Gauge fields on a lattice. III. Strong-coupling expansions and transition points

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(Received 6 November 1974)

We discuss the principles of the high-temperature expansion leading to a variation-perturbation method. For pure gauge fields, diagrams are two-dimensional manifolds. As an application, we compute the critical coupling constants for discrete, Abelian, and SU(2) gauge groups and compare them with some earlier results.

I. INTRODUCTION

In previous papers,[1] we described a gauge theory on a lattice following Wilson's idea.[2] The motivations and the notations are discussed at length in papers I and II. This paper presents some numerical calculations in the disordered, fully symmetric, high-temperature phase of the gauge field system (i.e., in the strong coupling limit).

In Sec. II, we recall the formalism of high-temperature expansions and the analysis of diagrams in terms of their strongly irreducible parts. This leads to a variational method, associated with a perturbative expansion of the generalized free energy to be varied.[3] To lowest order this procedure yields results equivalent to the mean-field approximation. The outline is general and we apply it to the two models discussed in Ref. 1: the scalar model (Sec. III) and the gauge model (Sec. IV).

II. FORMALISM OF HIGH-TEMPERATURE EXPANSIONS

A. The partition function in terms of diagrams

Consider the generating functional (or partition function)

$$Z(\{h_i\}) = \int \prod_i D\phi_i \exp\left(\sum h_i \phi_i + \beta S \right). \tag{2.1}$$

Here ϕ_i are (possibly multicomponent) fields assigned to the site i of a discrete lattice in dimension d. The values assumed by ϕ_i can be continuous or discrete. In the latter case, integration is replaced by summation. Whenever the range of ϕ is compact, we shall normalize the measure to unity. The action βS, assumed to be translationally invariant, is written

$$\beta S = \beta \sum_k \frac{1}{k!} \sum_{i_1, \ldots, i_k} V_{(k)}^{i_1 \cdots i_k} \phi_{i_1} \cdots \phi_{i_k}, \tag{2.2}$$

and the "potential" $V_{(k)}$ is totally symmetric in its indices.

At infinite temperature ($\beta = 0$), βS vanishes and fields at different sites are independent. Thus Z factorizes into $Z_0 = \prod_i z(h_i)$, with

$$z(h) \equiv \exp u(h) = \int D\varphi \exp(h\varphi). \tag{2.3}$$

The unperturbed average $\langle X \rangle_0$ of a functional $X(\phi_i)$ is defined as

$$\langle X \rangle_0 = Z_0^{-1} \int \prod_i D\phi_i \exp\left(\sum_i h_i \varphi_i \right) X(\phi_i). \tag{2.4}$$

For small β, we expand the exponential $e^{\beta S}$ in powers of β; thus we are led to the unsophisticated high-temperature series

$$Z(\{h_i\}) = Z_0(\{h_i\}) \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \langle S^n \rangle_0. \tag{2.5}$$

It is convenient[3,4] to interpret the terms in the series (2.5) as corresponding to *graphs*. Each graph consists of a finite subset of distinguished sites on the lattice and a set of vertices representing interactions. Lines are drawn joining sites to vertices. Each site is linked to at least one vertex. A contribution to Z is assigned to each graph in the following way:

(i) A factor $\beta V_{(k)}^{i_1 \cdots i_k}$ corresponds to a vertex linking the k sites $i_1, \ldots, i_k$.

(ii) Each site i linked to n_i vertices yields a factor

$$\langle \phi_i^{n_i} \rangle_0 = z(h_i)^{-1} \frac{d^{n_i}}{dh_i^{n_i}} z(h_i). \tag{2.6}$$

This formula is generalized in an obvious way if ϕ_i is a multicomponent field.

(iii) If a graph remains invariant by an interchange of some of its *vertices*, its contribution has to be divided by the order of its symmetry group.

The sum over all possible distinct graphs (including the empty one, which contributes a term equal to 1) reproduces the expansion (2.5) for $Z(\{h_i\})/Z_0(\{h_i\})$.

If we set all h_i equal to h, we may group the contributions of a family of graphs differing only by the locations of the sites, but yielding identical contributions due to the properties of the poten-

tials $V_{(k)}$. We represent such a family of graphs by a nonlabeled *diagram*. A diagram can thus be drawn independently of the structure of the lattice (for instance, of its dimensionality). Besides the previous rules, we then assign to a given diagram an extra factor, equal to the number of distinct corresponding graphs. Such a factor can be thought of as a number of configurations (n.c.), i.e., the number of ways a diagram can be mapped on the lattice. It may be evaluated by relabeling its sites under the condition that two sites cannot receive the same label.

In this type of calculations one is faced with two problems:

(i) Enumerate all diagrams of a given order in β, and

(ii) find the corresponding n.c. for each diagram.

A discussion of these points is given by Domb[4] for the case of Ising-type interactions ($V_{(k)}=0$ if $k>2$). In Sec. IV we shall present the more intricate case of the gauge model.

Because of translational invariance (with periodic boundary conditions), the n.c. is a polynomial in N, the number of sites, which vanishes for $N=0$. The degree of this polynomial is equal to the number of connected parts of the diagram. In calculating $F=\lim_{N\to\infty}(1/N)\ln Z$, we make use of the existence of the infinite-volume limit. As a result, the high-temperature series for $F-u(h)$ is defined in terms of the same diagrams and same rules as before, except for the replacement of the n.c. by the coefficient of order 1 in its power expansion in N. We call this coefficient the reduced number of configurations (r.n.c.).

With the above definition of diagrams, connected as well as disconnected ones contribute to F. In the subsequent sections, we shall also use more elaborate expansions in terms of suitably modified connected diagrams, which we now recall.

B. Connected, irreducible graphs

The unperturbed averages $\langle\phi^n\rangle_0=e^{-u}(d^n/dh^n)e^u$ can be expressed in terms of cumulants defined as

$$\langle\phi^n\rangle_c=\frac{d^n u}{dh^n}. \tag{2.7}$$

The relation is

$$\langle\phi^n\rangle_0=\sum\langle\phi^{n_1}\rangle_c\cdots\langle\phi^{n_j}\rangle_c, \tag{2.8}$$

where the summation extends over all partitions of n distinct objects. Substituting (2.8) in the expansion of Z/Z_0 defines new graphs, with the property that the lines arriving at each site are tied together in all possible ways, each one corresponding to one term in (2.8). This is to be contrasted with the previous rules, where a single factor $\langle\phi^n\rangle_0$ was assigned to a given site linked to n vertices. With these new rules, when summing over all diagrams, we are allowed to let the sites overlap. Consequently, summations over all connected parts are independent, and $\ln(Z/Z_0)$ is expressed entirely in terms of connected graphs. If we set all h_i equal to h, we obtain

$$F=u(h)+\sum(\text{connected diagrams}). \tag{2.9}$$

To be precise, let us repeat how one computes the contributions of the diagrams entering (2.9). Such a diagram consists of points, vertices, and lines. Each point is joined to at least one vertex through a line. The diagram is to be connected. A vertex of k lines yields a factor $\beta V_{(k)}$, a point linked to n vertices gives a factor $\langle\phi^n\rangle_c$. This contribution is completed by including two weights:

(i) a symmetry factor $p(D)^{-1}$, where $p(D)$ is the order of the symmetry group of the diagrams by interchange of its vertices, and

(ii) a r.n.c., computed by dividing by N the number of distinct mappings of the diagram points onto the lattice sites (note that for connected diagrams the n.c. is simply equal to N times the r.n.c.).

The restriction that two points cannot be mapped on the same site is removed. Two maps differing only by a relabeling of the points of the diagram are not distinguished. Examples illustrating these rules will be worked out shortly.

A connected graph is *reducible* if it is cut out into k parts by removing one of its k-vertices. One allows among the possible parts a single site without any vertex. We might of course have enlarged the definition of the graphs to include such a case, to which corresponds the term $u(h_i)$ in the expression (2.9). A general connected graph is thus a tree of lines joining vertices to irreducible "bubbles" (see Fig. 1).

Call $B(\{h_i\})$ the sum over connected irreducible

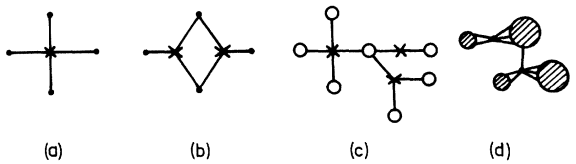


FIG. 1. Examples of graphs: sites are represented as points, vertices by crosses, irreducible parts by circles, strongly irreducible parts by hatched circles. (a) Reducible graph. (b) Irreducible graph. (c) General decomposition of a graph into a tree of irreducible parts. (d) Same decomposition as before but in strongly irreducible parts.

graphs. For a general connected graph, adding a k-vertex increases the number n_k of k-vertices by one unit and the number n_B of bubbles by $k-1$ units. We then get the topological identity

$$1 = n_B - \sum_k (k-1) n_k . \tag{2.10}$$

Set

$$\langle \phi_i \rangle \equiv m_i = \frac{\partial}{\partial h_i} \ln Z \tag{2.11}$$

and average the relation (2.10) by multiplying both sides by the contribution of each graph. The following equality is obtained:

$$\ln Z(\{h_i\}) = B\left(\left\{h_i + \beta \sum_k \frac{1}{(k-1)!} \sum_{j_2 \cdots j_k} V_{(k)}^{ij_2 \cdots j_k} m_{j_2} \cdots m_{j_k}\right\}\right) - \beta \sum_k \frac{k-1}{k!} \sum_{i_1 \cdots i_k} V_{(k)}^{i_1 \cdots i_k} m_{i_1} \cdots m_{i_k} . \tag{2.12}$$

The combination of (2.11) and (2.12) yields a variational principle for the calculation of the total free energy. Namely, consider the right-hand side of (2.12) as a function Φ of the two independent sets of variables h_i and m_i; the h_i's being kept fixed, $\ln Z$ is the stationary value of Φ when varying the m_i's:

$$\ln Z(\{h_i\}) = \Phi(\{h_i\}, \{m_i\}) , \tag{2.13}$$

$$\frac{\partial \Phi}{\partial m_i} = 0 . \tag{2.14}$$

The proof may rely on the topological relation (2.10).[5] We may also check (2.14) directly. From the definition of Φ, we have

$$\frac{\partial \Phi}{\partial m_l} = \sum_j \frac{\partial B}{\partial h_j}\left(\left\{h_i + \beta \sum_k \frac{1}{(k-1)!} \sum_{j_2 \cdots j_k} V_{(k)}^{ij_2 \cdots j_k} m_{j_2} \cdots m_{j_k}\right\}\right) \beta \sum_k \frac{1}{(k-2)!} \sum_{j_3 \cdots j_k} V_{(k)}^{jlj_3 \cdots j_k} m_{j_3} \cdots m_{j_k}$$

$$-\beta \sum_k \frac{1}{(k-2)!} \sum_{j_2 \cdots j_k} V_{(k)}^{lj_2 \cdots j_k} m_{j_2} \cdots m_{j_k} . \tag{2.15}$$

On the other hand, making use of (2.12) where m_i takes its actual value (2.11), we obtain

$$m_j = \frac{\partial B}{\partial h_j} + \sum_l \frac{\partial \Phi}{\partial m_l} \frac{\partial m_l}{\partial h_j} . \tag{2.16}$$

We now eliminate the explicit B term of (2.15) by use of (2.16), which yields

$$0 = \sum_l \left[\delta_{jl} + \beta \sum_k \frac{1}{(k-2)!} \times \sum_{j_2 \cdots j_k} V_{(k)}^{jj_2 \cdots j_k} \frac{\partial m_l}{\partial h_{j_2}} m_{j_3} \cdots m_{j_k}\right] \frac{\partial \Phi}{\partial m_l} .$$

The factor in brackets is equal to a unit matrix for $\beta = 0$, and hence its determinant does not vanish for β small enough. Thus, as expected, $\partial\Phi/\partial m_l$ vanishes and (2.14) is proved.

If all h_i are set equal to h and $F(h) = (1/N) \ln Z$, then all m_i are equal to m, $B = Nb(h)$, $\Phi = N\phi(h, m)$, and $m = dF/dh$. All previous formulas apply after omitting a factor N. In later applications, however, N will stand for the number of lattice nodes, whereas the field variables will be defined on the links, the number of which is Nd. The relation between m and dF/dh will therefore be modified into $md = dF/dh$.

Although the lowest-order approximation to B in (2.13) yields, as we shall see below, the mean-field results at the stationary point, it should be noticed that the stationary value of Φ is not necessarily a maximum in m.

If one sets the external field h equal to zero, the extremum lies at the point $m = 0$ for $\beta = 0$. If there is a second-order transition at β_c, m smoothly departs from $m = 0$ at that point. The critical value β_c is then obtained by requiring that

$$\frac{\partial^2}{\partial m^2} \phi(0, 0) = 0 . \tag{2.17}$$

If, on the other hand, the transition is a first-order one with a jump from 0 to m_c at β_c, these two quantities follow from the system of equations:

$$\phi(0, 0) = \phi(0, m_c) ,$$

$$\frac{\partial \phi}{\partial m}(0, m_c) = 0 . \tag{2.18}$$

Of course the whole method allows one to com-

pute in a systematic way various other quantities of interest. In the sequel we shall mainly be interested in the calculation of the critical value of β.

C. Strong irreducibility

It is possible to go further in the classification of graphs by strengthening the concept of irreducibility.[3] Up to now, irreducibility was defined in terms of vertices. The new type, called strong irreducibility, is defined with respect to sites. A connected graph is *strongly irreducible* if omission of any one of its sites (and of the corresponding lines) leaves it connected. A strongly irreducible graph with more than one vertex is *a fortiori* irreducible. A general connected graph is thus a tree of strongly irreducible parts [Fig. 1(d)].

Considering this decomposition of a graph into $n_{\mathfrak{G}}$ strongly irreducible parts, connected by n_L sites, let n_M be equal to the sum of these connecting sites, each one counted as many times as the number of strongly irreducible parts to which it is linked. The following topological identity holds[3,5]:

$$1 = n_{\mathfrak{G}} + n_L - n_M . \tag{2.19}$$

Let $\mathfrak{G}(\{M_i^{(n)}, h_i\})$ be the sum of contributions over strongly irreducible graphs computed according to the preceding rules, except for the replacement of $\langle \phi_i^n \rangle_c$ by the variables $M_i^{(n)}$ defined consistently through

$$M_i^{(n)} = \left[\exp\left(\sum_{k=1}^{\infty} G_i^{(k)} \frac{\partial^k}{\partial h_i{}^k} \right) \right] \langle \phi_i^n \rangle_c ,$$

$$G_i^{(k)} = \frac{\partial \mathfrak{G}}{\partial M_i^{(k)}} (\{M_j^{(n)}, h_j\}) \tag{2.20}$$

(for the first of these equations, no summation over i is implied). Let $\mathfrak{L}_i$ be the extension of formula (2.20) to $n=0$, in the form

$$\mathfrak{L}_i = \left[\exp\left(\sum_{k=1}^{\infty} G_i^{(k)} \frac{\partial^k}{\partial h_i{}^k} \right) \right] u(h_i) . \tag{2.21}$$

If one returns to (2.19) and averages both sides using the contribution of all connected graphs, one finds

$$\ln Z(\{h_i\}) = \mathfrak{G}(\{M_i^{(n)}, h_i\}) + \sum_i \mathfrak{L}_i - \sum_i \sum_{k=1}^{\infty} M_i^{(k)} G_i^{(k)} . \tag{2.22}$$

For a detailed justification of this result, see Refs. 3 and 5. As before, the solution of the self-consistent equations (2.20) can be cast into a variational principle. Namely, the right-hand side of (2.22) is considered as a function of independent h's, M's, and G's. Keeping h's fixed and varying M's and G's yields at the stationary point the value of $\ln Z$. All the discussion at the end of Sec. II B may be carried over to the present extension.

III. SCALAR MODEL

In this section, we apply the previous machinery to the scalar model with global symmetry $O(n)$ described in paper I. For $n=1$, this is nothing but the Ising model and for $n \geq 2$ it is the classical Heisenberg model. We shall also recover the Stanley model in the $n \to \infty$ limit using the technique described in Sec. II C.

A. Critical coupling constant as a power series in $1/d$

Fields are n-dimensional unit vectors $\vec{k}_i$ located at the nodes of a d-dimensional hypercubical lattice and interactions are between nearest neighbors. Thus, we have

$$S = \sum_{(i,j)} \vec{k}_i \cdot \vec{k}_j . \tag{3.1}$$

In the graphs, vertices join two neighboring sites and can just be represented by this bond. The function $z(\vec{h})$ in (2.3) is now ($h = |\vec{h}|$)

$$z(h) \equiv e^{u(h)} = \int d^{n-1}\hat{k} \exp(\vec{k} \cdot \vec{h})$$

$$= a_n h^{1-n/2} I_{n/2-1}(h) , \tag{3.2}$$

where $I_\rho(x)$ is the modified Bessel function, and where a_n is adjusted in such a way that $u(0)=0$.

To find β_c, we use the method of Sec. II B, where $b(h)$ is expanded in powers of the number of bonds. It is convenient to replace the variable m to be determined variationally by

$$H \equiv h + 2\beta d m .$$

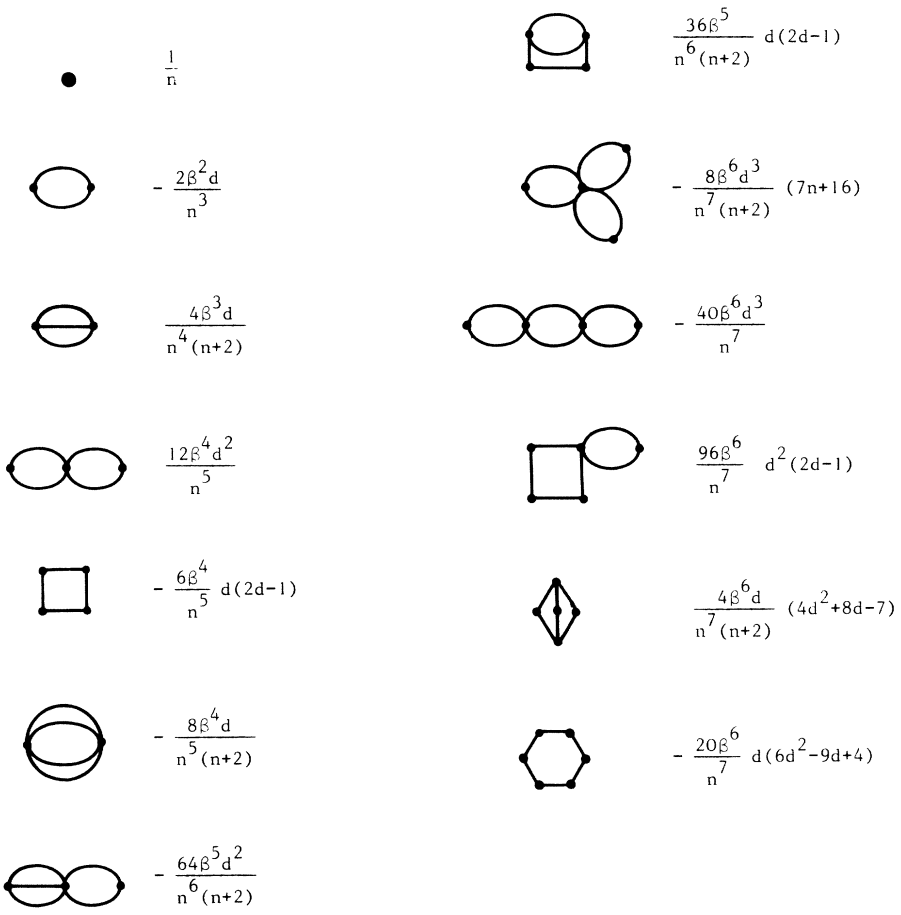
In the absence of external field, Eqs. (2.13) and (2.14) then reduce to

$$F = b(H) - \frac{H^2}{4\beta d},$$

$$\frac{db}{dH} = \frac{H}{2\beta d} . \tag{3.3}$$

To lowest order, we have $b_0(h) = u(h)$, and (3.3) reduces to the mean-field result (see, for instance, paper I), with a second-order transition. The critical β_c, obtained by requiring the vanishing of the H^2 term in F, is found to be equal to $\beta_c = n/2d$ to this order. We can now consider in a systematic fashion the expansion of $b(h)$ in powers of β. In Table I, we have collected the contributions to the coefficient of $\frac{1}{2}H^2$ up to sixth order. A diagram of order p yields a contribution to b which is a polynomial of degree $[p/2]$ in d,

TABLE I. Coefficients of $\frac{1}{2}H^2$ in the contributions to $b(H)$, up to sixth order in β, for the scalar model.



given by the n.c. As β_c is of order $1/d$ from the zeroth-order result, the correction to b and therefore to $\beta_c d$ is of order $d^{-p+[p/2]}$. Consequently the series for β_c appears as a $1/d$ expansion. Thus our calculation yields

$$(\beta_c^*)^{-1} = \left(\frac{2\beta_c d}{n}\right)^{-1} = 1 - \frac{1}{2d} - \frac{2-2/(n+2)}{(2d)^2} - \frac{7-8/(n+2)}{(2d)^3} + O\left(\frac{1}{d^4}\right). \tag{3.4}$$

We recover the results of Ref. 6 for the case $n=1$. We shall also check this formula in the limit $n \to \infty$ at the end of this section. Some numerical values are listed in Table II for β_c^* to sixth order in the expansion of $b(H)$.

TABLE II. Values for $\beta_c^* = 2\beta_c d/n$ in the scalar model obtained through the $1/d$ expansion. The numbers in parentheses are obtained using the alternative method of Sec. III B. Exact results are listed in brackets, either for $n=1$, $d=2$, or for the limit $n \to \infty$ (Sec. III C).

n \ d	2	3	4	5	∞
1	1.6696 [1.763]	1.2883	1.1824	1.1334	1
2		1.3012 (1.3170)	1.1879 (1.1924)	1.1364 (1.1357)	1
3		1.3091	1.1913	1.1382	1
∞	[∞]	1.3416 [1.4808]	1.2047 [1.2385]	1.1455 [1.1563]	1 [1]

B. Free energy and two-point function ($n=2$)

Picking the special case $n=2$ as a typical example, we now perform various high-temperature expansions. They will provide a cross-check for the value of β_c in reasonable agreement with the previous one.

Up to twelfth order, the high-temperature expansion of the free energy (2.9) in zero external field yields

$$F(\beta) = d\beta^2 + (d^2 - \tfrac{5}{4}d)\beta^4 + (\tfrac{16}{3}d^3 - 16d^2 + \tfrac{97}{9}d)\beta^6 + (54d^4 - 284d^3 + \tfrac{22747}{48}d^2 - \tfrac{15613}{64}d)\beta^8$$
$$+ (\tfrac{3968}{5}d^5 - 6328d^4 + \tfrac{55537}{3}d^3 - \tfrac{92585}{4}d^2 + \tfrac{6101009}{600}d)\beta^{10}$$
$$+ (\tfrac{45280}{3}d^6 - 167836d^5 + \tfrac{6677002}{9}d^4 - \tfrac{14484205}{9}d^3 + \tfrac{5457782167}{3240}d^2 - \tfrac{17218423249}{25920}d)\beta^{12} + O(\beta^{14}) . \quad (3.5)$$

The curve $F = F(\beta)$ is drawn in Fig. 2 for $d=4$. The zero-momentum propagator $\chi(\beta)$ (the susceptibility), computed up to seventh order, reads

$$\chi(\beta) = 1 + 2\beta d + (4d^2 - 2d)\beta^2 + (8d^3 - 8d^2 + d)\beta^3 + (16d^4 - 24d^3 + 4d^2 + 4d)\beta^4 + (32d^5 - 64d^4 + 20d^3 + 18d^2 - \tfrac{19}{3}d)\beta^5$$
$$+ (64d^6 - 160d^5 + 80d^4 + 24d^3 + \tfrac{146}{3}d^2 - \tfrac{341}{6}d)\beta^6 + (128d^7 - 384d^6 + 272d^5 + 8d^4 + 246d^3 - \tfrac{2423}{6}d^2 + \tfrac{1071}{8}d)\beta^7 + O(\beta^8) . \quad (3.6)$$

The corresponding curve is also drawn in Fig. 2 for $d=4$. At $\beta=\beta_c$ a singularity occurs. Its position may be determined by computing either the limiting ratio of two consecutive terms of the series or the smallest positive zero of a Padé-approximant denominator. Both methods give the numbers quoted in parentheses in Table II.

For large distances, the propagator behaves as $e^{-\mu r}$ up to a power of r. The first few terms of the expansion of μ (in units of the inverse lattice spacing) are

$$\mu(S_2)^{1/2} = \ln\beta + (S_{-1} - 2d + \tfrac{1}{2})\beta^2$$
$$+ [5S_{-1}^2 - S_{-1}S_{-2} - \tfrac{1}{2}S_{-3} + (\tfrac{5}{2} - 12d)S_{-1} + (2d - \tfrac{1}{2})S_{-2} - \tfrac{1}{2}S_2 - 10d^2 + 12d - \tfrac{67}{24}]\beta^4 + O(\beta^6) , \quad (3.7)$$

with

$$\alpha_i = |r_i| \Big/ \sum_{j=1}^{d} |r_j|$$

and

$$S_n = \sum_{i=1}^{d} \alpha_i^n .$$

Away from the critical point, no Euclidean invariance appears, even in the large-distance limit. It is, nevertheless, generally believed that this invariance shows up near the critical point when the correlation distance becomes infinite.

C. The Stanley model ($n\to\infty$ limit)

It is well known that, in the $n\to\infty$ limit, the scalar model is soluble.[7] We briefly outline the use of the present techniques to find this result. For n large, we need an estimate of $z(h) = e^{u(h)}$ for h of order n in formula (3.2). This is given by

$$u(h) \simeq \frac{1}{2}n\left\{\left(1 + \frac{4h^2}{n^2}\right)^{1/2} - 1 - \ln\left[\frac{1}{2} + \frac{1}{2}\left(1 + \frac{4h^2}{n^2}\right)^{1/2}\right]\right\}, \quad (3.8)$$

where h^2 stands for $\sum_{\alpha=1}^{n} h_\alpha^2$. Returning to (2.22) and (2.21), it is easily shown that $G^{(k)}$, $k \geq 3$, can

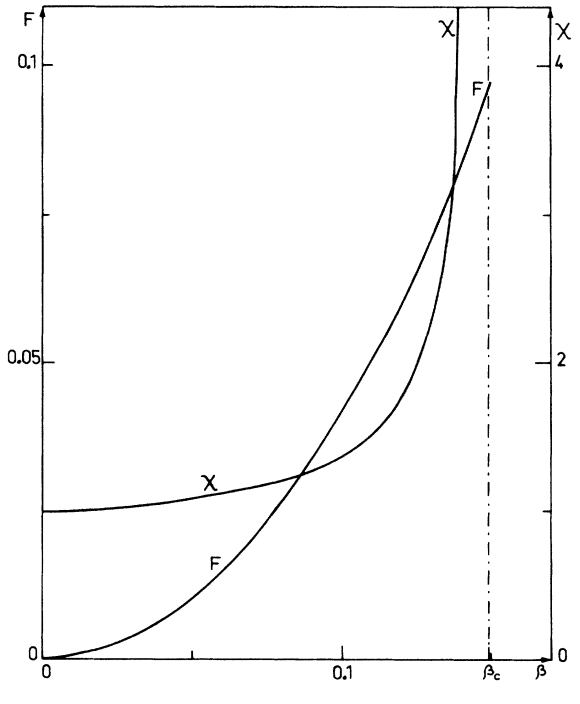


FIG. 2. The free energy and susceptibility of a scalar, $n=2$, model in dimension 4, as computed from formulas (3.5) and (3.6) of the text.

be dropped for $n \to \infty$ so that

$$\mathcal{L}(\vec{G}^{(1)}, \underline{G}^{(2)}, \vec{h}) = \left[\exp\left(\frac{\partial}{\partial h_a}\underline{G}^{(2)}{}_{ab}\frac{\partial}{\partial h_b}\right)\right] \times u(|\vec{h}+\vec{G}^{(1)}|)\,. \tag{3.9}$$

The notation $\underline{G}^{(2)}$ is to recall that we are dealing with a $n\times n$ matrix. To leading order in n, (3.9) yields the approximation

$$\mathcal{L} \simeq u((|\vec{h}+\vec{G}^{(1)}|^2 + 2\,\mathrm{tr}\underline{G}^{(2)})^{1/2})\,. \tag{3.10}$$

Finally the relevant set of diagrams for the computation of $\mathcal{G}$ turns out to be those with no more than one loop (Fig. 3). Thus

$$\mathcal{G}(M) = \beta d(\vec{M}^{(1)})^2 + \sum_{p=2}^{\infty}\frac{1}{2p}\mathrm{tr}[(\underline{M}^{(2)})^p]\frac{1}{N}\mathrm{Tr}[(\beta V)^p]\,. \tag{3.11}$$

Here $\underline{M}^{(2)}$ is a $n\times n$ matrix, while V is an $N\times N$ matrix with elements V_{ij} equal to one or zero according to whether (ij) are nearest neighbors or not. Using a Fourier series to compute the trace over $(\beta V)^p$ and noting that this quantity vanishes for $p=1$, we arrive at

$$\mathcal{G}(M) = \beta d(\vec{M}^{(1)})^2 - \tfrac{1}{2}\mathrm{tr}\int_0^{2\pi}\frac{dq_1}{2\pi}\cdots \times\int_0^{2\pi}\frac{dq_d}{2\pi}\ln\left(1-2\beta\underline{M}^{(2)}\sum_{\alpha=1}^{d}\cos q_\alpha\right),$$

which can conveniently be recast in the form

$$\mathcal{G}(M) = \beta d(\vec{M}^{(1)})^2 + \tfrac{1}{2}\mathrm{tr}\left\{\int_0^{\infty}\frac{ds}{s}e^{-s}[I_0(2\beta_s\underline{M}^{(2)})^d - 1]\right\}. \tag{3.12}$$

We now have to maximize $\mathcal{G}+\mathcal{L}-\vec{G}^{(1)}\cdot\vec{M}^{(1)} - \mathrm{tr}\underline{G}^{(2)}\underline{M}^{(2)}$ with respect to M and G to obtain the free energy. This yields, for $\beta<\beta_c$,

$$\frac{2F}{n} = \frac{2\beta\nu}{n} - 1 - \ln\frac{2\beta\nu}{n} + \int_0^{\infty}\frac{ds}{s}e^{-s\nu}[I_0(s)^d - 1], \tag{3.13}$$

with ν defined implicitly in terms of β by

$$\frac{2\beta}{n} = \int_0^{\infty} ds\, e^{-s\nu}I_0(s)^d\,. \tag{3.14}$$

This equation expresses the stationarity of (3.13) with respect to ν.

The critical β_c, marking the onset of spontaneous magnetization, is obtained either from the variational equations giving $\vec{M}^{(1)}$ or by setting ν to its lowest possible value equal to d in (3.14). This gives

$$\beta_c^* = \frac{2\beta_c d}{n} = d\int_0^{\infty} ds[e^{-s}I_0(s)]^d\,. \tag{3.15}$$

Figure 4 shows the behavior of β_c^*, which blows up for $d=2$. An expansion in powers of $1/d$ is obtained by using the Taylor series of the Bessel function, and this agrees with the series (3.4), where one omits the terms in $1/(n+2)$. The values of β_c^* for $d=3,4,5$ are indicated between brackets in Table II.

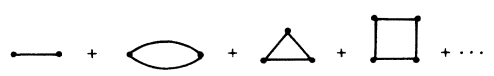


FIG. 3. The set of leading strongly irreducible diagrams for the calculation of the $n\to\infty$ limit.

IV. YANG-MILLS MODEL

We shall first discuss some technical points for obtaining the high-temperature expansion for the pure Yang-Mills field in the simple case of a discrete Z_2 gauge group. We further extend the analysis to a continuous group, Abelian [U(1)] or non-Abelian [SU(2)]. Finally, we present some numerical results.

A. Diagrammatic rules

We consider the action as a sum over all plaquettes

$$S = \sum_p A_{12}A_{23}A_{34}A_{41}\,, \tag{4.1}$$

where the fields $A_{ij}=\pm 1$ and sources h are attached to each link of the hypercubical lattice. The unperturbed partition function relative to a link is then

$$z(h) \equiv e^{u(h)} = \tfrac{1}{2}\sum_{A=\pm 1} e^{Ah} = \cosh h\,. \tag{4.2}$$

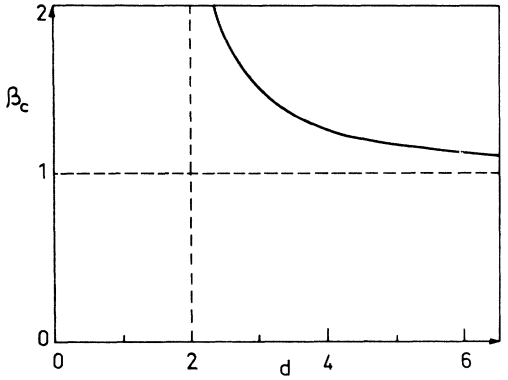


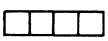
FIG. 4. The critical coupling β_c as a function of d in the $n\to\infty$ limit.

The expansion of Z in powers of β is performed as in Sec. II A. Graphs are drawn as *two-dimensional surfaces built of plaquettes* which now represent the four-vertices. When the graph is considered independently from the lattice by preserving the relation between links and plaquettes, it is called a diagram, as previously. The problem now is to analyze these diagrams and to compute the associate r.n.c.

We shall disregard the simplification introduced in the present case by the remark that a series in powers of $\tanh\beta$ obeys simpler rules. Hence we study the plain β expansion (2.5) of Z. To do this, we first introduce *skeleton diagrams*, which never contain the same plaquette of the lattice twice. Given a skeleton, one can reconstruct all associated diagrams by dressing each of its plaquettes once, twice, three times, ...

In Table III, we present all the connected skeletons up to four plaquettes. Computations in zero

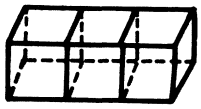
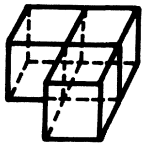
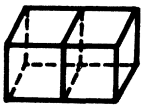
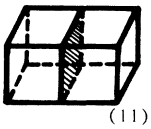
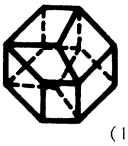
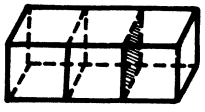
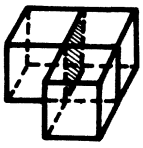
TABLE III. The number of configurations for all connected skeletons up to fourth order.

1		$\frac{1}{2}\,d(d-1)$			
2		$d(d-1)(2d-3)$	4		$8d(d-1)(d-2)(2d^2-8d+9)$
3		$d(d-1)(2d-3)^2$	4		$\frac{1}{6}\,d(d-1)(d-2)(2d-3)(2d-5)$
3		$2d(d-1)(4d^2-14d+13)$	4		$2d(d-1)(d-2)(2d-3)^2$
3		$\frac{2}{3}\,d(d-1)(d-2)(2d-3)$	4		$4d(d-1)(d-2)(4d^2-16d+17)$
3		$\frac{4}{3}\,d(d-1)(d-2)$	4		$\frac{1}{2}\,d(d-1)(d-2)$
4		$\frac{1}{2}\,d(d-1)(16d^3-72d^2+107d-52)$	4		$16d(d-1)(d-2)^2$
4		$4d(d-1)(2d-3)(4d^2-14d+13)$	4		$8d(d-1)(d-2)^2$
4		$2d(d-1)(8d^3-44d^2+84d-55)$	4		$\frac{1}{2}\,d(d-1)(4d^2-16d+17)$
4		$2d(d-1)(8d^3-44d^2+84d-55)$	4		$2d(d-1)(d-2)$

external field require the consideration of closed skeletons (because each link must occur an even number of times). Connected ones are classified up to sixteen plaquettes in Table IV.

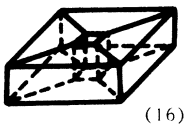
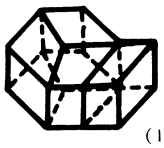
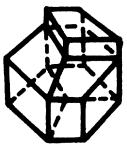
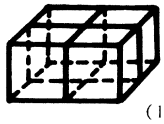
On these same two tables appear the r.n.c. The latter is identical for a skeleton and for its dressed counterparts. We were unable to find a general rule to write the r.n.c. for the connected, strongly irreducible skeletons, and had to calculate them case by case. However, once these were obtained, the r.n.c. of an arbitrary diagram can readily be computed. Let us illustrate the reasoning on some

TABLE IV. The number of configurations for strongly irreducible skeletons contributing to zero-external-field expansions up to sixteenth order.

 (1)	$\frac{1}{2}\,d(d-1)$	 (14)	$\frac{1}{2}\,d(d-1)(d-2)(4d^2-20d+25)$
 (6)	$\frac{1}{6}\,d(d-1)(d-2)$	 (14)	$2d(d-1)(d-2)(4d^2-20d+25)$
 (10)	$\frac{1}{6}\,d(d-1)(d-2)(4d-9)$	 (14)	$d(d-1)(d-2)(d-3)$
 (11)	$\frac{1}{6}\,d(d-1)(d-2)(4d-9)$	 (14)	$4d(d-1)(d-2)(d-3)$
 (12)	$\frac{4}{3}\,d(d-1)(d-2)(d-3)$	 (15)	$d(d-1)(d-2)(4d^2-20d+25)$
 (12)	$\frac{1}{3}\,d(d-1)(d-2)(d-3)$	 (15)	$4d(d-1)(d-2)(4d^2-22d+31)$

(Continued on following page)

Table IV (*continued*)

 (16)	$8d(d-1)(d-2)(d-3)$	 (16)	$16d(d-1)(d-2)(d-3)^2$
 (16)	$16d(d-1)(d-2)(d-3)^2$	 (16)	$\frac{1}{2}d(d-1)(d-2)(4d^2-24d+37)$
 (16)	$8d(d-1)(d-2)(d-3)^2$		

simple examples.

(a) Single plaquette. There are $N[d(d-1)/2]$ plaquettes on the lattice.

(b) Two plaquettes sharing one link. Choosing the link leads to Nd possibilities. The first plaquette can then be set in $2(d-1)$ directions, the second one in $2(d-1)-1$. Dividing by a factor 2, because of the indiscernability between the two plaquettes, finally leads to $Nd(d-1)(2d-3)$.

(c) Three plaquettes at the corner of a cube. There are $N[d(d-1)(d-2)/3!]$ cubes on the lattice and 8 corners on each cube. Hence the r.n.c. is $\frac{4}{3}d(d-1)(d-2)$.

It is easy to compute the r.n.c. of a disconnected skeleton S. We start from the set theoretic formula[4]

$$\{S_1\}\circ\{S_2\}=\sum_{S=S_1\cup S_2} n_s\{S\}, \tag{4.3}$$

where $\{S\}$ denotes the n.c. of a diagram S, and where the summation runs over all skeletons which can be decomposed as $S_1\cup S_2$ in n_S different ways. The r.n.c. $[S]$, i.e., the coefficient of N in $\{S\}$, then satisfies

$$\sum_{S=S_1\cup S_2} n_s[S]=0\,. \tag{4.4}$$

Equation (4.4) allows a recursive calculation of $[S]$ if we isolate on its left-hand side the term corresponding to the maximal number of disconnected parts.

This method can be refined to compute the r.n.c. for connected, but not strongly irreducible skeletons. Let us distinguish a link of a given connected skeleton S, and denote by k the symmetry number pertaining to this link (meaning that there are k links on the skeleton playing the same topological

role). Every link of the lattice will be occupied by the distinguished one of the skeleton an equal number of times (up to a choice among the k identical ones). This number of mappings is therefore, again up to the symmetry of order k,

$$(Nd)^{-1}k\{S\} = \frac{k[S]}{d}.$$

If S can be decomposed into $S_1 \cup S_2$ (both sharing the distinguished link) in n_s ways, formula (4.3) becomes

$$\frac{k_1[S_1]}{d}\frac{k_2[S_2]}{d} = \sum_{S=S_1\cup S_2} n_s \frac{k[S]}{d}. \tag{4.5}$$

Let us consider, for instance, the example (b) above. The distinguished link for the considered skeleton, made of two adjacent squares, is the central one ($k=1$), while it is any one for the single square ($k=4$). Equation (4.5) gives the diagrammatic equation shown in Fig. 5, where the factor $n_s=2$ in front of the first term of the right-hand side represents the two ways in which the skeleton can be obtained as a union of two distinct plaquettes. We thus recover the r.n.c. $d(d-1)(2d-3)$ of the reducible diagram of example (b).

B. Continuous gauge groups

We discuss in turn the extra factors contributed by the gauge groups U(1) and SU(2) (see paper I).

1. *Abelian group* U(1)

The action is now

$$S = \sum_p \cos(\psi_{12}+\psi_{23}+\psi_{34}+\psi_{41})$$

$$= \sum_p{}' \zeta_{12}\zeta_{23}\zeta_{34}\zeta_{41}, \tag{4.6}$$

where $\sum_p'$ means a sum over *oriented* plaquettes, $\zeta_{ij}=\zeta_{ji}^*=e^{i\psi_{ij}}$. The sum over configurations involves integration over every angle ψ in an interval of 2π with a measure $(2\pi)^{-1}d\psi$. The source term is written

$$\sum_l{}' h_{ij}\zeta_{ji},$$

where again the sum runs over oriented links and $h_{ij}=h_{ji}^*$.

In the diagrammatic expansion, a vertex is now associated with an oriented plaquette, field variables being associated with the oriented links. The contribution of any link described n_1 times in the direction $i\to j$ and n_2 times in the reverse direction $j\to i$ is

$$\frac{\partial^{n_1+n_2}}{\partial h_{ij}^{*n_1}\partial h_{ij}^{n_2}} z(h_{ij}), \tag{4.7}$$

where

$$z(h) = \int \frac{d\zeta}{2i\pi\zeta}\exp(h^*\zeta + h\zeta^*)$$

$$= I_0(2|h|). \tag{4.8}$$

The general procedure of Sec. II then applies in a straightforward way, diagrams being now built with oriented plaquettes.

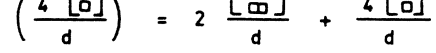


FIG. 5. The diagrammatic equivalent of Eq. (4.5) for two plaquettes sharing one link.

2. *Non-Abelian group* SU(2)

On the example of SU(2) as a gauge group, we illustrate the new technical points arising from the noncommutativity of the gauge field. The action is

$$S = \sum_p \chi(A_{12}A_{23}A_{34}A_{41}). \tag{4.9}$$

The matrices $A_{ij}=A^{-1}{}_{ji}$ belong to SU(2) and can be parametrized with an angle ϕ ($0\le\phi\le 2\pi$) and a unit three-dimensional vector $\vec{n}$ as

$$A = \cos\phi + i\sin\phi\,\vec{n}\cdot\vec{\sigma}, \tag{4.10}$$

where $\vec{\sigma}$ stands for the Pauli matrices. The normalized measure on SU(2) is

$$dA = \frac{\sin^2\phi\, d\phi\, d^2n}{2\pi^2}.$$

We choose for χ the character associated with the spin-$\frac{1}{2}$ representation:

$$\chi(A) = \mathrm{tr}A = 2\cos\phi. \tag{4.11}$$

If we were to display the matrix elements of A and expand $\chi(AAAA)$ in terms of those, we could blindly use the formalism for multicomponent fields described previously. It is, however, wiser to try to take as much advantage as possible of the group-theoretic framework. For instance, in the case of a zero external field, we found it convenient to compute the contribution of a given diagram by using recursive formulas such as

$[A_1, A_2 \in \mathrm{SU}(2)]$

$$\rho_{pq}(A_1, A_2) \equiv \int dA [\chi(A_1 A)]^p [\chi(A^\dagger A_2)]^q = \sum_r C_r^{pq} [\chi(A_1 A_2)]^r . \quad (4.12)$$

A generating function for ρ_{pq} is

$$R(A_1, A_2) = \sum_{p,q} \rho_{pq}(A_1, A_2) \frac{u^p}{p!} \frac{v^q}{q!} = \int dA \exp[u \operatorname{tr}(A_1 A) + v \operatorname{tr}(A^\dagger A_2)] = \frac{I_1(2x)}{x}, \quad x = [u^2 + v^2 + uv \operatorname{tr}(A_1 A_2)]^{1/2} \quad (4.13)$$

from which we derive the coefficients C_r^{pq}. The result is

$$C_r^{pq} = \frac{p!q!}{[1 + \frac{1}{2}(p+q)]![\frac{1}{2}(p-r)]![\frac{1}{2}(q-r)]!r!} \quad (4.14)$$

or $C_r^{pq} = 0$ if any of the arguments of the factorials is not a non-negative integer. (This implies in particular that $p = q = r \bmod 2$.)

Let us illustrate the use of formulas (4.12) and (4.14) by exhibiting the contribution to the free energy of a diagram consisting of a cube. It reads

$$\beta^6 \tfrac{1}{6}[d(d-1)(d-2)] \int \prod_1^{12} dA \prod_1^6 \chi(AAAA) .$$

We have used a short-hand notation to indicate the integration over the twelve A_{ij} attached to the links and the six factors contributed by the plaquettes. This integral is calculated by use of the previous formula and yields

$$\beta^6 \tfrac{1}{6}[d(d-1)(d-2)] \sum_{pqrst} C_p^{11} C_q^{p1} C_r^{q1} C_s^{r1} C_t^{s1} (\mathrm{tr}\underline{1})^t = \beta^6 \tfrac{1}{6}[d(d-1)(d-2)] \tfrac{1}{16} .$$

To give a slightly more tricky case, let us look at the eighth-order contribution arising from a cube dressed with two extra plaquettes on one of its six faces (observe that there is no such term of order seven, since each link variable must appear an even number of times, and this also explains why two extra plaquettes have to be added on the same face):

$$6\beta^8 \frac{d(d-1)(d-2)}{6} \int \prod_1^{12} dA \frac{[\chi(AAAA)]^3}{3!} \times \prod_1^5 \chi(AAAA) .$$

The extra factor 6 arises from the choice of the distinguished plaquette repeated three times, and the denominator 3! is the symmetry number associated with permutations of these three plaquettes. The calculation of the integral proceeds as before and gives

$$\sum_{pqrst} C_p^{31} C_q^{p1} C_r^{q1} C_s^{r1} C_t^{s\,1} (\mathrm{tr}\underline{1})^t = \tfrac{1}{8} .$$

The generalization to a cube dressed in an arbitrary way and to various other diagrams is obvious.

In order to be able to perform computations in a nonzero external field as was indicated in Sec. II, for instance to obtain the critical value of the coupling constant, it is necessary to introduce in the action a source term of the form

$$\sum_i \chi(A_{ij} h_{ji}) .$$

A source should be an element of a real vector space; on the other hand, it is obvious from the above form of the source term that h_{ij} should be (like the average value of A_{ij}) a linear combination of elements of SU(2). Thus, each h_{ij} is a 2×2 matrix of the form

$$h = u_0 + i\vec{u} \cdot \vec{\sigma} , \quad u_0, \vec{u} \text{ real} . \quad (4.15)$$

Moreover, like the field A itself, the source satisfies $h_{ij} = h_{ji}^\dagger$. If A_{ij} and h_{ij} are parametrized according to (4.10) and (4.15), the source term is

$$\chi(A_{ij} h_{ji}) = 2(u_0 \cos\phi + \sin\phi\, \vec{n} \cdot \vec{u}) ,$$

i.e., twice the associated scalar product. In particular, from (4.13), we get

$$z(h) = \int dA \exp[\chi(Ah^\dagger)] = \frac{I_1(2\eta)}{\eta}, \quad (4.16)$$

where

$$\eta^2 = u_0^2 + \vec{u}^2 = \det h .$$

C. Numerical results

A straightforward application of the high-temperature expansion provides us with a formula for the *free energy* without external field. It turned out that the most economical method was also the least sophisticated one of Sec. II A. We present the results up to sixteenth order for the three groups discussed above.

For Z_2,

$$F = d(d-1)\left[\tfrac{1}{4}\beta^2 - \tfrac{1}{24}\beta^4 + (\tfrac{1}{6}d - \tfrac{29}{90})\beta^6 + (-\tfrac{1}{3}d + \tfrac{3343}{5040})\beta^8 + (d^2 - \tfrac{184}{45}d + \tfrac{118\,471}{28\,350})\beta^{10} + (-\tfrac{8}{3}d^2 + \tfrac{121\,153}{11\,340}d - \tfrac{20\,022\,781}{1871\,100})\beta^{12}\right.$$
$$\left. + (10d^3 - \tfrac{208}{3}d^2 + \tfrac{935\,561}{5670}d - \tfrac{5\,647\,451\,354}{42\,567\,525})\beta^{14} + (-\tfrac{74}{3}d^3 + \tfrac{120\,761}{840}d^2 - \tfrac{345\,869\,921}{1\,247\,400}d + \tfrac{3\,612\,986\,481\,191}{20\,432\,412\,000})\beta^{16} + O(\beta^{18})\right]. \tag{4.17}$$

We might also have written in this case an expansion in powers of $\tanh\beta$ involving fewer terms. Namely, the expression

$$Z = 2^{-Nd}\sum_{\{A_{ij}=\pm 1\}}\exp\left(\beta\sum_{p}AAAA\right)$$
$$= 2^{-Nd}(\cosh\beta)^{Nd(d-1)/2}\sum_{\{A_{ij}\}}\left[\prod_{p}(1 + AAAA\tanh\beta)\right]$$

is expanded as a sum over closed skeleton diagrams only, each plaquette carrying now a factor $\tanh\beta$. This remark provides a check on (4.17).

For the continuous gauge groups considered above, we find for U(1),

$$F = d(d-1)\left[\tfrac{1}{2}(\tfrac{1}{2}\beta)^2 - \tfrac{1}{8}(\tfrac{1}{2}\beta)^4 + (\tfrac{1}{3}d - \tfrac{11}{18})(\tfrac{1}{2}\beta)^6 + (-d + \tfrac{757}{384})(\tfrac{1}{2}\beta)^8\right.$$
$$+ (2d^2 - \tfrac{85}{12}d + \tfrac{2473}{400})(\tfrac{1}{2}\beta)^{10} + (-\tfrac{26}{3}d^2 + \tfrac{6569}{192}d - \tfrac{1\,750\,513}{51\,840})(\tfrac{1}{2}\beta)^{12}$$
$$\left. + (20d^3 - \tfrac{749}{6}d^2 + \tfrac{197\,803}{720}d - \tfrac{44\,476\,939}{211\,680})(\tfrac{1}{2}\beta)^{14} + (-96d^3 + \tfrac{4777}{8}d^2 - \tfrac{43\,844\,513}{34\,560}d + \tfrac{9\,463\,083\,949}{10\,321\,920})(\tfrac{1}{2}\beta)^{16} + O(\beta^{18})\right], \tag{4.18}$$

and for SU(2),

$$F = d(d-1)\left[\tfrac{1}{4}\beta^2 - \tfrac{1}{48}\beta^4 + (\tfrac{1}{96}d - \tfrac{7}{576})\beta^6 + (-\tfrac{1}{96}d + \tfrac{7}{384})\beta^8 + (\tfrac{1}{256}d^2 - \tfrac{49}{4608}d + \tfrac{10\,481}{1\,612\,800})\beta^{10}\right.$$
$$+ (-\tfrac{3}{512}d^2 + \tfrac{85\,189}{4\,976\,640}d - \tfrac{965\,807}{87\,091\,200})\beta^{12} + (\tfrac{5}{2048}d^3 - \tfrac{155}{12\,288}d^2 + \tfrac{3229}{122\,880}d - \tfrac{288\,747\,853}{13\,412\,044\,800})\beta^{14}$$
$$\left. + (-\tfrac{107}{24\,576}d^3 + \tfrac{25\,657}{1\,179\,648}d^2 - \tfrac{15\,024\,019}{371\,589\,120}d + \tfrac{1\,199\,262\,152\,197}{41\,845\,579\,776\,000})\beta^{16} + O(\beta^{18})\right]. \tag{4.19}$$

The corresponding curves are drawn in dimension four in Fig. 6.

Let us now investigate the *critical couplings*. Variational expansions generalizing (3.3) may be written for each group, and will yield to lowest order in β the result of mean field theory, namely, the existence of a first-order transition for $\beta_c = O(1/d)$. However, higher-order terms do not provide a natural expansion of β_c in powers of $1/d$, as was the case for the scalar model. Indeed, the maximum dimensionality occuring in pth-order terms of the expansion (2.12) is now p. Since β_c is of order $1/d$, this yields a contribution to Φ, and hence to $\beta_c d^2$, of order 1. Successive terms of the perturbation expansion therefore all contribute to the corrections of order d^{-2} to the critical coupling as given by mean field theory.

Furthermore, due to gauge invariance, setting all m_i to an equal value m is, in fact, dangerous. This was discussed at length in paper I, where we proposed to integrate first over a subset of field variables in order to break formal gauge invariance. A possible strategy was to eliminate all "vertical" variables, resulting in an action

$$S = \sum_{p_\perp}\chi(A_{12}A_{23}A_{34}A_{41}) + \sum_{p_\parallel}\chi(A_{12}A_{34}). \tag{4.20}$$

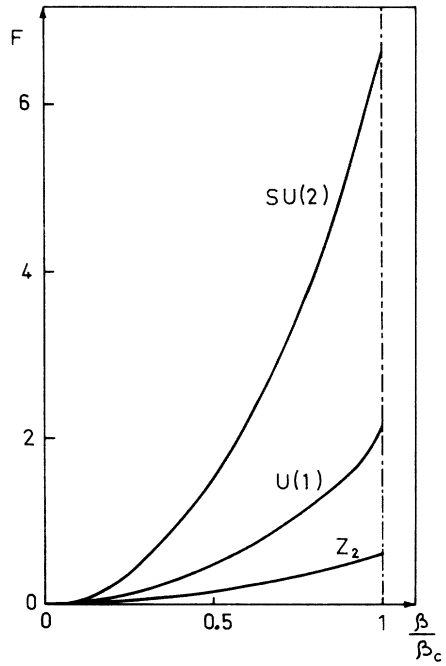


FIG. 6. The free energy of the gauge models for the three groups Z_2, U(1), and SU(2), plotted against β/β_c for $d=4$. We have used for β_c the best values of Table V.

Here $p_\perp$ denotes the plaquettes perpendicular to the "vertical," timelike direction, $p_\parallel$ denotes the parallel ones, and the variables A_{ij} only refer to perpendicular, spacelike links. This was called the action in the *Coulomb gauge*. Of course some invariance still remains, but it is now a surface effect rather than a volume effect. With this choice, we may set all remaining spacelike m_i variables equal to m in the variational procedure.

We have now all the necessary tools to perform the variational calculation of β_c along the lines of Sec. II B. We have pushed the evaluation of the B function occurring in (2.12) up to third order in β in the case of the Z_2 gauge group, and to second order in the cases of U(1) and SU(2). This was only designed to demonstrate the feasibility of these calculations, which could of course be pursued. The analytical expressions soon become rather cumbersome. As an example, let us display the simplest of them, relative to the Z_2 group. We express the variational function $\Phi(h,m)$ and the irreducible kernel $B(h)$ occurring in (2.13) and (2.12):

$$\varphi(h,m) = \frac{1}{N(d-1)}\Phi(h,m)$$

$$= b(h + 2\beta(d-2)m^3 + 2\beta m) - \tfrac{3}{2}(d-2)\beta m^4 - \beta m^2 , \qquad (4.21)$$

with $b(h)$ given, in terms of $u = u(h) = \ln\cosh h$ and its derivatives $u_i = d^i u(h)/dh^i$, by

$$b(h) = \frac{1}{N(d-1)}B(h)$$

$$= u + \tfrac{1}{2}\beta^2\left[\tfrac{1}{2}(d-2)(u_2^4 + 4u_2^3u_1^2 + 6u_2^2u_1^4) + u_2^2\right] + \beta^3\left[\tfrac{1}{2}(d-2)\left(\tfrac{1}{6}u_3^4 + 2u_1u_2u_3^3 + \tfrac{2}{3}u_1^3u_3^3 + 9u_1^2u_2^2u_3^2 + 6u_1^4u_2u_3^2 + u_1^6u_3^2 + 12u_1^5u_2^2u_3 + 16u_1^3u_2^3u_3 + 9u_1^4u_2^4 + 4u_1^6u_2^3\right) + \tfrac{1}{6}u_3^2 + \tfrac{4}{3}(d-2)(d-3)u_1^6u_2^3\right] + O(\beta^4) . \qquad (4.22)$$

We recall from paper I that, to lowest order, we find a first-order transition in F when varying m. Thus we use Eqs. (2.18), setting $h = 0$ in (4.21) and in its analogs for the other groups, and looking for the stationary value of the approximate φ, up to various orders of the expansion (4.22). The results are displayed in Tables V and VI.

For the gauge group Z_2, the values obtained for $\beta_c^* = 2\beta_c d$ in dimensions $d = 3$ and $d = 4$ may be compared with the exact results predicted in paper II. The agreement seems to be excellent in dimension 4, and seems to indicate that the transition is indeed a first-order one. However, for $d = 3$, where one knows that the Yang-Mills field has a second-order transition (while the present calculation is performed in the framework of a first-order one), some discrepancy occurs. It was indeed expected that $d = 3$ would appear as a limiting dimensionality. This has also been checked by treating with the present formalism the case $d = 2$, known to be equivalent to a one-dimensional Ising model. Whereas the zeroth order (mean field) predicts a spurious (second-order) transition, the second-order terms are sufficient to rule out this transition. The perturbation variation treatment we are using thus does not seem to suffer from the

TABLE V. Critical couplings $\beta_c^* = 2\beta_c d$ for gauge models.

Group	Approx.	$d=3$	$d=4$	$d=5$	$d=6$	$d=\infty$
Z_2	0	2.6028	2.6840	2.7104	2.7229	2.7552
	2	4.2970	3.5272	3.2905	3.1677	
	3	4.2941	3.5257	3.2897	3.1672	
	exact	4.5678	3.5254	?	?	
U(1)	0	5.6563	6.2133	6.4793	6.6384	7.2934
	2		11.3761	9.0530	8.4929	
	Padé	10.0934	8.9781	8.4968	8.2298	
SU(2)	0	5.8708	6.7270	7.1520	7.4096	8.4787
	2			12.821	10.5948	
	Padé		11.4160	10.333	9.8510	

TABLE VI. The discontinuity m_c of m at the critical point for gauge models.

Group	Approx.	$d=3$	$d=4$	$d=5$	$d=6$	$d=\infty$
Z_2	0	0.8643	0.9416	0.9616	0.9704	
	2	0.9921	0.9868	0.9865	0.9869	0.990 61
	3	0.9913	0.9858	0.9859	0.9865	
U(1)	0	0.6558	0.7913	0.8296	0.8479	
	2		0.8949	0.8793	0.8707	0.9004
	Padé	0.7829	0.8272	0.8469	0.8582	
SU(2)	0	0.4983	0.7042	0.7619	0.7892	
	2			0.8683	0.8400	0.8682
	Padé		0.7726	0.7938	0.8081	

defect of mean-field theory, which may predict spurious transitions.

Table V also exhibits the convergence of the expansion. For large d, β_c^* should be close to the mean field value (exact for d infinite), and convergence is expected to be rapid. Values obtained for β_c^*, either by expanding b up to second order in β [as in Eq. (4.22)] or by using the Padé approximant for b to the same order, are indeed close to each other. Furthermore, inclusion of third-order terms does not seem to improve the second-order results significantly. Such a fact already appeared for the scalar field model of Sec. III; a possible explanation is the occurrence of new irreducible skeletons at even orders only.

The discontinuity of the parameter m, which jumps from 0 to m_c at β_c is given in Table VI. Since m is the average value of the field A which varies on a unit sphere, it is bounded by 1. It is quite striking that m_c is close to this maximum. Consider for instance the case of the group Z_2 in the limit $d\to\infty$. Using the expression (4.21), in the equations (2.18), one is led to

$$\begin{aligned} &\ln\cosh(\beta_c^* m_c^3) - \tfrac{3}{4}\beta_c^* m_c^4 = 0\,, \\ &\tanh(\beta_c^* m_c^3) - m_c = 0\,, \end{aligned} \tag{4.23}$$

which yield $m_c = 0.9906$, $\beta_c^* = 2.755$.

In the case of U(1), the transition disappears for dimension 3 when we include second-order corrections. This is similar to the disappearance of the transition for the group Z_2 in 2 dimensions. It may be that the transition, which is a first-order one for high dimensionalities, becomes a second-order transition around $d=4$, then disappears. It is also possible that, for a continuous group, the low-lying excitations destroy the order at low dimension. A subtle kind of phase transition might remain at the limiting dimension. This phenomenon is known[8] to occur for the scalar models of Sec. III: whereas the two-dimensional Ising model ($d=2$, $n=1$) exhibits a usual second-order transition (like the Yang-Mills model for $d=3$, $n=1$), a transition without ordering takes place for $d=2$, $n\geq 2$.

For gauge fields, the limiting dimension is not unambiguously ascertained. In particular, we see in Table V that for SU(2), the predictions of the plain expansion and the Padé expansion differ for $d=4$. Note finally that in this case m is replaced by a 2×2 matrix, equal to the average value $\langle A_{ij}\rangle$ of spacelike field variables. Whereas A_{ij} is unitary, its average belongs to the algebra of SU(2), and has the form $u_0 + i\vec{u}\cdot\vec{\sigma}$. Hence $(u_0 + i\vec{u}\cdot\vec{\sigma})(u_0 + i\vec{u}\cdot\vec{\sigma})^\dagger = m^2$ is a c number bounded by 1. When looking for the stationary value of F, one finds $\vec{u}=0$. The corresponding numbers for m_c are listed in Table VI.

V. CONCLUSION

A diagrammatic approach, combining perturbative and variational techniques, allows one to compute various quantities of interest, such as critical couplings. A good numerical convergence seems to be achieved even in low dimension, and a check is provided by some known results. In particular, for the case of a Z_2 gauge group for $d=4$, we have shown in paper II that a transition occurs at $\beta_c^* = 3.5254$, to be compared with the value 3.5257 obtained with the present techniques developed to third order (Table V).

The method is based on a self-consistent determination of m, the average value of the field for a vanishing source. For the scalar model of Sec. III, with its second-order transition, m is a natural physical order parameter. It is somehow surprising to obtain quite satisfactory results for gauge fields, using the same techniques. Indeed, the analysis given in paper II shows that m has not the usual interpretation of an order parameter, and that no local order parameter exists in the Yang-Mills models. It was therefore important to test the validity of the method.

We have taken care to perform the calculations in the Coulomb gauge, suppressing the timelike field variables A_{ij}, and taking m as the (common) average value of the spacelike ones. This procedure was essential. We have tried to perform the same calculations in a straightforward fashion, without breaking formal gauge invariance, m being then the average of all variables A_{ij}. Although both approaches agree qualitatively to order zero, higher-order terms yield nonsensical results when no gauge condition is imposed. On the contrary, Tables V and VI show the success of second-order calculations in the Coulomb gauge, even in low dimensions.

The nature of the transition remains to some extent an open question. Since terms beyond second order yield $1/d$ corrections, the mean field approximation becomes very likely exact for $d \to \infty$. The transition of the Yang-Mills model is therefore a first-order one in high enough dimension, the function F having a discontinuous derivative. Not only is the transition a first-order one, but m jumps from 0 to a value close to its maximum, equal to 1. Ordering thus appears bluntly if we characterize it by m. It may, however, be that the order parameter associated with the average $\langle \prod A_{ij} \rangle$ of the product of fields along a large loop has a smoother behavior.

Finally, as seen from the variations of β_c^* and m_c exhibited in Tables II, V, and VI, the transition sets in with more and more difficulty when the dimension decreases (and also when the number of components of the field increases). A limiting dimension exists, below which no ordering takes place. For this limiting dimension, a phase transition may still exist, possibly of second order (as for the group Z_2 in 3 dimensions), with a critical behavior.

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