Static fluid spheres in Einstein-Cartan theory

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Following the work of Trautman we have described briefly the Einstein-Cartan equations with special reference to a perfect fluid distribution and then obtained three solutions adopting Hehl's approach and Tolman's technique. We have found that a space-time metric similar to the Schwarzschild solution (interior) will no longer represent a homogeneous fluid sphere in the presence of spin density, and the corresponding equation of state has the form $8\pi p = 8\pi\rho - 6/R^2 + (B_2/2\pi A R^2)(8\pi\rho - 3/R^2)^{1/2}$, where R, B_2, and A are constants. At the boundary of the fluid sphere the hydrostatic pressure p is discontinuous.

I. INTRODUCTION

Modifying Einstein's equations of general relativity has been one of the techniques followed to avoid space-time singularities. Recently, Trautman[1] has proposed that spin and torsion may avert gravitational singularities, by considering a Friedmann type of universe in the framework of Einstein-Cartan theory and obtaining a minimum radius R_0 at $t=0$. Isham, Salam, and Strathdee[2] have shown that if one considers the Trautman model in the framework of their two-tensor theory then the minimum radius would increase substantially, giving a reasonable density for the universe in the early stages. Applying the same arguments for finite collapsing objects, the present author has shown[3] that it is possible to get a minimum critical mass for black holes. Having seen that the new idea regarding prevention of catastrophic collapse could have an interesting role in astrophysical situations, we wish to understand the full implications of the Einstein-Cartan theory for finite distributions like fluid spheres with nonzero pressure. Also, since spin is a very important property of a particle, it is very relevant to consider its role in the study of such configurations as one may find in the interior of a star.

As a first step in such a study, we now consider the problem of static fluid spheres in the framework of Einstein-Cartan theory. Moreover, if a collapsing fluid sphere stops collapsing because of spin and torsion and then stays as a static body, its interior will no longer be described by the Schwarzschild solution for a homogeneous fluid sphere of general relativity. Finding a proper solution to describe this situation is one of the motivations of the present work. We adopt Tolman's technique[4] to solve the field equations and thus obtain two more solutions and their corresponding equations of state.

The plan of the paper is as follows. In Sec. II we describe briefly the Einstein-Cartan theory and the governing equations following the notation and treatment of Trautman.[5] In Sec. III the metric and the curvature are presented along with the components of the Einstein tensor. Section IV deals with the energy-momentum tensor and the field equations. Three solutions are given in Sec. V, and the paper is concluded along with some discussions in Sec. VI.

II. EINSTEIN-CARTAN THEORY

Let M be a C^∞ four-dimensional, oriented, connected Hausdorff differential manifold with a Lorentz metric g defined on it. All geometric objects other than the forms are defined by their components with respect to a field of coframes Θ^i (in the cotangent space of M) which are linearly independent at each point of M. Since we are interested in spinor fields we take the Θ^i to be in general anholonomic and the associated tetrad to be orthonormal. Since the manifold is paracompact there exists a connection ω on it which we assume to be a metric linear connection. The metric g and the connection ω are described with respect to the chosen coframe Θ^i by the metric components g_{ij} and by a set of one-forms $\omega^i{}_j$ defining the covariant derivative, respectively. Hence we have

$$g = ds^2 = g_{ij}\Theta^i \otimes \Theta^j \,. \qquad (2.1)$$

$\omega^i{}_j$ themselves are completely determined by 64 functions $\Gamma^i{}_{kj}$ such that

$$\omega^i{}_j = \Gamma^i{}_{kj}\Theta^k \,. \qquad (2.2)$$

The torsion and the curvature two-forms on M are respectively given by

$$\begin{aligned} \mathbf{\Theta}^i &= D\Theta^i \\ &= d\Theta^i + \omega^i{}_j \wedge \Theta^j \\ &= \tfrac{1}{2}Q^i{}_{jk}\Theta^j \wedge \Theta^k \,, \end{aligned} \qquad (2.3)$$

$$\Omega^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j$$
$$= \tfrac{1}{2} R^i{}_{jkl} \Theta^k \wedge \Theta^l , \qquad (2.4)$$

where D denotes the exterior covariant derivative. For a tensor-valued zero-form, say, φ_A, $D\varphi_A = \Theta^i \nabla_i \varphi_A$, where ∇_i is the usual covariant derivative. $Q^i{}_{jk}$ and $R^i{}_{jkl}$ are the torsion and the curvature tensors, respectively.

If we introduce a completely antisymmetric tensor η_{ijkl} such that $\eta_{1234} = |\det g_{ij}|^{1/2}$, this zero-form along with the forms

$$\eta_{ijk} = \Theta^l \eta_{ijkl} , \quad \eta_{ij} = \tfrac{1}{2}\Theta^k \wedge \eta_{ijk} ,$$
$$\eta_i = \tfrac{1}{3}\Theta^j \wedge \eta_{ij} , \quad \eta = \tfrac{1}{4}\Theta^i \wedge \eta_i \qquad (2.5)$$

spans the Grassmann algebra of M.

The field equations are obtained from the variational principle

$$\delta \int (S + \kappa L) = 0 , \qquad (2.6)$$

where $L = L(\psi_A, D\psi_A, \Theta^i, g_{ij})$ is the material Lagrangian four-form depending locally on the spinor or tensor fields ψ_A, their covariant derivatives $D\psi_A$, and the metric; κ is the gravitational constant and S is the Ricci four-form defined globally as

$$S = \tfrac{1}{2}\eta_k{}^l \wedge \Omega^k{}_l = \tfrac{1}{2}R\eta , \qquad (2.7)$$

where $R = g^{ln}\delta^m{}_k R^k{}_{lmn}$; η is the volume four-form. Varying the total action with respect to the metric, i.e., Θ^i since g_{ij} are fixed, the connection $\omega^i{}_j$, and the fields ψ_A independently, we get the equations

$$e_i = \kappa t_i , \quad c^j{}_i = \kappa s^j{}_i , \quad \frac{\delta L}{\delta \psi_A} = 0 , \qquad (2.8)$$

wherein

$$e_i = \tfrac{1}{2}\eta_{ikl} \wedge \Omega^{kl} , \quad c^j{}_i = -D\eta^j{}_i , \quad t_i = \frac{\delta L}{\delta \Theta^i}, \quad s^j{}_i = \tfrac{1}{2}\frac{\delta L}{\delta \omega^i{}_j} . \qquad (2.9)$$

The orthonormality of the frames together with the fact that the connection is a metric connection ($Dg_{ij} = 0$) tells us that an infinitesimal variation in connection induces tetrad rotation. Hence one identifies $s^j{}_i$ as the spin density of the system.[6] t_i is the energy-momentum vector-valued three-form. In the general case (when the variation in metric is induced through the variation in g_{ij}) we have the energy-momentum symmetric four-form, $T^{ij} = \tfrac{1}{2}\delta L/\delta g_{ij}$, which, along with t_i and $s^j{}_i$, satisfies the identity

$$T_i{}^j = \Theta^j \wedge t_i - \tfrac{1}{2}Ds_i{}^j . \qquad (2.10)$$

Using (2.3) and (2.4) in (2.8) and (2.9) we can write the Einstein-Cartan equations as

$$R^j{}_i - \tfrac{1}{2}R\delta^j{}_i = -\kappa t^j{}_i , \qquad (2.11)$$

$$Q^i{}_{jk} - \delta^i{}_j Q^l{}_{lk} - \delta^i{}_k Q^l{}_{jl} = -\kappa s^i{}_{jk} , \qquad (2.12)$$

where $t^j{}_i$ and $s^i{}_{jk}$ are defined through the relations

$$t_i = \eta_j t^j{}_i , \quad s_{jk} = \eta_i s^i{}_{jk} . \qquad (2.13)$$

In order to derive the conservation laws we make use of the Bianchi identities

$$D\Theta^i = \Omega^i{}_j \wedge \Theta^j , \quad D\Omega^i{}_j = 0 . \qquad (2.14)$$

From (2.9) and (2.14) we get

$$De_j = \tfrac{1}{2}\Theta^l \wedge \Omega^{pq}\eta_{jpql} ,$$

i.e.

$$De_j = \eta[-Q^k{}_{jm}(R^m{}_k - \tfrac{1}{2}\delta^m{}_k R) + \tfrac{1}{2}(Q^k{}_{lm}R^{lm}{}_{jk} + 2Q^k{}_{kl}R^l{}_j)] , \qquad (2.15)$$

$$Dc_{ij} = \eta_{ik} \wedge \Omega^k{}_j - \eta_{jk} \wedge \Omega^k{}_i$$
$$= e_i \wedge \Theta_j - e_j \wedge \Theta_i . \qquad (2.16)$$

If we use the field equations and simplify, the conservation laws for the spin and the energy-momentum are respectively given by

$$Ds_{ij} = t_i \wedge \Theta_j - t_j \wedge \Theta_i , \qquad (2.17)$$

$$Dt_j = \eta(Q^k{}_{jm}t^m{}_k - \tfrac{1}{2}s^k{}_{lm}R^{lm}{}_{jk}) . \qquad (2.18)$$

Simplifying (2.18) we get

$$\nabla_k t^k{}_j = s^l{}_{jk}R^k{}_l + \tfrac{1}{2}s^m{}_{pq}R^{pq}{}_{mj} + s^p{}_{kp}(R^k{}_j - \tfrac{1}{2}\delta^k{}_j R) . \qquad (2.19)$$

In order to consider solutions of Einstein-Cartan equations we use a classical description of spin[1] as given by

$$s^i{}_{jk} = u^i S_{jk} , \text{ with } u^k S_{jk} = 0 , \qquad (2.20)$$

wherein u^i is the velocity four-vector and S_{jk} is the intrinsic angular momentum tensor. If we have a perfect fluid distribution with isotropic pressure then the canonical tensor for such a distribution is given by[7]

$$t_{jk} = h_j u_k - p g_{jk} , \qquad (2.21)$$

where h_j is the density of enthalpy. The conservation law (2.17) then gives us

$$h_j = (\rho + p)u_j - u^i \nabla_k (u^k S_{ij}) , \qquad (2.22)$$

where $\rho = t_{ij}u^i u^j$ is the energy density in the rest frame. Using the expression for $t^k{}_j$ from (2.21) and (2.22) we get the equations

$$\nabla_k[(\rho + p)u^k - g^{ki}u^l \nabla_m (u^m S_{li})] - u^j \nabla_j p = 0 \qquad (2.23)$$

and

$$[(\rho+p)u^k - g^{ki}u^l\nabla_m(u^m S_{li})]\nabla_k u_j = -\nabla_{lp}(u^l u_j - \delta^l{}_j) + u^k S_{jm}R^m{}_k - \tfrac{1}{2}u^k S_{lm}R^{lm}{}_{jk}. \tag{2.24}$$

If $\overline{T}_i{}^j$ denotes the usual symmetric energy-momentum tensor then we have for a classical description of spin (2.20) the relation between the canonical tensor $t^j{}_i$, the spin tensor $s^i{}_{jk}$, and $\overline{T}_i{}^j$ obtained from the identity (2.10) as

$$\eta\overline{T}_i{}^j = \Theta^j \wedge \eta_k t^k{}_i - \tfrac{1}{2}D(\eta_k s^k{}_i{}^j), \tag{2.25}$$

which on simplification yields

$$\overline{T}_i{}^j = t^j{}_i - \tfrac{1}{2}g^{jm}\nabla_k(s^k{}_{im}). \tag{2.26}$$

III. METRIC AND CURVATURE

We consider a static spherically symmetric matter distribution represented by the space-time metric

$$ds^2 = -e^{2\mu}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\varphi^2 + e^{2\nu}dt^2, \tag{3.1}$$

where μ and ν are C^1 functions of r alone. If Θ^i represents an orthonormal coframe we have from (2.1) and (3.1)

$$\Theta^1 = e^\mu dr, \quad \Theta^2 = rd\theta, \quad \Theta^3 = r\sin\theta d\varphi, \quad \Theta^4 = e^\nu dt, \tag{3.2}$$

so that

$$g_{ij} = \mathrm{diag}\{-1,-1,-1,1\}.$$

Assuming that the spins of the individual particles composing the fluid are all aligned in the r direction we get for the spin tensor S_{ij} the only independent nonzero component to be $S_{23}=K$, say. Since the fluid is supposed to be static we have the velocity four-vector $u^i = \delta^i{}_4$. Thus the nonzero components of $s^i{}_{jk}$ are

$$s^4{}_{23} = -s^4{}_{32} = K. \tag{3.3}$$

Hence from the Cartan equation (2.12) we get for $Q^i{}_{jk}$ the components

$$Q^4{}_{23} = -Q^4{}_{32} = -\kappa K; \tag{3.4}$$

the others are zero.

Using (3.4) in (2.3) we can obtain the torsion two-form Θ^k to be

$$\Theta^1 = 0, \quad \Theta^2 = 0, \quad \Theta^3 = 0, \quad \Theta^4 = -\kappa K\Theta^2 \wedge \Theta^3. \tag{3.5}$$

Once we have the torsion form we can use it in (2.3) along with (3.2) and solve for the components of $\omega^k{}_l$, which in the present case turn out to be

$$\begin{aligned}
&\omega^1{}_4 = \omega^4{}_1 = e^{-\mu}\nu'\Theta^4, \quad \omega^2{}_1 = -\omega^1{}_2 = \frac{e^{-\mu}}{r}\Theta^2,\\
&\omega^2{}_4 = \omega^4{}_2 = \tfrac{1}{2}\kappa K\Theta^3, \quad \omega^3{}_1 = -\omega^1{}_3 = \frac{e^{-\mu}}{r}\Theta^3,\\
&\omega^3{}_4 = \omega^4{}_3 = -\tfrac{1}{2}\kappa K\Theta^2, \quad \omega^3{}_2 = -\omega^2{}_3 = -\tfrac{1}{2}\kappa K\Theta^4 + \frac{\cot\theta}{r}\Theta^3.
\end{aligned} \tag{3.6}$$

Using them in (2.4) we get the curvature form $\Omega^k{}_l$ with the nonzero components

$$\begin{aligned}
\Omega^1{}_4 &= [e^{-2\mu}(\nu'' + \nu'^2 - \mu'\nu')](\Theta^1 \wedge \Theta^4) - \frac{\kappa K}{r}e^{-\mu}(\Theta^2 \wedge \Theta^3),\\
\Omega^2{}_4 &= \tfrac{1}{2}\kappa e^{-\mu}\left(K' + \frac{K}{r}\right)(\Theta^1 \wedge \Theta^3) + \left(\frac{e^{-2\mu}}{r}\nu' + \tfrac{1}{4}\kappa^2K^2\right)(\Theta^2 \wedge \Theta^4),\\
\Omega^3{}_4 &= -\tfrac{1}{2}\kappa e^{-\mu}\left(K' + \frac{K}{r}\right)(\Theta^1 \wedge \Theta^2) + \left(\frac{e^{-2\mu}}{r}\nu' + \tfrac{1}{4}\kappa^2K^2\right)(\Theta^3 \wedge \Theta^4),\\
\Omega^1{}_2 &= \frac{e^{-2\mu}}{r}\mu'(\Theta^1 \wedge \Theta^2) - \tfrac{1}{2}\kappa K e^{-\mu}\left(\nu' - \frac{1}{r}\right)(\Theta^4 \wedge \Theta^3),\\
\Omega^1{}_3 &= \frac{e^{-2\mu}}{r}\mu'(\Theta^1 \wedge \Theta^3) - \tfrac{1}{2}\kappa K e^{-\mu}\left(\nu' - \frac{1}{r}\right)(\Theta^4 \wedge \Theta^2),\\
\Omega^2{}_3 &= \left(\frac{1-e^{-2\mu}}{r^2} + \tfrac{1}{4}\kappa^2K^2\right)(\Theta^2 \wedge \Theta^3) + \tfrac{1}{2}\kappa e^{-\mu}(K' + K\nu')(\Theta^1 \wedge \Theta^4).
\end{aligned} \tag{3.7}$$

Equations (2.4) and (3.7) together give

$$\begin{aligned}
&R^1{}_{414} = e^{-2\mu}(\nu'' + \nu'^2 - \mu'\nu'),\\
&R^2{}_{424} = R^3{}_{434} = \tfrac{1}{4}\kappa^2K^2 + \frac{e^{-2\mu}\nu'}{r},\\
&R^1{}_{212} = R^1{}_{313} = \frac{e^{-2\mu}\mu'}{r},\\
&R^2{}_{323} = \frac{1-e^{-2\mu}}{r^2} + \tfrac{1}{4}\kappa^2K^2,\\
&R^1{}_{423} = -\frac{\kappa K}{r}e^{-\mu},\\
&R^2{}_{413} = -R^3{}_{412} = \tfrac{1}{2}\kappa e^{-\mu}\left(K' + \frac{K}{r}\right),\\
&R^1{}_{243} = -R^1{}_{342} = \tfrac{1}{2}\kappa K e^{-\mu}\left(\nu' - \frac{1}{r}\right),\\
&R^2{}_{314} = \tfrac{1}{2}e^{-\mu}(K' + K\nu').
\end{aligned} \tag{3.8}$$

The Ricci tensor R_{ij} and the scalar of curvature R are therefore given by

$$R_{11}=-e^{-2\mu}\left(\nu''+\nu'^2-\mu'\nu'-\frac{2\mu'}{r}\right),$$

$$R_{22}=R_{33}=-\frac{e^{-2\mu}}{r^2}[1+r(\nu'-\mu')]+\frac{1}{r^2}, \qquad (3.9)$$

$$R_{44}=e^{-2\mu}\left(\nu''+\nu'^2-\mu'\nu'+\frac{2\nu'}{r}\right)+\tfrac{1}{2}\kappa^2K^2,$$

$$R=-2\left\{\frac{1}{r^2}-e^{-2\mu}\left[\frac{1}{r^2}+\nu''+\nu'^2-\mu'\nu'+\frac{2}{r}(\nu'-\mu')\right]\right\}+\tfrac{1}{2}\kappa^2K^2, \qquad (3.10)$$

with $R_{ij}=0$, $i\neq j$. Hence the Einstein tensor $G_{ij}=R_{ij}-\frac{1}{2}Rg_{ij}$ is found to have the components

$$G_{11}=-\frac{1}{r^2}+e^{-2\mu}\left(\frac{2\nu'}{r}+\frac{1}{r^2}\right)+\tfrac{1}{4}\kappa^2K^2,$$

$$G_{22}=G_{33}=e^{-2\mu}\left[\nu''+\nu'^2-\mu'\nu'+\frac{1}{r}(\nu'-\mu')\right]+\tfrac{1}{4}\kappa^2K^2, \qquad (3.11)$$

$$G_{44}=\frac{1}{r^2}+e^{-2\mu}\left(\frac{2\mu'}{r}-\frac{1}{r^2}\right)+\tfrac{1}{4}\kappa^2K^2.$$

IV. ENERGY-MOMENTUM TENSOR AND FIELD EQUATIONS

Since we are considering a perfect fluid distribution with isotropic pressure p and matter density ρ we have from (2.21) and (2.22) for $t^j{}_i$

$$t^j{}_i=g^{jk}\{[(\rho+p)u_k-u^l\nabla_m(u^mS_{lk})]u_i-pg_{ki}\}. \qquad (4.1)$$

Using (3.4) we get then the nonzero components

$$t^1{}_1=t^2{}_2=t^3{}_3=-p, \quad t^4{}_4=\rho. \qquad (4.2)$$

Hence the field equations (2.11) may be written, using (3.11) and (4.2), as

$$-\frac{1}{r^2}+e^{-2\mu}\left(\frac{2\nu'}{r}+\frac{1}{r^2}\right)+\tfrac{1}{4}\kappa^2K^2=-\kappa p, \qquad (4.3)$$

$$e^{-2\mu}\left[\nu''+\nu'^2-\mu'\nu'+\frac{1}{r}(\nu'-\mu')\right]+\tfrac{1}{4}\kappa^2K^2=-\kappa p, \qquad (4.4)$$

$$-\frac{1}{r^2}-e^{-2\mu}\left(\frac{2\mu'}{r}-\frac{1}{r^2}\right)-\tfrac{1}{4}\kappa^2K^2=\kappa\rho. \qquad (4.5)$$

The conservation laws governed by Eqs. (2.23) and (2.24) give us the relations

$$\nabla_i[(\rho+p)u^i]=0 \text{ (matter conservation)}, \qquad (4.6)$$

$$\nabla_i(Ku^i)=0 \text{ (spin conservation)}, \qquad (4.7)$$

and

$$\frac{dp}{dr}+(\rho+p)\nu'+\tfrac{1}{2}\kappa K(K'+K\nu')=0. \qquad (4.8)$$

If we assume the equation of hydrostatic equilibrium to hold as in general relativity, namely

$$\frac{dp}{dr}+(\rho+p)\nu'=0, \qquad (4.9)$$

we get the additional equation

$$K'+K\nu'=0. \qquad (4.10)$$

Solving for K we get

$$K=Ae^{-\nu}, \qquad (4.11)$$

where A is a constant of integration to be determined. Setting $\kappa=-8\pi G/c^2$ with $G=1$, $c=1$ we can write the field equations as

$$8\pi p=16\pi^2K^2-\frac{1}{r^2}+e^{-2\mu}\left(\frac{2\nu'}{r}+\frac{1}{r^2}\right), \qquad (4.12)$$

$$8\pi\rho=16\pi^2K^2+\frac{1}{r^2}+e^{-2\mu}\left(\frac{2\mu'}{r}-\frac{1}{r^2}\right), \qquad (4.13)$$

$$e^{-2\mu}\left[\left(\frac{\nu''}{r}-\frac{\nu'}{r^2}-\frac{1}{r^3}\right)-\mu'\left(\frac{2\nu'}{r}+\frac{1}{r^2}\right)+\nu'\left(\frac{\mu'+\nu'}{r}\right)\right]+\frac{1}{r^3}=0. \qquad (4.14)$$

In principle we now have a completely determined system if an equation of state is specified. However, it is well known that in practice this set of equations is formidable to solve using a preassigned equation of state, except perhaps for the case $\rho=p$, which may not be physically meaningful. Secondly, we have the question of boundary conditions to be chosen for fitting the solutions in the interior and the exterior of the star. A very interesting aspect of the Einstein-Cartan theory is that outside the fluid distribution the equations reduce to Einstein's equations for empty space, *viz.*, $R_{ij}=0$, since there is no spin density.

Following Hehl's[8] approach, if we transform the terms in K^2 to the left-hand side of the equations and redefine the pressure and density as

$$\tilde{p}=p-2\pi K^2, \quad \tilde{\rho}=\rho-2\pi K^2, \qquad (4.15)$$

we find that the equations take the usual general-relativistic form for a static fluid sphere as given by

$$8\pi\tilde{p}=-\frac{1}{r^2}+e^{-2\mu}\left(\frac{2\nu'}{r}+\frac{1}{r^2}\right), \qquad (4.16)$$

$$8\pi\tilde{\rho}=\frac{1}{r^2}+e^{-2\mu}\left(\frac{2\mu'}{r}-\frac{1}{r^2}\right), \qquad (4.17)$$

with (4.14) remaining the same. The equation of continuity (4.8) now becomes

$$\frac{d\tilde{p}}{dr}+(\tilde{\rho}+\tilde{p})\nu'=0. \qquad (4.18)$$

Equations (4.16), (4.17), and (4.13) are the same as obtained by Tolman,[4] so we can use the same solutions for our discussions. Assuming that the sphere has a finite radius $r=a$ for $r>a$, since the equations are $R_{ij}=0$, we have by Birkhoff's theorem the space-time metric represented by the Schwarzschild solution

$$ds^2=-\left(1-\frac{2m}{r}\right)^{-1}dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+\left(1-\frac{2m}{r}\right)dt^2, \quad (4.19)$$

where m is a constant associated with the mass of the sphere. With this we use the boundary conditions

$$[e^{-2\mu}]_{r=a}=[e^{2\nu}]_{r=a}=\left(1-\frac{2m}{a}\right) \quad (4.20)$$

and

$$\tilde{p}=0 \text{ at } r=a. \quad (4.21)$$

V. SOLUTIONS

Case (1). Let us consider the case corresponding to the well-known Schwarzschild solution

$$ds^2=-\left(1-\frac{r^2}{R^2}\right)^{-1}dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+\left[B_2-B_1\left(1-\frac{r^2}{R^2}\right)^{1/2}\right]^2dt^2, \quad (5.1)$$

with

$$B_2=\tfrac{3}{2}(1-a^2/R^2)^{1/2},\ B_1=\tfrac{1}{2},\ 2m/a=a^2/R^2. \quad (5.2)$$

Unlike in the case of general relativity, the fluid sphere is now no longer of uniform density. The pressure and the density are given by

$$8\pi p=\frac{16\pi^2A^2+(1/R^2)[3B_1(1-r^2/R^2)^{1/2}-B_2][B_2-B_1(1-r^2/R^2)^{1/2}]}{[B_2-B_1(1-r^2/R^2)^{1/2}]^2}, \quad (5.3)$$

$$8\pi\rho=\frac{16\pi^2A^2+(3/R^2)[B_2-B_1(1-r^2/R^2)^{1/2}]^2}{[B_2-B_1(1-r^2/R^2)^{1/2}]^2}. \quad (5.4)$$

The constant A can be evaluated in terms of the central density ρ_0 to be

$$A=\frac{1}{8\pi}\left(8\pi\rho_0-\frac{3}{R^2}\right)^{1/2}\left[3\left(1-\frac{a^2}{R^2}\right)^{1/2}-1\right]. \quad (5.5)$$

As in the case of Einstein's theory we find that a singularity at $r=0$ occurs only for the case $B_2=B_1$, i.e., $m/a=\frac{4}{9}$. From (5.4) we can compute r in terms of ρ and substituting the value so obtained in (5.3) we get the equation of state

$$8\pi p=8\pi\rho-\frac{6}{R^2}+\frac{B_2}{2\pi AR^2}\left(8\pi\rho-\frac{3}{R^2}\right)^{1/2}. \quad (5.6)$$

Case (2). Assuming $e^{2\nu}\nu'=Cr$, where C is a constant, and solving (4.14) for μ we get the space-time metric

$$ds^2=-\frac{(D+2Cr^2)}{(1+B_1r^2)(D+Cr^2)}dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+(D+Cr^2)dt^2, \quad (5.7)$$

with

$$B_1=-\frac{m}{a^3},\ C=\frac{m}{a^3},\ D=\left(1-\frac{3m}{a}\right). \quad (5.8)$$

The pressure and density are given by

$$8\pi p=\frac{16\pi^2A^2}{(1-3m/a+mr^2/a^3)}+\frac{3m^2}{a^4}\frac{(1-r^2/a^2)}{(1-3m/a+2mr^2/a^3)}, \quad (5.9)$$

$$8\pi\rho=\frac{16\pi^2A^2}{(1-3m/a+mr^2/a^3)}+\frac{m}{a^3}\frac{(6-9m/a+mr^2/a^3)}{(1-3m/a+2mr^2/a^3)}-\frac{4m^2r^2}{a^6}\frac{(1-mr^2/a^3)}{(1-3m/a+2mr^2/a^3)^2}. \quad (5.10)$$

At $r=0$ we have now

$$8\pi\rho_0=8\pi p_0+\frac{6m}{a^3}\left(1-\frac{2m}{a}\right)\left(1-\frac{3m}{a}\right)^{-1}, \quad (5.11)$$

and we can again express the constant A associated with the spin density in terms of ρ_0 as

$$A=\frac{1}{4\pi}\left[8\pi\rho_0\left(1-\frac{3m}{a}\right)-\frac{6m}{a^3}+\frac{9m^2}{a^4}\right]^{1/2}. \quad (5.12)$$

Eliminating r^2 between p and ρ we can get the equation of state.

Case (3). If we assume $e^{2\nu}=C_1r^{2n}$, the complete solution is

$$ds^2=-\frac{(1+2n-n^2)}{1+(1+2n-n^2)B_1r^{2N}}dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+C_1r^{2n}dt^2, \quad (5.13)$$

with

$$N=\frac{(1+2n-n^2)}{(1+n)},\quad B_1=\left(1-\frac{2m}{a}-\frac{1}{1+2n-n^2}\right)a^{-2N},\quad C_1=a^{-2n}\left(1-\frac{2m}{a}\right). \quad (5.14)$$

The pressure and the density are evaluated to be

$$8\pi p=\frac{16\pi^2A^2}{C_1r^{2n}}+\frac{1}{r^2}\left(\frac{n^2}{1+2n-n^2}\right)+B_1(1+2n)r^{2n(1-n)/(1+n)}, \quad (5.15)$$

$$8\pi\rho=\frac{16\pi^2A^2}{C_1r^{2n}}+\frac{1}{r^2}\left(\frac{2n-n^2}{1+2n-n^2}\right)-\left(\frac{3+5n-2n^2}{1+n}\right)B_1r^{2n(1-n)/(1+n)}. \quad (5.16)$$

At $r=a$, $\tilde{p}=0$ gives us n in terms of m and a, as

$$n=(m/a)(1-2m/a)^{-1}. \quad (5.17)$$

It is obvious from the expressions above that as $r\to 0$ both p and ρ tend to infinity. However, we can study for various values of n how the ratio p_0/ρ_0 behaves. For

$0<n<1$, i.e., $m/a<\frac{1}{3}$, $p_0/\rho_0\to n/(2-n)$,

$n\geq 1$, i.e., $m/a\geq\frac{1}{3}$, $p_0/\rho_0\to 1$,

$0>n>-1$ and $1+2n-n^2>0$, $p_0/\rho_0\to n/(2-n)$,

$0>n>-1$ and $1+2n-n^2<0$, $p_0/\rho_0\to\dfrac{(1+2n)(1+n)}{(2n^2-5n-3)}$,

$n<-1$, $p_0/\rho_0\to n/(2-n)$.

$n=-1$, i.e., $m=a$, ρ is infinite, and the metric is degenerate. As in the case of Tolman's solution if we consider the case $n=\frac{1}{2}$, equivalently $m/a=\frac{1}{4}$, we get the space-time metric

$$ds^2=-\frac{7}{4-\frac{1}{2}(r/a)^{7/3}}dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2+\tfrac{1}{2}(r/a)dt^2, \quad (5.18)$$

with the pressure and density given by

$$8\pi p=32\pi^2A^2(a/r)+(1/7r^2)[1-(r/a)^{7/3}], \quad (5.19)$$

$$8\pi\rho=32\pi^2A^2(a/r)+(3/7r^2)[1+\tfrac{5}{9}(r/a)^{7/3}]. \quad (5.20)$$

At the center $r=0$ we have $p_0/\rho_0=\frac{1}{3}$. Since we do not have any further relation to determine A, we can set a limit on it by assuming that at the boundary $r=a$, the ratio of pressure to density is $<\frac{1}{3}$. This gives us the condition

$$A<(1/8\pi)(\tfrac{2}{3})^{1/2}(1/a). \quad (5.21)$$

The expressions for the pressure and density suggest the relation

$$(3\pi\rho+5\pi p)r^2-32\pi^2A^2ar-\tfrac{1}{4}=0, \quad (5.22)$$

from which solving for r and substituting in either of the relations (5.19) or (5.20) we get the equation of state

$$8\pi p(1\pm z)^2=\tfrac{2}{7}(3\pi\rho+5\pi p)(6+2z^2\pm 7z)-\frac{2^{7/3}A^{2/3}\pi^{2/3}(1\pm z)^{7/3}}{14a^2(3\pi\rho+5\pi p)^{1/3}}, \quad (5.23)$$

where

$$z=\left[1+\frac{(3\pi\rho+5\pi p)}{1024\pi^4A^4a^2}\right]^{1/2}. \quad (5.24)$$

VI. DISCUSSIONS AND CONCLUSIONS

At the outset we observe that the continuity of $\tilde{p}$ (not of p) across the surface $r=a$ ensures the continuity of ν' as required by Eq. (4.13), whereas μ' is discontinuous. The discontinuity in μ' is due to the curvature coordinates employed and hence the same as in general relativity. However, since the spin density is discontinuous the pressure p is discontinuous across $r=a$. Thus we find that the usual general-relativistic boundary conditions, namely that (1) the metric potentials are C^1 and that (2) the hydrostatic pressure is continuous, are not satisfied. This, in our opinion, should not be surprising, as in this theory spin does not influence the geometry outside the distribution.

As could be seen the presence of spin density induces nonuniformity in density in a Schwarzschild sphere, and consequently the equation of state is changed. The other three cases considered by Tolman,[4] (i) $e^{2\nu}=$ constant, (ii) $e^{-2\mu-2\nu}=$ constant, and (iii) $e^{2\mu}=$ constant, do not give us any interesting distributions. Case (i) represents the static Einstein universe and cases (ii) and (iii) represent fluid spheres with singularities, without suggesting any interesting equation of state.

Adopting the scheme of Hehl[8] as done here one can always use any known solution of general relativity and study the corresponding equation of state under the influence of spin density. It might be very interesting to consider the recent general solution as given by Adler[9] for static fluid sphere and that of Bowers and Liang[10] for anisotropic fluid sphere.

In conclusion we note that the assumption of spin alignment will have to be justified by extending the study to the case when a magnetic field is present. To have a radially outgoing magnetic flux one may have to assume the existence of a magnetic monopole at the center of the sphere. On the other hand, if we relax spherical symmetry and consider the case of axially symmetric fluid distributions the physical picture will be much more plausible.

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