-(\partial_\mu e^\Lambda)(\partial_\mu e^{-\Lambda})\,. \tag{3.9}$$

It can easily be shown that there are two tensors, X and Y, that can be formed from M and N. These are defined as

$$X=M-N_\mu N^\mu\,, \tag{3.10}$$

$$Y_{\mu\nu}=\partial_\mu N_\nu-\partial_\nu N_\mu+[N_\mu,N_\nu]\,. \tag{3.11}$$

Both X and Y have the transformation law

$$T\to\bar{T}=e^\Lambda Te^{-\Lambda}\,. \tag{3.12}$$

One can define the covariant derivative of any tensor T that transforms as in (3.12),

$$T_{.\mu}=T_{,\mu}+[N_\mu,T]\,, \tag{3.13}$$

where

$$T_{.\mu}\to\bar{T}_{.\mu}=e^\Lambda T_{.\mu}e^{-\Lambda}\,.$$

Lastly, the covariant derivative of h is defined as

$$h_{.\mu}=\partial_\mu h+N_\mu h\,. \tag{3.14}$$

By use of (3.10) and (3.14) $\mathcal{L}$ can be rewritten in the manifestly covariant form

$$\mathcal{L}=\tfrac{1}{2}(h^i{}_{.\mu}h_{i.\mu}-h^iX_{ij}h^j)\,. \tag{3.15}$$

Let us now turn our attention to obtaining a solution to Eq. (3.4). We wish to obtain an exact solution to this equation by placing appropriate restrictions on the background fields M and N. However, we must now be careful to ensure that these conditions, if they are to lead to an invariant effective Lagrangian, must themselves be manifestly covariant. Thus we choose the covariant analog of Eq. (2.9),

$$X_{.\rho.\sigma.\tau}=0\,, \tag{3.16}$$

whereas the natural condition to impose on the N field is

$$Y_{\mu\nu.\rho}=0. \tag{3.17}$$

[In the example of electrodynamics (3.6), for instance, this corresponds to a constant external field, i.e., $\partial_\rho F_{\mu\nu}=0$.]

Inherent in the method for the solution of the Green's function in Sec. II was the substitution of a Taylor-series expansion for any quantities dependent upon the background field that occurred in the defining equation of the Green's function. As they stand, conditions (3.16) and (3.17) are not readily amenable to such an approach. Thus we must first recast (3.16) and (3.17) into a more tractable form. To this end let us first examine the consequences of (3.17).

We first quote two lemmas, the proofs of which are to be found in Appendix A.

Lemma 1.

$$Y_{\mu\nu.\rho}=0 \Rightarrow [Y_{\mu\nu}, Y_{\kappa\lambda}]=0. \tag{3.18}$$

Lemma 2. The most general form for N_μ that satisfies $Y_{\mu\nu}\equiv\partial_\mu N_\nu-\partial_\nu N_\mu+[N_\mu, N_\nu]$ when $Y_{\mu\nu.\rho}=0$ is given by

$$N_\mu=-\tfrac{1}{2}Y_{\mu\tau}(x)x^\tau+e^{\Omega(x)}\partial_\mu e^{-\Omega(x)}, \tag{3.19}$$

where $\Omega(x)$ is an arbitrary antisymmetric matrix.

Thus $Y_{\mu\nu.\rho}=0$ implies that we can write N_μ in the form given by (3.19). We now ask whether we can find a gauge transformation $N_\mu\to\overline{N}_\mu$ such that $\overline{N}_\mu$ takes on a simple form. Clearly we must make the transformation

$$N_\mu\to\overline{N}_\mu=e^{-\Omega}N_\mu e^{\Omega}-(\partial_\mu e^{-\Omega})e^{\Omega}. \tag{3.20}$$

Under the transformation (3.20)

$$Y_{\mu\nu}\to\overline{Y}_{\mu\nu}=e^{-\Omega}Y_{\mu\nu}e^{\Omega}. \tag{3.21}$$

Substituting for N_μ in (3.20) from (3.19), we find

$$\overline{N}_\mu=-\tfrac{1}{2}\overline{Y}_{\mu\nu}x^\nu. \tag{3.22}$$

Moreover, in the barred system $Y_{\mu\nu.\rho}=0$ becomes

$$\partial_\rho\overline{Y}_{\mu\nu}=0 \tag{3.23}$$

as a consequence of (3.18) and (3.22).

To summarize our results so far, when $Y_{\mu\nu.\rho}=0$, there exists a gauge in which $Y_{\mu\nu}(x)$ is constant $[=\overline{Y}_{\mu\nu}(0)]$ and $\overline{N}_\mu(x)=-\frac{1}{2}\overline{Y}_{\mu\nu}(0)x^\nu$.

Secondly we have to deal with (3.16). Here again we begin by quoting another lemma (proof in Appendix A).

Lemma 3.

$$(X_{.\rho\sigma\tau}=Y_{\mu\nu.\rho}=0)\Rightarrow([Y_{\mu\nu},X_{.\rho}]=0,\ [Y_{\mu\nu},X_{.\rho\sigma}]=0,\ [Y_{\mu\nu},[Y_{\kappa\lambda},X]]=0). \tag{3.24}$$

If we now continue to work in the barred system, the following results are easily established:

$$\begin{aligned}\overline{X}_{.\tau}&=\overline{X}_{.\tau}+\tfrac{1}{2}[\overline{Y}_{\tau\sigma},\overline{X}]x^\sigma,\\ \overline{X}_{.\tau\rho}&=\overline{X}_{.\tau\rho}+\tfrac{1}{2}[\overline{Y}_{\tau\rho},\overline{X}],\\ \overline{X}_{.\tau\rho\sigma}&=\overline{X}_{.\tau\rho\sigma}=0.\end{aligned} \tag{3.25}$$

Thus we may solve $\overline{X}_{.\rho\sigma\tau}=0$ by the Taylor series:

$$\overline{X}(x)=\overline{X}(0)+\overline{X}_{.\tau}(0)x^\tau+\tfrac{1}{2}\{(\overline{X}_{.\tau\rho}(0)+\tfrac{1}{2}[\overline{Y}_{\tau\rho}(0),\overline{X}(0)]\}x^\tau x^\rho. \tag{3.26}$$

We are now in a position to substitute (3.26) and (3.22) into (3.4), whereupon we are led to an equation that is essentially of the same form as that encountered in Sec. II. In so doing, (3.4) becomes

$$\begin{aligned}(\partial^2+X(x')+X_{.\tau}(x')(x-x')^\tau+(x-x')^\sigma Y_{\sigma\tau}(x')\partial_\tau\\ +(x-x')^\sigma\{\tfrac{1}{2}X_{.\sigma\tau}(x')+\tfrac{1}{4}[Y_{\sigma\tau}(x'),X(x')]-\tfrac{1}{4}Y^2{}_{\sigma\tau}(x')\}(x-x')^\tau)G(x,x')=\delta(x,x').\end{aligned} \tag{3.27}$$

Note that we have shifted the origin in Eqs. (3.22) and (3.26) to the point x', and dropped the bars from the X's. It is now ensured that N and M will enter the effective Lagrangian only via the tensor combinations X and Y. This is of course as it must be when one imposes covariant restrictions on the background field.

With the exception of the term $(x-x')^\sigma Y_{\sigma\tau}\partial_\tau$, Eq. (3.27) is of the same form as our basic equation of Sec. II, (2.11), where α, β, and γ are given by

$$\begin{aligned}\alpha&=X,\\ \beta_\mu&=X_{.\mu},\\ \gamma^2{}_{\mu\nu}&=2X_{.\mu\nu}+[Y_{\mu\nu},X]-Y^2{}_{\mu\nu}.\end{aligned} \tag{3.28}$$

But there is now a difference in that in general

$$[\beta_\mu, \beta_\nu] \neq 0, \quad [\gamma^2{}_{\mu\nu}, \gamma^2{}_{\kappa\lambda}] \neq 0, \text{ etc.} \tag{3.29}$$

The number of invariants (formed from these nonvanishing commutators) upon which the effective Lagrangian can depend is now without limit unless we specify the underlying group. In consequence the complete solution to (3.27) becomes very much more complicated than in Sec. II.

However, if one is interested solely in the infinite part of the effective Lagrangian, these difficulties do not manifest themselves. Now although we have already emphasized that the isolation of such infinite counterterms is not an essential step in the calculation (at least for renormalizable theories), it is often of interest to display the divergent parts explicitly, especially if one is interested in testing the renormalizability of a given theory. By way of a diversion, therefore, we shall show how these one-loop counterterms now follow almost immediately.

B. Counterterms

Returning to Sec. II, if one isolates from Eq. (2.31) for $\mathcal{L}^{(1)}$ those terms which yield divergences at $n=4$, we have

$$\mathcal{L}_{\text{div}} = \frac{\hbar}{2(4\pi)^{n/2}} \operatorname{Tr} \int_0^\infty \frac{ds}{s^{1+n/2}} \left[e^{-\alpha s}\left(1 + \tfrac{1}{12}\operatorname{tr}\gamma^2 s^2\right) - e^{-\alpha(0)s}\right], \tag{3.30}$$

where we have supplied the trace symbol necessary when α and γ carry i, j indices. (Tr implies a trace over i, j labels and tr over μ, ν labels.) Thus,

$$\mathcal{L}_{\text{div}} = \frac{\hbar}{2(4\pi)^{n/2}} \operatorname{Tr}\left\{\Gamma(-\tfrac{1}{2}n)[\alpha^{n/2} - \alpha(0)^{n/2}] + \Gamma(2 - \tfrac{1}{2}n)\alpha^{n/2-2}\tfrac{1}{12}\operatorname{tr}\gamma^2\right\}. \tag{3.31}$$

To cancel the infinity at $n=4$, therefore, we need a counterterm

$$\Delta\mathcal{L} = \frac{\hbar}{32\pi^2(n-4)} \operatorname{Tr}[\alpha^2 - \alpha^2(0) + \tfrac{1}{6}\operatorname{tr}\gamma^2], \tag{3.32}$$

which in the general theory becomes

$$\Delta\mathcal{L} = \frac{\hbar}{32\pi^2(n-4)} \operatorname{Tr}[X^2 - X^2(0) - \tfrac{1}{6}\operatorname{tr}Y^2] \tag{3.33}$$

on using Eqs. (3.28), and remembering that $X_{.\mu}{}^{\mu}$ is a total divergence. This expression was first obtained by 't Hooft and Veltman[7] using different methods.

Alternatively, $\Delta\mathcal{L}$ may be derived by simply iterating the basic Green's-function equation (3.4), the first few terms of the iteration providing the divergent part of $\mathcal{L}^{(1)}$.

C. The effective Lagrangian and another example

We now return to the problem at hand, deriving $\mathcal{L}^{(1)}$ in the general case.[8] As was mentioned in the Introduction, rather than try to determine $\mathcal{L}^{(1)}$ in terms of X and Y for a particular group, we shall instead keep the group arbitrary but make the simplification

$$X_{ij} = m^2\delta_{ij}. \tag{3.34}$$

This would be the case, for example, when the quantum field of mass m interacts with an external gauge field.

The effective Lagrangian is given by

$$\frac{\partial\mathcal{L}^{(1)}}{\partial m^2} = -\tfrac{1}{2}\langle h^i(x)h_i(x)\rangle = -\frac{\hbar}{2i}\operatorname{Tr}G(x,x), \tag{3.35}$$

and Eq. (3.27) for G is now

$$[\partial^2 + m^2 + Y_{\mu\nu}(x')(x-x')^\mu\partial_\nu - \tfrac{1}{4}Y^2{}_{\mu\nu}(x')(x-x')^\mu(x-x')^\nu]\,G(x,x') = \delta(x,x'). \tag{3.36}$$

The only matrix which need concern us now is $Y^{ij}_{\mu\nu}$, which, moreover, satisfies (lemma 1) $[Y_{\mu\nu}, Y_{\kappa\lambda}]=0$. Equation (3.36) may now be solved in the same way as (2.11). The term in (3.36) which is linear in Y, and which has no analogue in (2.11), is of no consequence. The reader may convince himself of this by noting that $Y_{\mu\nu}$ is antisymmetric and that it commutes with $A_{\rho\sigma}$, B_ρ, and C of Eq. (2.18) since these depend only on Y.

Thus $\mathcal{L}^{(1)}$ is given simply by Eq. (2.31), with $\alpha = m^2$, $\beta_\mu = 0$, and $\gamma^2{}_{\mu\nu} = -Y_{\mu\nu}{}^2$, and by supplying the overall trace.

$$\mathcal{L}^{(1)} = \frac{\hbar}{2(4\pi)^{n/2}} \operatorname{Tr}\int_0^\infty \frac{ds}{s^{1+n/2}}\, e^{-m^2 s}(e^{-f(s)} - 1), \tag{3.37}$$

where

$$f(s)=\tfrac{1}{2}\,\mathrm{tr}\ln[(-Y^2)^{1/2}s]^{-1}\sin[(-Y^2)^{1/2}s]\,.$$

This is the one-loop effective Lagrangian for an arbitrary field theory, subject to $X=m^2$.

As an example, let us consider charged bosons in an external electromagnetic field, as in Eqs. (3.6). All the non-Abelian character of (3.37) now disappears and we have simply

$$Y^{2\,\,ij}_{\mu\nu}=-e^2F^2{}_{\mu\nu}\delta^{ij}\,,\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu\,. \tag{3.38}$$

Substituting into (3.37) and being careful to remember that

$$\mathrm{Tr}\,\delta^{ij}=2 \tag{3.39}$$

we obtain

$$\mathcal{L}^{(1)}=\frac{\hbar}{(4\pi)^{n/2}}\int_0^\infty\frac{ds}{s^{1+n/2}}\,e^{-m^2s}\{\exp[-\tfrac{1}{2}\,\mathrm{tr}\ln(eFs)^{-1}\sin(eFs)]-1\}. \tag{3.40}$$

After renormalization, the total Lagrangian is that given by Schwinger:

$$\begin{aligned}\mathcal{L}&=\mathcal{L}^{(0)}+\mathcal{L}^{(1)}\\&=\tfrac{1}{4}\,\mathrm{tr}F^2+\frac{\hbar}{16\pi^2}\int_0^\infty\frac{ds}{s^3}e^{-m^2s}\{\exp[-\tfrac{1}{2}\,\mathrm{tr}\ln(eFs)^{-1}\sin(eFs)]-1-\tfrac{1}{12}e^2s^2\,\mathrm{tr}F^2\}\\&=\tfrac{1}{4}\,\mathrm{tr}F^2+\frac{\hbar}{90m^4}\left(\frac{e^2}{4\pi}\right)^2[\tfrac{1}{4}\,\mathrm{tr}F^4+\tfrac{5}{16}(\mathrm{tr}F^2)^2]+\cdots\,.\end{aligned} \tag{3.41}$$

IV. CONCLUSIONS

In this paper we have concentrated on the evaluation of the exact effective Lagrangians induced by single closed loops in a prescribed background field, these exact expressions summarizing the contributions from all one-loop Feynman diagrams possessing arbitrarily many external legs.

Further improvements in the approximation may follow two courses: (i) extending the calculation to two or more loops, and (ii) allowing a greater degree of arbitrariness to the background field. There are also other extensions of the calculation, within the present approximation, and these include (iii) studying the situation where the loop particles are themselves gauge fields, and in the case of non-Abelian groups including the fictitious particle contributions, and (iv) allowing for the presence of the gravitational field. We intend to return to these improvements and extensions in a future publication, but shall include a brief discussion of them here and anticipate which are likely to be straightforward and which are likely to present problems.

A. Improvements and extensions

Let us firstly consider the extension to more loops. Since there is associated with each closed loop a factor $\hbar$ in the effective Lagrangian, the one-loop approximation corresponds to determining the Green's function $G_{ij}(x,x')$ to zeroth order in $\hbar$ and hence the Lagrangian to first order in $\hbar$. This is what we have done in this paper. If one is interested only in a non-self-interacting quantized field in an external classical field, of course, then one loop is all one will ever need. For more realistic situations, though, it is desirable to go to two or more loops. Now, provided we have an exact solution for the effective action to order $\hbar$, which we have in the quasilocal situation, then we can obtain exact results to any desired order of $\hbar$ by iteration of the Dyson-Schwinger equation[3] for the effective action Γ:

$$\begin{aligned}\Gamma_{,i}&=S_{,i}+\frac{1}{2!}\left(\frac{\hbar}{i}\right)S_{,ijk}G^{jk}\\&\quad+\frac{1}{3!}\left(\frac{\hbar}{i}\right)^2S_{,ijkl}G^{ja}G^{kb}G^{lc}\Gamma_{,abc}\\&\quad+\cdots\,.\end{aligned} \tag{4.1}$$

The Green's function G^{ij} is defined by

$$G^{ik}\Gamma_{,kj}=-\delta^i_j\,. \tag{4.2}$$

Here we have used DeWitt's condensed notation in which functional differentiation with respect to the background field ϕ^i is denoted by a comma and a sum over indices implies an integration over the appropriate space-time argument. For a prescribed background field, therefore, the problem of extending the calculation to more loops is merely one of calculational tedium rather than of principle.

As far as allowing a greater degree of arbitrariness to the background field is concerned, however, exact solutions will be more difficult to find. The appropriate Green's-function equation can always be set up, of course, no matter how many derivatives of the field are kept. Probably the best method of solving this equation is then to find an exact solution including the nth derivative (for $n=2$, say) and use this as a starting point to find the dependence on the $(n+1)$th.

We do not anticipate any further obstacles in the inclusion of closed loops of gauge particles, even in the non-Abelian case, since the fictitious particle Green's function may be evaluated in the same way as any other. One does expect, however, that the resulting effective Lagrangian will be gauge-dependent. Even though the background field method and dimensional regularization ensure that only covariant quantities appear, the numerical coefficients in front of these quantities will involve the gauge parameter. Nor is this surprising. The Green's function, and hence the effective Lagrangian, are off-mass-shell quantities, and there is no reason to expect them to be gauge-invariant. On-mass-shell S-matrix elements determined from this Lagrangian, on the other hand, ought to be satisfactorily independent of one's choice of gauge.

Of particular interest is the application of our methods to gravity, whether it be in the form of an external field or itself quantized. Here we need to study the case where the W of Eq. (3.1) is itself space-time-dependent, $W^{\mu\nu}_{ij}=g^{\mu\nu}(x)\delta_{ij}$. Apart from this one (very important) difference, the calculation proceeds along the lines we have already indicated.

B. Applications

We do not, of course, claim to be the first to recognize the power and elegance of the effective-Lagrangian approach to field theory, but we do feel that its usefulness cannot be overstressed, and here we mention again some of its many applications since they have, perhaps, received insufficient publicity in the literature.

As an example, consider the problem of perturbation-theory anomalies. To test whether a theory is anomalous, one simply looks to see whether the effective Lagrangian contains terms which do not respect the symmetry present in the original Lagrangian. The anomalous amplitude for a particular process then follows after functionally differentiating these terms with respect to the appropriate fields. There is no need for lengthy computation of Feynman diagrams, nor tracing over strings of Dirac matrices.

Symmetry violation of a different kind—spontaneous symmetry breakdown—may also arise if the effective action has a minimum away from the origin even when the classical action does not, i.e., if

$$S_{,i}=0 \text{ for } \phi^i=0\,, \tag{4.3}$$

but

$$\Gamma_{,i}=0 \text{ for some } \phi^i\neq 0\,.$$

This has been discussed by Coleman and Weinberg and we shall not repeat the details here.

There is another very interesting application to a subject which has recently become fashionable, especially in relativity and cosmology circles, namely, particle creation by a prescribed background field. As Schwinger[1] displayed so elegantly, to extend one's results to pair-producing fields it is merely necessary to add an infinitesimal negative imaginary constant to the inverse of the Green's function. Since the vacuum-to-vacuum amplitude in the presence of the background field is related to the quantum action functional, W,[9] by

$${}_{+}\langle 0|0\rangle_{-}=e^{iW}\,, \tag{4.4}$$

the positive imaginary contribution to W thus obtained is interpreted by the statement that

$$|e^{iW}|^2=e^{-2\,\mathrm{Im}(W)} \tag{4.5}$$

represents the probability that no pair creation occurs in the history of the field. Thus the probability, per unit volume and per unit time, that a pair is created by the field is given simply by twice the imaginary part of the effective Lagrangian. For a constant pure electric field, $\mathcal{E}$, for example, Schwinger derived [from Eq. (B13)] his famous result

$$2\,\mathrm{Im}\mathcal{L}=\frac{\alpha^2}{\pi^2}\mathcal{E}^2\sum_{n=1}^{\infty}n^{-2}\exp\left(-\frac{n\pi^2m^2}{e\mathcal{E}}\right). \tag{4.6}$$

Note that from Sec. III, we can now extend this result to the case of an external Yang-Mills field. A much more interesting, and, from the cosmological point of view, more realistic extension would be the evaluation of particle production rates in a prescribed background gravitational field, that is to say, in a given geometry. Indeed, with our method, one might expect to obtain *exact* results in locally symmetric space-times for which the Riemann tensor obeys

$$\nabla_\lambda R_{\mu\nu\rho\sigma}=0 \tag{4.7}$$

since this equation is the gravitational analog of the condition

$$\nabla_\lambda Y^{ij}_{\mu\nu}=0\quad(\text{i.e., } Y^{ij}_{\mu\nu.\lambda}=0) \tag{4.8}$$

employed in Sec. III. We note here, in passing, that in just the same way as the energy-momentum tensor is given by functionally differentiating the classical action with respect to the metric,

$$\frac{\delta S}{\delta g_{\mu\nu}(x)}=-\tfrac{1}{2}g^{1/2}(x)T^{\mu\nu}(x)\,, \tag{4.9}$$

so the "regularized energy-momentum tensor" is given by the functional derivative of the effective action

$$\frac{\delta W}{\delta g_{\mu\nu}(x)} = -\tfrac{1}{2}\langle g^{1/2}(x) T^{\mu\nu}(x)\rangle . \qquad (4.10)$$

We also remark that, in the context of particle production, it is desirable that the effective Lagrangian be exact; the nonanalyticity in the coupling constant of Eq. (4.6) means that perturbative methods would be altogether inappropriate.

Note added in proof. Another example of the usefulness of effective actions in field theory has recently been provided by Abdus Salam and J. Strathdee [ICTP, Trieste, Reports Nos. IC/74/133 and 140 (unpublished)], who indicate that there may exist critical values of constant external magnetic fields above which a spontaneously broken symmetry might be restored.

With regard to the hope of obtaining effective Lagrangians for space-times obeying (4.7), P. Candelas and D. J. Raine [Phys. Rev. D (to be published)] have since obtained exact results for de Sitter space.

APPENDIX A

(i) Proof of lemma 1 [*Eq. (3.18)*].

$$Y_{\mu\nu.\rho} = 0 \Rightarrow Y_{\mu\nu.\kappa\lambda} - Y_{\mu\nu.\lambda\kappa} = 0 . \qquad (A1)$$

By the definition of the covariant derivative (3.13) (A1) reads

$$[(\partial_\kappa N_\lambda - \partial_\lambda N_\kappa), Y_{\mu\nu}] + [N_\kappa, [N_\lambda, Y_{\mu\nu}]] - [N_\lambda, [N_\kappa, Y_{\mu\nu}]] = 0$$

$$\Rightarrow [Y_{\kappa\lambda}, Y_{\mu\nu}] + [N_\kappa, [N_\lambda, Y_{\mu\nu}]] + [N_\lambda, [Y_{\mu\nu}, N_\kappa]] + [Y_{\mu\nu}, [N_\kappa, N_\lambda]] = 0$$

$$\Rightarrow [Y_{\kappa\lambda}, Y_{\mu\nu}] = 0 \text{ by the Jacobi identity.}$$

(ii) Proof of lemma 2 [*Eq. (3.19)*]. Let us write

$$N_\mu = -\tfrac{1}{2} Y_{\mu\tau} x^\tau + A_\mu(x), \qquad (A2)$$

where A_μ is arbitrary for the moment.

Then N_μ is constrained to satisfy

$$Y_{\mu\nu} \equiv \partial_\mu N_\nu - \partial_\nu N_\mu + [N_\mu, N_\nu],$$

$$Y_{\mu\nu.\rho} = 0 .$$

Substituting (A2) into the above equations we obtain

$$Y_{\mu\nu} \equiv Y_{\mu\nu} - \tfrac{1}{2}(Y_{\nu\tau,\mu} - Y_{\mu\tau,\nu}) x^\tau + \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] - \tfrac{1}{2}[Y_{\mu\tau}, A_\nu] x^\tau - \tfrac{1}{2}[A_\mu, Y_{\nu\tau}] x^\tau , \qquad (A3)$$

$$\partial_\rho Y_{\mu\nu} + [A_\rho, Y_{\mu\nu}] = 0 . \qquad (A4)$$

In deriving Eq. (A4) it is necessary to employ lemma 1. Equations (A3) and (A4) can now be combined to yield

$$Y_{\mu\nu}(A) \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \equiv 0 . \qquad (A5)$$

Equation (A5) simply states that the Riemann tensor (for the internal-symmetry group) constructed from the A field vanishes. It follows[10] that there exists a gauge transformation $A \to A'$ such that

$$A'_\mu = 0 . \qquad (A6)$$

Thus the most general form for A_μ is given by a gauge transformation of (A6), viz.

$$A_\mu = e^{\Omega} \partial_\mu e^{-\Omega} . \qquad (A7)$$

(A7) and (A2) now yield the most general form for N_μ,

$$N_\mu = -\tfrac{1}{2} Y_{\mu\nu} x^\nu + e^{\Omega}\partial_\mu e^{-\Omega} .$$

(iii) Proof of lemma 3 [*Eq. (3.24)*].

$$X_{.\tau\rho\sigma} = 0 \Rightarrow (X_{.\tau\rho} - X_{.\rho\tau})_{.\sigma} = 0 , \qquad (A8)$$

$$X_{.\tau\rho} - X_{.\rho\tau} = [Y_{\rho\tau}, X] . \qquad (A9)$$

(A8) follows immediately from the definition of the covariant derivative (3.13) and the Jacobi identity. Combining (A8) and (A9) and using the condition $Y_{\rho\tau.\sigma} = 0$ yields

$$[Y_{\rho\tau}, X_{.\sigma}] = 0 . \qquad (A10)$$

$$\text{Equation (A9)} \Rightarrow [Y_{\rho\tau}, X_{.\kappa\lambda}] = 0 . \qquad (A11)$$

$$\text{Equations (A11) and (A9)} \Rightarrow [Y_{\rho\tau}, [Y_{\kappa\lambda}, X]] = 0 . \qquad (A12)$$

APPENDIX B: EXTENSION TO FERMIONS

The text of this paper confined itself to bosons. As an illustration of how fermions are included, consider charged fermions of mass m in an external electromagnetic field. Reasoning akin to (2.13) leads to

$$\frac{\partial \mathcal{L}^{(1)}}{\partial m} = -\langle \bar{\psi}^{\alpha}(x)\psi_{\alpha}(x)\rangle = -\frac{\hbar}{i}\,\mathrm{Tr}G(x,x) \tag{B1}$$

(trace over spinor indices), and, in the fermion case, $G(x,x')$ obeys

$$[\partial^{\mu}(-i\partial_{\mu}+eA_{\mu})+m]\,G(x,x')=\delta(x,x'). \tag{B2}$$

When $\partial_{\rho}F_{\mu\nu}=0$, we have from (3.22)

$$A_{\mu}(x)=-\tfrac{1}{2}F_{\mu\nu}(x')(x-x')^{\nu}, \tag{B3}$$

and therefore

$$\{\gamma^{\mu}[-i\partial_{\mu}-\tfrac{1}{2}eF_{\mu\nu}(x')(x-x')^{\nu}]+m\}\,G(x,x')=\delta(x,x') \tag{B4}$$

or, in p space,

$$\left[-\gamma^{\mu}\left(p_{\mu}+\tfrac{1}{2}ieF_{\mu\nu}\frac{\partial}{\partial p_{\nu}}\right)+m\right]G(p)=1. \tag{B5}$$

To determine $G(x,x)$,

$$G(x,x)=\int\frac{d^4p}{(2\pi)^4}G(p) = \int\frac{d^4p}{(2\pi)^4}G^{+}(p), \tag{B6}$$

we need only the even part of $G(p)$, denoted $G^{+}(p)$, i.e.,

$$G^{+}(p)=\tfrac{1}{2}[G(p)+G(-p)], \tag{B7}$$

and G^{+} obeys the second-order equation

$$\left\{-\left[\gamma^{\mu}\left(p_{\mu}+\tfrac{1}{2}ieF_{\mu\nu}\frac{\partial}{\partial p_{\nu}}\right)\right]^2+m^2\right\}G^{+}(p)=m. \tag{B8}$$

With

$$\tfrac{1}{2}\{\gamma_{\mu},\gamma_{\nu}\}=\delta_{\mu\nu}, \quad \tfrac{1}{2}i[\gamma_{\mu},\gamma_{\nu}]=\sigma_{\mu\nu}, \tag{B9}$$

this becomes

$$\left[-p^2+m^2-\tfrac{1}{2}\sigma_{\mu\nu}F^{\mu\nu}-ieF_{\mu\nu}p^{\mu}\frac{\partial}{\partial p_{\nu}} -\tfrac{1}{4}e^2F^2_{\ \mu\nu}\frac{\partial^2}{\partial p_{\mu}\partial p_{\nu}}\right]G^{+}(p)=m. \tag{B10}$$

As in Sec. III, the term $F_{\mu\nu}p^{\mu}\partial/\partial p_{\nu}$ is of no consequence and (B10) may be solved in the same way as our basic equation (2.11) with

$$\alpha=m^2-\tfrac{1}{2}e\sigma\cdot F, \quad \beta_{\mu}=0, \quad \gamma^2_{\ \mu\nu}=e^2F^2_{\ \mu\nu}. \tag{B11}$$

Thus,

$$G(x,x)=\frac{m}{i(4\pi)^{n/2}}\int_0^{\infty}\frac{ds}{s^{n/2}}e^{-m^2s}e^{e\sigma\cdot Fs/2}e^{-f(s)}, \tag{B12}$$

with

$$f(s)=\tfrac{1}{2}\,\mathrm{tr}\ln(eFs)^{-1}\sin(eFs),$$

leading, via (B1), to the effective Lagrangian

$$\mathcal{L}^{(1)}=-\frac{\hbar}{2(4\pi)^{n/2}}\int_0^{\infty}\frac{ds}{s^{1+n/2}}e^{-m^2s}e^{-f(s)}\mathrm{Tr}\,e^{e\sigma\cdot Fs/2} \tag{B13}$$

in agreement (at $n=4$) with that obtained by Schwinger.[1]

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[1]J. Schwinger, Phys. Rev. 82, 664 (1951). See also W. Heisenberg and H. Euler, Z. Phys. 98, 714 (1936).

[2]A thorough treatment of this approach is given by B. S. DeWitt, Phys. Rev. 162, 1195 (1967).

[3]M. R. Brown and M. J. Duff, Nuovo Cimento Lett. 11, 80 (1974); 11, 544 (1974). This paper was prompted by an earlier paper of Abdus Salam and J. Strathdee, Phys. Rev. D 9, 1129 (1974).

[4]To within a sign error. The sign in front of $\frac{1}{2}\lambda m^2\phi^2$ is minus, not plus as given in Ref. 5.

[5]S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973).

[6]See also R. Jackiw, Phys. Rev. D 9, 1686 (1974).

[7]G. 't Hooft, Nucl. Phys. B62, 444 (1973); G. 't Hooft and M. Veltman, CERN Report No. TH 1723 (unpublished).

[8]If one is interested solely in the effective potential ($N=0$, M=constant), (3.4) yields the result

$$V^{(1)}[\phi]=\frac{-\hbar}{2(4\pi)^{n/2}}\mathrm{Tr}\int_0^{\infty}\frac{ds}{s^{1+n/2}}e^{-M[\phi]s}+\text{constant}.$$

Were we to introduce a fermion interaction $\bar{\psi}^{\alpha}\mu_{\alpha\beta}[\phi]\psi^{\beta}$, this would become

$$V^{(1)}[\phi]=\frac{-\hbar}{2(4\pi)^{n/2}}\mathrm{Tr}\int_0^{\infty}\frac{ds}{s^{1+n/2}}[e^{-Ms}-e^{-\mu^{\dagger}\mu s}] +\text{constant}.$$

[9]Equation (4.4) is valid at the one-loop level. For more than one loop, one must distinguish between the vacuum-to-vacuum amplitude which is the generating functional for the connected Green's functions and the effective action which generates the one-particle irreducible vertices.

[10]J. L. Anderson, *Principles of Relativity Physics* (Academic, New York, 1967), Chap. 2.