Static cylinder of perfect fluid with nonzero spin density

A. R. Prasanna*
Institut für Theoretische Kernphysik der Universität Bonn, Nussallee-16, D 5300 Bonn, Federal Republic of Germany
(Received 13 August 1974)

In this paper we have studied the interior field of an infinite static cylinder of a perfect fluid having the spins of the individual particles aligned along the axis of symmetry. We have obtained a solution with an assumed equation of state and expressed the pressure and the density as departures from their values on the symmetry axis. As in the case of spherical symmetry, the pressure is discontinuous across the boundary of the cylinder.

I. INTRODUCTION

The discovery of pulsars gave a big impetus to the developments in relativistic astrophysics, particularly in the study of neutron stars. Since the rotating-neutron-star model for a pulsar seems to be almost undisputed[1] it is now essential to study the possible internal structure of neutron stars with respect to gravitation and the associated geometry of space-time therein. Observations indicate a very high magnetic field associated with them in comparison with any other compact objects in the universe. It is not unlikely that this magnetic field might induce a spin polarization of the nucleons composing the fluid of a neutron star.[2] If the spins are aligned then it is probable that there would be a substantial nonzero spin density which would then play, along with the mass density, a dynamical role in influencing the geometry of space-time containing the fluid. In general relativity as given by Einstein there is no way of considering the spin effects on the geometry of space-time. On the other hand, it is clear that one could study such configurations in the framework of the Einstein-Cartan theory. In fact as an attempt to investigate whether the Einstein-Cartan theory admits self-gravitating fluid systems Prasanna had recently considered a study of static fluid spheres in its framework.[3] However, since we had assumed therein a radial alignment of spins (implying the presence of a magnetic monopole at the center) the picture is not very physical. Further, since a rotating system cannot be spherical it is necessary to consider axisymmetric distributions which are more physical. In this connection we now propose to study the simplest axisymmetric system, namely a static cylinder of perfect fluid composed of particles having their spins aligned along the symmetry axis.

We follow the notation of our earlier paper[3] (now onwards referred to as paper I) wherein we have given in Sec. II the necessary details regarding Einstein-Cartan equations. While referring to the equations from paper I, we use the notation (I2) appropriately. In the following we label the coordinates r, φ, z, and t by 1, 2, 3, and 4.

II. METRIC AND CURVATURE

We start from the static cylindrically symmetric metric

$$ds^2 = -e^{2\mu-2\nu}(dr^2+dz^2) - r^2 e^{-2\nu} d\varphi^2 + e^{2\nu}dt^2 , \tag{2.1}$$

wherein μ and ν are functions of r alone, so that we have the orthonormal tetrad

$$\Theta^1 = e^{\mu-\nu}dr\ ,\ \Theta^2 = re^{-\nu}d\varphi,\ \Theta^3 = e^{\mu-\nu}dz\ ,\ \Theta^4 = e^{\nu}dt\ , \tag{2.2}$$

along with the metric tensor

$$g_{ij} = \mathrm{diag}\{-1,-1,-1,\ 1\}\ .$$

Since we have assumed that the spins of the individual particles are aligned along the symmetry axis (z axis) we will have for the spin tensor S_{ij} the only nonzero components

$$S_{12} = -S_{21} = K\ (\text{say}). \tag{2.3}$$

As in the previous case, since the fluid distribution is static, the velocity vector $u^i = \delta^i{}_4$ and hence the nonzero components of $s^i{}_{jk}$ are

$$s^4{}_{12} = -s^4{}_{21} = K\ . \tag{2.4}$$

Consequently, from Cartan equations (I2.12) we get the torsion tensor $Q^i{}_{jk}$ to be

$$Q^4{}_{12} = -Q^4{}_{21} = -\kappa K\ , \tag{2.5}$$

where the other components are zero. The torsion two-form Θ^i is therefore given by

$$\Theta^1 = 0,\ \Theta^2 = 0,\ \Theta^3 = 0,\ \Theta^4 = -\kappa K(\Theta^1 \wedge \Theta^2). \tag{2.6}$$

One can now easily compute the connection one-form $\omega^i{}_j$ and the curvature two-form $\Omega^i{}_j$, which are found to have the components

$$\omega^1{}_2 = -\omega^2{}_1 = -\frac{e^{\nu-\mu}}{r}(1-r\nu')\Theta^2 + \tfrac{1}{2}\kappa K\Theta^4,\ \ \omega^1{}_4 = \omega^4{}_1 = e^{\nu-\mu}\nu'\Theta^4 + \tfrac{1}{2}\kappa K\Theta^2,$$
$$\omega^1{}_3 = -\omega^3{}_1 = -e^{\nu-\mu}(\mu'-\nu')\Theta^3,\ \ \omega^2{}_4 = \omega^4{}_2 = -\tfrac{1}{2}\kappa K\Theta^1, \tag{2.7}$$
$$\omega^2{}_3 = -\omega^3{}_2 = 0,\ \ \omega^3{}_4 = \omega^4{}_3 = 0,$$

$$\Omega^1{}_2 = -\Omega^2{}_1 = \left[e^{2(\nu-\mu)}\left(\nu'' + \frac{\mu'}{r} - \mu'\nu' + \frac{\nu'}{r}\right) + \tfrac{1}{4}\kappa^2K^2\right](\Theta^1\wedge\Theta^2) + \tfrac{1}{2}\kappa[e^{\nu-\mu}(K'+2K\nu')](\Theta^1\wedge\Theta^4),$$
$$\Omega^1{}_3 = -\Omega^3{}_1 = [e^{2(\nu-\mu)}(\nu''-\mu'')](\Theta^1\wedge\Theta^3),$$
$$\Omega^2{}_3 = -\Omega^3{}_2 = \left[e^{2(\nu-\mu)}\left(\nu' - \frac{1}{r}\right)(\mu'-\nu')\right](\Theta^2\wedge\Theta^3) + [\tfrac{1}{2}\kappa Ke^{\nu-\mu}(\mu'-\nu')](\Theta^4\wedge\Theta^3), \tag{2.8}$$
$$\Omega^1{}_4 = \Omega^4{}_1 = [e^{2(\nu-\mu)}(\nu''+2\nu'^2-\nu'\mu') + \tfrac{1}{4}\kappa^2K^2](\Theta^1\wedge\Theta^4) + \tfrac{1}{2}\kappa e^{\nu-\mu}K'(\Theta^1\wedge\Theta^2),$$
$$\Omega^2{}_4 = \Omega^4{}_2 = \left[e^{2(\nu-\mu)}\nu'\left(\frac{1}{r} - \nu'\right) + \tfrac{1}{4}\kappa^2K^2\right](\Theta^2\wedge\Theta^4),$$
$$\Omega^3{}_4 = \Omega^4{}_3 = [e^{2(\nu-\mu)}\nu'(\nu'-\mu')](\Theta^4\wedge\Theta^3) - [\tfrac{1}{2}\kappa Ke^{\nu-\mu}(\mu'-\nu')](\Theta^2\wedge\Theta^3).$$

Using (2.8) in (I2.4) we can now read out the nonzero components of $R^i{}_{jkl}$ from which the Ricci tensor R_{ij} and the scalar of curvature R may be easily evaluated. We have thus

$$R^1{}_{212} = e^{2(\nu-\mu)}\left(\nu'' - \mu'\nu' + \frac{\mu'+\nu'}{r}\right) + \tfrac{1}{4}\kappa^2K^2,\ \ R^1{}_{214} = \tfrac{1}{2}\kappa e^{\nu-\mu}(K'+2K\nu'),$$
$$R^1{}_{313} = e^{2(\nu-\mu)}(\nu''-\mu''),\ \ R^2{}_{323} = e^{2(\nu-\mu)}\left(\nu' - \frac{1}{r}\right)(\mu'-\nu'),$$
$$R^2{}_{343} = \tfrac{1}{2}\kappa Ke^{\nu-\mu}(\mu'-\nu'),\ \ R^4{}_{112} = \tfrac{1}{2}\kappa e^{\nu-\mu}K', \tag{2.9}$$
$$R^4{}_{114} = e^{2(\nu-\mu)}(\nu''+2\nu'^2-\nu'\mu') + \tfrac{1}{4}\kappa^2K^2,\ \ R^4{}_{224} = e^{2(\nu-\mu)}\left(\frac{\nu'}{r} - \nu'^2\right) + \tfrac{1}{4}\kappa^2K^2,$$
$$R^4{}_{343} = e^{2(\nu-\mu)}\nu'(\nu'-\mu'),\ \ R^4{}_{323} = -\tfrac{1}{2}\kappa Ke^{\nu-\mu}(\mu'-\nu'),$$

$$R_{11} = e^{2(\nu-\mu)}\left(\nu'' - \mu'' - 2\nu'^2 + \frac{\nu'+\mu'}{r}\right),\ \ R_{22} = e^{2(\nu-\mu)}\left(\nu'' + \frac{\nu'}{r}\right),$$
$$R_{33} = e^{2(\nu-\mu)}\left(\nu'' - \mu'' + \frac{\nu'-\mu'}{r}\right),\ \ R_{24} = \tfrac{1}{2}\kappa e^{\nu-\mu}[K' + K(\nu'+\mu')], \tag{2.10}$$
$$R_{44} = e^{2(\nu-\mu)}\left(\nu'' + \frac{\nu'}{r}\right) + \tfrac{1}{2}\kappa^2K^2,\ \ R_{42} = \tfrac{1}{2}\kappa e^{\nu-\mu}[K' + K(\mu'-\nu')],$$

where the others are identically zero:

$$R = 2e^{2(\nu-\mu)}\left(\mu'' - \nu'' + \nu'^2 - \frac{\nu'}{r}\right) + \tfrac{1}{2}\kappa^2K^2. \tag{2.11}$$

Finally we have the Einstein tensor G_{ij} as given by

$$G_{11} = e^{2(\nu-\mu)}\left(\frac{\mu'}{r} - \nu'^2\right) + \tfrac{1}{4}\kappa^2K^2,\ \ G_{22} = e^{2(\nu-\mu)}(\mu''+\nu'^2) + \tfrac{1}{4}\kappa^2K^2,$$
$$G_{33} = e^{2(\nu-\mu)}\left(\nu'^2 - \frac{\mu'}{r}\right) + \tfrac{1}{4}\kappa^2K^2,\ \ G_{24} = \tfrac{1}{2}\kappa e^{\nu-\mu}[K' + K(\nu'+\mu')], \tag{2.12}$$
$$G_{44} = e^{2(\nu-\mu)}\left(2\nu'' - \mu'' + \frac{2\nu'}{r} - \nu'^2\right) + \tfrac{1}{4}\kappa^2K^2,\ \ G_{42} = \tfrac{1}{2}\kappa e^{\nu-\mu}[K' + K(\mu'-\nu')].$$

III. FIELD EQUATIONS

We assume the material distribution to be that of a perfect fluid with anisotropic pressure represented by the symmetric tensor $\overline{T}_i{}^j$,

$$\overline{T}_i{}^j = \mathrm{diag}\{-p_r, -p_\varphi, -p_z, \rho\} . \tag{3.1}$$

Considering this along with (2.4) in (I2.26) we get the nonzero components of the canonical tensor $t^j{}_i$ to be

$$\begin{aligned} &t^1{}_1 = \overline{T}_1{}^1 = -p_r , \quad t^4{}_4 = \overline{T}^4{}_4 = \rho , \\ &t^2{}_2 = \overline{T}_2{}^2 = -p_\varphi , \quad t^4{}_2 = \tfrac{1}{2}Ke^{\nu-\mu}\nu' , \\ &t^3{}_3 = \overline{T}_3{}^3 = -p_z , \quad t^2{}_4 = \tfrac{1}{2}Ke^{\nu-\mu}\nu' . \end{aligned} \tag{3.2}$$

If we use (2.12) and (3.2), the field equations (I2.11) may be explicitly written as

$$e^{2(\nu-\mu)}\left(2\nu'' - \mu'' + \frac{2\nu'}{r} - \nu'^2\right) + \tfrac{1}{4}\kappa^2K^2 = -\kappa\rho , \tag{3.3}$$

$$e^{2(\nu-\mu)}\left(\nu'^2 - \frac{\mu'}{r}\right) - \tfrac{1}{4}\kappa^2K^2 = \kappa p_r , \tag{3.4}$$

$$e^{2(\nu-\mu)}(-\mu'' - \nu'^2) - \tfrac{1}{4}\kappa^2K^2 = \kappa p_\varphi , \tag{3.5}$$

$$e^{2(\nu-\mu)}\left(-\nu'^2 + \frac{\mu'}{r}\right) - \tfrac{1}{4}\kappa^2K^2 = \kappa p_z , \tag{3.6}$$

$$e^{(\nu-\mu)}(K' + K\mu' - K\nu') = -Ke^{\nu-\mu}\nu' , \tag{3.7}$$

$$e^{(\nu-\mu)}(K' + K\mu' + K\nu') = Ke^{\nu-\mu}\nu' . \tag{3.8}$$

From (3.7) and (3.8) it directly follows that

$$K' + K\mu' = 0 , \tag{3.9}$$

which on integration gives

$$K = Be^{-\mu} , \tag{3.10}$$

where B is an arbitrary constant to be determined. The conservation equations (I2.19) give for $j = 1$ the continuity equation

$$\frac{dp_r}{dr} + (\rho + p_r)\nu' - (p_r - p_\varphi)\left(\nu' - \frac{1}{r}\right) - (\nu' - \mu')(p_r - p_z) = -\tfrac{1}{2}\kappa K(K' + K\nu') , \tag{3.11}$$

where the rest are identically satisfied. It can be easily verified that Eq. (3.11) may be obtained directly as a consequence of the field equations.

IV. SOLUTION

If we adopt the similar procedure as in the case of spherical symmetry and use

$$\tilde{p} = p - 2\pi K^2 , \quad \tilde{\rho} = \rho - 2\pi K^2 \tag{4.1}$$

(we have set $\kappa = -8\pi G/c^2$ with $G = 1$, $c = 1$), the field equations reduce to the form

$$8\pi\tilde{\rho} = e^{2(\nu-\mu)}\left(2\nu'' - \mu'' + \frac{2\nu'}{r} - \nu'^2\right) , \tag{4.2}$$

$$8\pi\tilde{p}_r = -8\pi\tilde{p}_z = e^{2(\nu-\mu)}\left(\frac{\mu'}{r} - \nu'^2\right) , \tag{4.3}$$

$$8\pi\tilde{p}_\varphi = e^{2(\nu-\mu)}(\mu'' + \nu'^2) , \tag{4.4}$$

and the continuity equation becomes

$$\frac{d\tilde{p}_r}{dr} + (\tilde{\rho} + \tilde{p}_r)\nu' - (\tilde{p}_r - \tilde{p}_\varphi)\left(\nu' - \frac{1}{r}\right) - 2\tilde{p}_r(\nu' - \mu') = 0 . \tag{4.5}$$

We have only three independent equations to determine five unknowns. In fact these equations are the same as those obtained by Marder[4] in connection with the study of the static fluid cylinder in general relativity, and thus one could use the solution as given by him and determine the corresponding equation of state. However, we now assume an equation of state of the form $\tilde{\rho} = \gamma\tilde{p}_\varphi$, where γ is a constant. This gives us an additional equation.

$$2\nu'' + \frac{2\nu'}{r} - (1+\gamma)\nu'^2 = (1+\gamma)\mu'' . \tag{4.6}$$

Since our set of equations is still incomplete we will assume a particular form for one of the metric potentials and thus determine the system. Assuming $\nu' = Xr^n$ where X and n are constants, so that

$$\nu = (Xr^{n+1})/(n+1) + C_1 , \tag{4.7}$$

we can solve for μ from (4.6) which we find to be

$$\mu = \frac{2\times r^{(n+1)}}{n(1+\gamma)} - \frac{X^2r^{2(n+1)}}{(2n+1)(2n+2)} + D_1r + D_2 . \tag{4.8}$$

We have now four arbitrary constants X, C_1, D_1, and D_2 which are to be determined through the boundary conditions. Assuming that the cylinder has a radius $r = a$, we have for $r > a$ the field equations $R_{ij} = 0$. A well-known solution for Einstein equations for empty space with cylindrical symmetry is that given by Levi-Civita,[5] which is expressed as

$$ds^2 = -A^2r^{-2C(1-C)}(dr^2 + dz^2) - r^{2(1-C)}d\varphi^2 + r^{2C}dt^2 , \tag{4.9}$$

where C and A are constants. Since Eqs. (4.2)–(4.5) are similar to the Einstein equations in form, we can use the Lichnerowicz boundary conditions, namely that the metric potentials are C^1 across the surface $r = a$. Thus the continuity of μ, μ' and ν, ν' gives us

$$X=C/a^{n+1},\quad C_1=C\ln a-C/(n+1),$$

$$D_1=\frac{2C(n+1)}{a(1+\gamma)}\left[\frac{C(1+\gamma)}{(2n+1)}-\frac{1}{n}\right], \tag{4.10}$$

$$D_2=\ln A+C^2\ln a+\frac{2C}{1+\gamma}-\frac{C^2(3+8n+4n^2)}{(2n+1)(2n+2)}.$$

Hence we have for the interior of the cylinder the solution

$$\mu=\frac{2C}{1+\gamma}\left[\frac{R}{n}(R^n-1)-(R-1)\right]-\frac{C^2}{2n+1}\left[\frac{R^{2n+2}}{2n+2}-(2n+2)R\right]+\ln A+C^2\ln a-\frac{C^2(3+8n+4n^2)}{(2n+1)(2n+2)}, \tag{4.11}$$

$$\nu=\frac{C}{n+1}(R^{n+1}-1)+C\ln a,\quad R=r/a \tag{4.12}$$

with the pressure and density given by

$$\begin{aligned}8\pi p_r&=16\pi^2B^2e^{-2\mu}+e^{2(\nu-\mu)}\left[\frac{2C(n+1)}{(1+\gamma)nar}(R^n-1)+\frac{2C^2(n+1)}{(2n+1)ar}(1-R^{2n+1})\right],\\ 8\pi p_z&=16\pi^2B^2e^{-2\mu}-e^{2(\nu-\mu)}\left[\frac{2C(n+1)}{(1+\gamma)nar}(R^n-1)+\frac{2C^2(n+1)}{(2n+1)ar}(1-R^{2n+1})\right],\\ 8\pi p_\varphi&=16\pi^2B^2e^{-2\mu}+e^{2(\nu-\mu)}\left[\frac{2C(n+1)}{(1+\gamma)a^2}R^{n-1}\right],\\ 8\pi\rho&=16\pi^2B^2e^{-2\mu}+e^{2(\nu-\mu)}\left[\frac{2C(n+1)\gamma}{(1+\gamma)a^2}R^{n-1}\right].\end{aligned} \tag{4.13}$$

From the expressions above it is clear that along the axis of the cylinder $r=0$, the central values of the pressure and density are all equal and expressible in terms of the constant B, provided we avoid the singularities by choosing $n\geqslant 1$ and

$$\frac{C(1+\gamma)}{2n+1}-\frac{1}{n}=0,\ \text{i.e. } D_1=0.$$

Hence we have

$$\gamma=\frac{2n+1-nC}{nC}. \tag{4.14}$$

The system is thus determined by the pressure and density

$$8\pi p_r=F(8\pi p_r)_0+\frac{FG}{N}R^{n-1}\left[\frac{1}{n}-\frac{C(1+\gamma)}{2n+1}R^{n+1}\right], \tag{4.15}$$

$$8\pi p_\varphi=F(8\pi p_\varphi)_0+\frac{FG}{N}R^{n-1}, \tag{4.16}$$

$$8\pi p_z=F(8\pi p_z)_0-\frac{FG}{N}R^{n-1}\left[\frac{1}{n}-\frac{C(1+\gamma)}{2n+1}R^{n+1}\right], \tag{4.17}$$

$$8\pi\rho=F(8\pi\rho)_0+\frac{FG}{N}\gamma R^{n-1}, \tag{4.18}$$

$$\begin{aligned}F&=\exp\left\{-\frac{4C}{1+\gamma}\left[\frac{R}{n}(R^n-1)-R\right]+\frac{2C^2}{2n+1}\left[\frac{R^{2n+2}}{2n+2}-(2n+2)R\right]\right\},\\ G&=\frac{2C(n+1)}{(1+\gamma)}a^{2(c-1)}\exp\left[\frac{2C}{n+1}(R^{n+1}-1)\right],\\ N&=A^2a^{2C^2}\exp\left[\frac{4C}{1+\gamma}-\frac{2C^2(3+8n+4n^2)}{(2n+1)(2n+2)}\right],\end{aligned} \tag{4.19}$$

satisfying the equation of state,

$$\frac{\rho-\gamma p_\varphi}{1-\gamma}=2\pi K^2, \tag{4.20}$$

with γ given by Eq. (4.14).

In order to identify the constant C we follow Marder[4] and consider the integral expression for the gravitational mass M_g of unit length of the cylinder as given by

$$M_g=\int_0^a\int_0^{2\pi}\int_0^1(\rho+p_r+p_\varphi+p_z)re^{2(\mu-\nu)}\,dr\,d\varphi\,dz. \tag{4.21}$$

Using the values from (4.13) we get

$$M_g=\frac{16\pi^2B^2}{a^{2C}}\int_0^a re^{[2C/(n+1)](1-R^{n+1})}\,dr+\tfrac{1}{2}C. \tag{4.22}$$

Expanding the exponential term up to first order in C and integrating we get

$$M_g = \tfrac{1}{2}C + 16\pi B^2 a^{2(1-C)}\left(\frac{1}{2} + \frac{C}{n+3}\right). \tag{4.23}$$

Thus we find that the value for the gravitational mass is the same as that obtained by Marder, together with the contributions from the spin density.

V. DISCUSSIONS AND CONCLUSIONS

As we had in the case of spherical symmetry here also we have discontinuity in the hydrostatic pressure across the surface $r=a$, due to the presence of the spin density. From the expression for γ it can be readily seen that, as $n \geq 1$ and $C<1$, $\gamma>1$. Hence from the equation of state (4.20) we get for real K $\rho - \gamma\, p_\varphi < 0$, i.e.,

$$\rho/p_\varphi < \gamma\ . \tag{5.1}$$

The only undetermined parameter is the constant A.

If we consider Marder's solution as given by

$$\mu = -\frac{C^2}{m+1}(1-R^{m+1}) + C^2 \ln a + \ln A\ ,$$
$$\nu = -\frac{C}{n+1}(1-R^{n+1}) + C\ln a\ , \tag{5.2}$$

with m and n as constants ≥ 1, the pressure and the density are then given by

$$8\pi p_r = 16\pi^2 B^2 e^{-2\mu} - \frac{C^2}{a^2} e^{2\nu-2\mu} R^{2n}(1-R^{m-2n-1})\ ,$$

$$8\pi p_\varphi = 16\pi^2 B^2 e^{-2\mu} + \frac{C^2}{a^2} e^{2\nu-2\mu} R^{2n}(1+mR^{m-2n-1})\ , \tag{5.3}$$

$$8\pi p_z = 16\pi^2 B^2 e^{-2\mu} + \frac{C^2}{a^2} e^{2\nu-2\mu} R^{2n}(1-R^{m-2n-1})\ ,$$

$$8\pi \rho = 16\pi^2 B^2 e^{-2\mu} + \frac{C}{a^2} e^{2\nu-2\mu} R^{2n}[2(n+1)R^{-n-1} - mCR^{m-2n-1} - C]\ .$$

In general it is formidable to calculate the equation of state. However, if we consider the special case $m=n$, then we get the equation of state

$$C(\rho + p_\varphi) - 2(p_r + p_\varphi) = 4\pi(C-2)K^2\ . \tag{5.4}$$

It may be recalled that in the case of spherical symmetry (paper I) the equation determining the spin density had to be assumed, whereas in the present case we obtained it from the field equations. It is interesting to see that because of the presence of the spin density we could determine the pressure and the density as deviations from their values on the axis of symmetry.

As could be seen from the two cases of symmetry considered, in the static models the major contribution of the spin density is to change the equation of state of the system, as one would expect. Further studies with rotating-fluid distributions might give us more interesting aspects regarding the effects of the spin density.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor K. Bleuler for his kind interest and encouragement and the A. v. Humboldt-Stiftung for the award of a Research Fellowship.

---

*Alexander von Humboldt Research Fellow.

[1]G. Börner, in *Springer Tracts in Modern Physics*, edited by G. Höhler (Springer, New York, 1973), Vol. 69.

[2]G. D. Kerlick [Astrophys. J. 185, 631 (1973)] has estimated that pure torsion effects in a neutron star are unobservable compared to those of magnetic-dipole interactions.

[3]A. R. Prasanna, preceding paper, Phys. Rev. D 11, 2076 (1975).

[4]L. Marder, Proc. R. Soc. A244, 524 (1958).

[5]T. Levi-Civita, R. C. Accad. Lincei 28, 101 (1919).