Vector potential and metric perturbations of a rotating black hole*

Paul L. Chrzanowski
Center for Relativity Theory, Department of Physics, The University of Texas at Austin, Austin, Texas 78712 and Department of Physics and Astronomy, The University of Maryland, College Park, Maryland 20742
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The assumption of factorized Green's functions together with the inhomogeneous Teukolsky equations are used to derive analytic expressions for homogeneous metric (and vector potential) perturbations of a Kerr black hole. These homogeneous solutions are used to construct solutions to the perturbation equations when sources are present. What one finds are particularly simple formulas for the energy and angular momentum flux in the asymptotic regions $r^* \to \pm\infty$.

I. INTRODUCTION

Ever since Teukolsky[1] discovered that the Newman-Penrose[2] equations describing certain components of the perturbed Weyl tensor (ψ_0 and ψ_4) can be decoupled and solved by separation of variables in a Kerr[3] background, the dynamics of rotating black holes has been a subject of considerable investigation. Significant work has been done by Press and Teukolsky,[4] who, by studying the separated radial functions, have demonstrated numerically that a rotating black hole is stable with respect to small perturbations. The separable equations have proved useful in many other computations, a few of which include the superradiant scattering calculations of Starobinsky,[5] Starobinsky and Churilov,[6] and Teukolsky and Press[7]; the gravitational spin-down work by Hartle[8] and Teukolsky[9]; and the point-charge computations of Cohen, Kegeles, Vishveshwara, and Wald.[10]

In the work cited above, knowledge of ψ_0 and/or ψ_4 (φ_0 and/or φ_2 in electromagnetic computations), two complex scalars formed by projecting the perturbed Weyl tensor along the legs of a suitably chosen tetrad, proved to be sufficient; the value of the other perturbed Newman-Penrose scalars did not need to be known. Such might be expected since the Weyl tensor components ψ_0 and ψ_4 are invariant under both gauge transformations and infinitesimal tetrad rotations. These scalars carry information in their real and imaginary parts about the two dynamical degrees of freedom of the perturbed field. In fact, Wald[11] has shown that well-behaved solutions to either the ψ_0 or the ψ_4 equation uniquely and completely specify a gravitational perturbation up to changes of the Kerr parameters a and M.

That either ψ_0 or ψ_4 determines the perturbation in full does not necessarily mean that full information about the perturbation is readily accessible from these scalars. Off hand, it would appear improbable that the entire perturbed Riemann tensor can be obtained with ease from ψ_0 or ψ_4, and it would appear even less likely that there is an easy way to find the perturbed metric given any or all the ψ's since the Riemann tensor must, in principle, be twice integrated to obtain the metric.

The purpose of this paper is to spell out in some detail an unexpected result that the perturbed metric potentials $h_{\mu\nu}$ given by

$$g_{\mu\nu} = g_{\mu\nu}^{(\mathrm{Kerr})} + h_{\mu\nu} \tag{1.1}$$

in fact are directly accessible from ψ_0 and ψ_4 by twice differentiating a particular combination of separated Teukolsky functions. The homogeneous (source-free) potentials $h_{\mu\nu}$ thus obtained are presented in two different gauges in Table I.

This construction of the perturbed Kerr metric potentials, together with the Teukolsky equations and Wald's result, completes the picture of Kerr metric perturbations. Any desired perturbed quantity may be found from the metric, which, in turn, may be determined from the separable Teukolsky functions as shown in Table I.

For many calculations of interest, however, this result is only of academic interest since formulas for the energy and angular momentum fluxes at infinity and at the horizon involve ψ_0 or ψ_4 without need of the perturbed metric. Notable exceptions to this general rule, that ψ_0 and ψ_4 are sufficient for computational purposes, lie in the areas of stationary perturbations and quantum processes in the neighborhood of black holes. In the first case, the potentials may be used to find the vectoral change in the angular momentum of a black hole when a nonaxially symmetric stationary perturbation is present. The results will be presented elsewhere together with an investigation of the properties of the perturbed event horizon. As for the study of quantum processes, the electromagnetic and gravitational

TABLE I. Homogeneous potentials.

Ingoing radiation gauge: $A_\mu l^\mu = h_{\mu\nu} l^\nu = h_\mu{}^\mu = 0$

$$A_\mu(x, lm\omega P = \pm) = [-l_\mu(\delta^* + 2\beta^* + \tau^*) + m_\mu^*(D + 2\epsilon^* + \rho^*)]\, {}_{-1}\tilde{R}_{lm\omega}(r)\, {}_{+1}Z^\omega_{lm}(\theta,\phi) e^{-i\omega\tau}$$
$$\pm [-l_\mu(\delta + 2\beta + \tau) + m_\mu(D + 2\epsilon + \rho)]\, {}_{-1}\tilde{R}_{lm\omega}(r)\, {}_{-1}Z^\omega_{lm}(\theta,\phi) e^{-i\omega\tau}$$

$$h_{\mu\nu}(x, lm\omega P = \pm) = \{-l_\mu l_\nu(\delta^* + \alpha + 3\beta^* - \tau^*)(\delta^* + 4\beta^* + 3\tau^*) - m_\mu^* m_\nu^*(D - \rho^* + 3\epsilon^* - \epsilon)(D + 3\rho^* + 4\epsilon^*)$$
$$+ l_{(\mu} m^*_{\nu)}[(D + \rho - \rho^* + \epsilon + 3\epsilon^*)(\delta^* + 4\beta^* + 3\tau^*) + (\delta^* + 3\beta^* - \alpha - \pi - \tau^*)(D + 3\rho^* + 4\epsilon^*)]\}$$
$$\times\, {}_{-2}\tilde{R}_{lm\omega}(r)\, {}_{+2}Z^\omega_{lm}(\theta,\phi) e^{-i\omega\tau}$$
$$\pm \{-l_\mu l_\nu(\delta + \alpha^* + 3\beta - \tau)(\delta + 4\beta + 3\tau) - m_\mu m_\nu(D - \rho + 3\epsilon - \epsilon^*)(D + 3\rho + 4\epsilon)$$
$$+ l_{(\mu} m_{\nu)}[(D + \rho^* - \rho + \epsilon^* + 3\epsilon)(\delta + 4\beta + 3\tau) + (\delta + 3\beta - \alpha^* - \pi^* - \tau)(D + 3\rho + 4\epsilon)]\}$$
$$\times\, {}_{-2}\tilde{R}_{lm\omega}(r)\, {}_{-2}Z^\omega_{lm}(\theta,\phi) e^{-i\omega\tau}$$

Outgoing radiation gauge: $A_\mu n^\mu = h_{\mu\nu} n^\nu = h_\mu{}^\mu = 0$

$$A_\mu(x, lm\omega P = \pm) = \rho^{*-2}[n_\mu(\delta + \pi^* - 2\alpha^*) - m_\mu(\Delta + \mu^* - 2\gamma^*)]\, {}_{+1}\tilde{R}_{lm\omega}(r)\, {}_{-1}Z^\omega_{lm}(\theta,\phi) e^{-i\omega\tau}$$
$$\pm \rho^{-2}[n_\mu(\delta^* + \pi - 2\alpha) - m_\mu^*(\Delta + \mu - 2\gamma)]\, {}_{+1}\tilde{R}_{lm\omega}(r)\, {}_{+1}Z^\omega_{lm}(\theta,\phi) e^{-i\omega\tau}$$

$$h_{\mu\nu}(x, lm\omega P = \pm) = \rho^{*-4}\{-n_\mu n_\nu(\delta - 3\alpha^* - \beta + 5\pi^*)(\delta - 4\alpha^* + \pi^*) - m_\mu m_\nu(\Delta + 5\mu^* - 3\gamma^* + \gamma)(\Delta + \mu^* - 4\gamma^*)$$
$$+ n_{(\mu} m_{\nu)}[(\delta + 5\pi^* + \beta - 3\alpha^* + \tau)(\Delta + \mu^* - 4\gamma^*) + (\Delta + 5\mu^* - \mu - 3\gamma^* - \gamma)(\delta - 4\alpha^* + \pi^*)]\}$$
$$\times\, {}_{+2}\tilde{R}_{lm\omega}(r)\, {}_{-2}Z^\omega_{lm}(\theta,\phi) e^{-i\omega\tau}$$
$$\pm \rho^{-4}\{-n_\mu n_\nu(\delta^* - 3\alpha - \beta^* + 5\pi)(\delta^* - 4\alpha + \pi) - m_\mu^* m_\nu^*(\Delta + 5\mu - 3\gamma + \gamma^*)(\Delta + \mu - 4\gamma)$$
$$+ n_{(\mu} m^*_{\nu)}[(\delta^* + 5\pi + \beta^* - 3\alpha + \tau^*)(\Delta + \mu - 4\gamma) + (\Delta + 5\mu - \mu^* - 3\gamma - \gamma^*)(\delta^* - 4\alpha + \pi)]\}$$
$$\times\, {}_{+2}\tilde{R}_{lm\omega}(r)\, {}_{+2}Z^\omega_{lm}(\theta,\phi) e^{-i\omega\tau}$$

potentials are crucial to extending the analysis of Unruh[12] to second quantization of electromagnetic and gravitational test fields in a Kerr background. The result, most expectedly, is that the photon and graviton spontaneous-emission formulas for the flux at infinity are the classical superradiant scattering formulas of Starobinsky and Churilov[6] and Teukolsky and Press.[7] In addition, these potentials may be used to repeat for higher-spin radiation fields the calculation of Hawking[13] to find the steady-state spectrum of electromagnetic and gravitational radiation resulting from a realistic collapse situation.

In general, however, the conceptual benefits of having found the perturbed Kerr metric potentials surpass the usefulness of these potentials for doing future computations. For one, the discussion in this paper reemphasizes the important lesson learned by Chrzanowski and Misner[14]: The amplitude of the $lm\omega P$ mode (P = polarization state) of the field generated by an arbitrary perturbative source with stress-energy $T^{\alpha\beta}$ depends on the strength of the direct coupling of $T^{\alpha\beta}$ to a solution of the homogeneous equation for the $lm\omega P$ mode. Specifically, what one finds for the gravitational potential generated outside of a bounded source is

$$h_{\mu\nu} = \int_{-\infty}^{\infty} d\omega \sum_{l,m} \sum_P \frac{2i\omega}{|\omega|} h^{\text{up}}_{\mu\nu}(lm\omega P) \times \langle h^{\text{out}}_{\alpha\beta}(lm\omega P), T^{\alpha\beta}\rangle, \qquad (1.2)$$

where the gauge-invariant inner product is defined by

$$\langle h_{\alpha\beta}, T^{\alpha\beta}\rangle = \int \sqrt{-g}\, d^4x\, h^*_{\alpha\beta} T^{\alpha\beta}, \qquad (1.3)$$

and "up" and "out" label properly normalized [see (5.6) for normalization of the radial function] homogeneous solutions which have the property that they vanish, respectively, at past null infinity and on the future horizon.

Equation (1.2) and an analogous expression valid inside the source radius lead to the particularly simple energy flux formulas

$$E = \sum_{l,m} \sum_P \int_0^\infty d\omega\, \omega |\langle h^{\text{out}}_{\alpha\beta}(lm\omega P), T^{\alpha\beta}\rangle|^2 \qquad (1.4)$$

near infinity and [with k defined in (5.21)]

$$E = \sum_{l,m} \sum_P \int_0^\infty d\omega \frac{k}{|k|} \omega |\langle h^{\text{down}}_{\alpha\beta}(lm\omega P), T^{\alpha\beta}\rangle|^2 \qquad (1.5)$$

near the horizon. The normalization of the "down" homogeneous solution, which vanishes near null future infinity, is given by (5.18). One sees that the energy radiated involves a sum over a complete set of modes; the source field coupling term $|\langle\,,\,\rangle|^2$ is the number of gravitons in a given mode of frequency ω. Equation (1.4) is mathematically equivalent to the formula of Teukolsky[1]

$$E=\lim_{r\to\infty}\int\frac{r^2}{4\pi\omega^2}|\psi_4|^2dt\,d\Omega, \tag{1.6}$$

but the former is far more suggestive as to how the perturbing source couples to the perturbed field in the physical process of the generation of radiation.

Also conceptually important is the fact that the gravitational potentials, and hence the perturbed Riemann tensor components, are found by differentiating a single scalar function ψ. In particular, in the gauge $h_{\alpha\beta}l^\alpha=h_\alpha{}^\alpha=0$, ψ is given by (see Sec. VI)

$$\psi=\int d\omega\sum_{l,m}{}_{-2}R_{lm\omega}(r)\,{}_{+2}S^{\omega}_{lm}(\theta)e^{im\varphi-i\omega t}$$
$$=[\rho^{-4}\psi_4]^*. \tag{1.7}$$

This represents a generalization of recent work by Cohen and Kegeles,[15] who find a Debye potential φ for electromagnetic perturbations of all algebraically special solutions. For the special case of the Kerr metric, ϕ satisfies $\varphi=[\rho^{-2}\,\phi_2]^*$. Clearly, for a Kerr background, ψ acts as a gravitational Debye potential, an object for which a general theory has not been developed. The results presented here, then, provide a foundation for future investigations into the subject of Debye potentials for gravitational fields.

There is yet another benefit arising from this derivation of the Kerr metric potentials. The Schwarzschild limits of the expressions in Table I lead to differential relationships (see Table III) between the $a=0$ Teukolsky functions and the radial functions which arise from studies[16-19] of metric perturbations of the Schwarzschild solution. Hence, this work serves as a link between these two distinct methods that have been used to investigate black hole perturbations.

In the succeeding sections, the perturbed metric potentials are derived and their basic properties are discussed. Specifically, what are considered in the second section are the two basic inputs into the derivation: the inhomogeneous Teukolsky equations and an assumed form, first used by Chrzanowski and Misner,[14] for Green's functions for the perturbed metric. Some justification for the assumption is given in this section and in Appendix A, where it is shown that the assumed form for the Green's function is valid at least in the Schwarzschild geometry.

The Green's functions and the Teukolsky equations are then used in Sec. III to derive the potentials listed in Table I. Verification that these formulas are correct is relegated to Appendixes B and C.

Following a discussion of the basic properties of the homogeneous potentials in Sec. IV, the inhomogeneous potentials are studied in Sec. V. Retarded Green's functions are constructed and the previously quoted formulas for the observed energy flux at infinity and at the horizon are derived. Angular momentum flux formulas also are found.

In the last section, the electromagnetic results are compared with the work of Cohen and Kegeles.[15] This leads to a speculation that the gravitational formulas are more general than their derivation warrants. It is suggested that the expressions for the perturbed Kerr metric, with minor modification, are valid for perturbations of any algebraically special vacuum solution.

II. FACTORIZED GREEN'S FUNCTIONS AND THE TEUKOLSKY EQUATIONS

The two inputs needed to find the metric potentials are the Teukolsky equations with sources and an assumption of the existence of factorized Green's functions for the perturbed potentials. For the purpose of establishing the notation to be used throughout the paper, the pertinent work of Teukolsky,[1] Teukolsky and Press,[7] and Chrzanowski and Misner[14] on these topics is reviewed here.

The Teukolsky equations are decoupled, separable differential equations for certain components of the perturbed Riemann tensor invariant under gauge transformations and infinitesimal tetrad rotations. Specifically, in an appropriately chosen tetrad, the equations for the Newman-Penrose quantities ψ_0 and ψ_4 (φ_0 and ϕ_2 for electromagnetic perturbations) are known to decouple, and the equations for

$$\Omega_s=\begin{cases}\rho^{-4}\psi_4, & s=-2\\ \rho^{-2}\phi_2, & s=-1\\ \Phi\ \text{(scalar field)}, & s=0\\ \phi_0, & s=1\\ \psi_0, & s=2\end{cases} \tag{2.1}$$

separate as follows:

$$\Omega_s=\sum\Omega_s(lm\omega)$$
$$=\sum{}_s\tilde{R}_{lm\omega}(r)\,{}_sZ^{\omega}_{lm}(\theta,\phi)e^{-i\omega\tau}, \tag{2.2}$$

where (ϕ, τ) is any one of the coordinate pairs (φ, t), $(\bar{\varphi}, u)$, or $(\tilde{\varphi}, v)$. The pair used depends on whether the perturbation is studied in, respectively, the Boyer-Lindquist, Kerr "outgoing," or Kerr "ingoing" coordinate system.[20] Throughout, the symbol $\sum$ will denote a sum over l and m (and polarization states P, where applicable) and an integral over all frequencies $(-\infty, \infty)$.

The angular functions in (2.2), normalized to give

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, {}_sZ_{lm}^{\omega}(\theta, \varphi)^* \, {}_sZ_{lm}^{\omega}(\theta, \phi) = 1\,, \tag{2.3}$$

are defined by

$${}_sZ_{lm}^{\omega}(\theta, \varphi) = (2\pi)^{-1/2} \, {}_sS_{lm}^{\omega}(\theta) e^{im\phi}, \tag{2.4}$$

with ${}_sS_{lm}^{\omega}(\theta)$ the solution to

$$\frac{1}{\sin\theta}\left(\frac{d}{d\theta}\sin\theta\frac{dS}{d\theta}\right) + \left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + E - s^2\right) S = 0, \tag{2.5}$$

which is regular on the interval $[0, \pi]$. Two useful symmetry relations,

$${}_sZ_{lm}^{\omega}(\pi - \theta, \phi + \pi) = {}_{-s}Z_{lm}^{\omega}(\theta, \phi), \tag{2.6}$$

$${}_sZ_{lm}^{\omega}(\theta, \varphi)^* = (-)^m \, {}_{-s}Z_{l-m}^{-\omega}(\theta, \phi), \tag{2.7}$$

follow from angular Eq. (2.5); the phase convention is chosen to agree with the spherical harmonic phase convention in the appropriate limit.

The differential equation satisfied by ${}_s\tilde{R}(r)$ depends both on the choice of tetrad and the choice of coordinate system. Written here in Boyer-Lindquist $[t, r, \theta, \varphi]$ coordinates, two commonly used tetrads are the Kinnersley tetrad[21]

$$\begin{aligned} l^{\alpha} &= [(r^2 + a^2)/\Delta_K, 1, 0, a/\Delta_K], \\ n^{\alpha} &= [r^2 + a^2, -\Delta_K, 0, a]/2\Sigma, \\ m^{\alpha} &= [ia\sin\theta, 0, 1, i/\sin\theta]/\sqrt{2}(r + ia\cos\theta), \end{aligned} \tag{2.8}$$

which is regular on the past horizon, and a tetrad with legs regular on the future horizon:

$$\begin{aligned} l^{\alpha} &= [-(r^2 + a^2)/\Delta_K, 1, 0, -a/\Delta_K], \\ n^{\alpha} &= [-(r^2 + a^2), -\Delta_K, 0, -a]/2\Sigma, \\ m^{\alpha} &= [-ia\sin\theta, 0, 1, -i/\sin\theta]/\sqrt{2}(r + ia\cos\theta). \end{aligned} \tag{2.9}$$

Here and throughout the notation $\Delta_K = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$ is used. The radial function ${}_s\tilde{R}(r)$ is then

$${}_s\tilde{R}(r) = \begin{cases} {}_sR(r), & \text{tetrad (2.8)}, \\ {}_sR(r)^*, & \text{tetrad (2.9)}, \end{cases} \tag{2.10}$$

where ${}_sR(r)$ satisfies the differential equation

$$\left[\Delta_K \frac{d^2}{dr^2} + 2(s+1)(r - M)\frac{d}{dr} + \frac{K^2 - 2is(r-M)K}{\Delta_K} + 4ir\omega s - \lambda\right]{}_sR = -T \tag{2.11}$$

when $(\phi, \tau) = (\varphi, t)$, the equation

$$\left(\Delta_K \frac{d^2}{dr^2} + 2[(s+1)(r - M) + iK]\frac{d}{dr} + 2(2s+1)i\omega r - \lambda\right){}_sR = -T \tag{2.12}$$

when $(\varphi, \tau) = (\bar{\varphi}, u)$, or the equation

$$\left(\Delta_K \frac{d^2}{dr^2} + 2[(s+1)(r - M) - iK]\frac{d}{dr} - \frac{4is(r-M)K}{\Delta_K} - 2(2s+1)i\omega r - \lambda\right){}_sR = -T \tag{2.13}$$

when $(\phi, \tau) = (\tilde{\varphi}, v)$. The eigenvalue λ is related to the angular equation eigenvalue by $\lambda = E - 2am\omega + a^2\omega^2 - s - s^2$, and $K = (r^2 + a^2)\omega - am$. The source term T is

$$T = \begin{cases} 2\int d\Omega\, d\tau\, \Sigma_s \boldsymbol{T}\, {}_sZ_{lm}^{\omega}(\theta, \phi)^* e^{i\omega\tau}, & \text{tetrad (2.8)}, \\ 2\int d\Omega\, d\tau\, \Sigma_s T^* \, {}_sZ_{lm}^{\omega}(\theta, \phi) e^{-i\omega\tau}, & \text{tetrad (2.9)}, \end{cases} \tag{2.14}$$

with ${}_sT$ given in Table II. The above differential equations admit the symmetries

$${}_sR_{lm\omega}(r) = (-)^m \, {}_sR_{l-m-\omega}(r)^*, \tag{2.15}$$

$${}_sR_{lm\omega}(r)^* = (1/\Delta_K)^s \, {}_{-s}R_{lm\omega}(r), \tag{2.16}$$

both of which prove to be useful.

Taken together with the inhomogeneous Teukolsky equations, the crucial idea which enables the expressions for the potentials to be derived is the

TABLE II. Source terms.

$$ {}_2T = 2\{(\delta+\pi^*-\alpha^*-3\beta-4\tau)[(D-2\epsilon-2\rho^*)T_{lm}-(\delta+\pi^*-2\alpha^*-2\beta)T_{ll}] + (D-3\epsilon+\epsilon^*-4\rho-\rho^*)[(\delta+2\pi^*-2\beta)T_{lm}-(D-2\epsilon+2\epsilon^*-\rho^*)T_{mm}]\} $$

$$ {}_1T = (\delta-\beta-\alpha^*-2\tau+\pi^*)J_l-(D-\epsilon+\epsilon^*-2\rho-\rho^*)J_m $$

$$ {}_0T = T^{\alpha}{}_{\alpha} $$

$$ {}_{-1}T = \rho^{-2}[(\Delta+\gamma-\gamma^*+2\mu+\mu^*)J_{m^*}-(\delta^*+\alpha+\beta^*+2\pi-\tau^*)J_n] $$

$$ {}_{-2}T = 2\rho^{-4}\{(\Delta+3\gamma-\gamma^*+4\mu+\mu^*)[(\delta^*-2\tau^*+2\alpha)T_{nm^*}-(\Delta+2\gamma-2\gamma^*+\mu^*)T_{m^*m^*}] + (\delta^*-\tau^*+\beta^*+3\alpha+4\pi)[(\Delta+2\gamma+2\mu^*)T_{nm^*}-(\delta^*-\tau^*+2\beta^*+2\alpha)T_{nn}]\} $$

concept of factorized Green's functions. The conjecture to be used as to the form taken by Green's functions for the perturbed electromagnetic and gravitational potentials is based on the result that

$$ G(x,x') = \begin{cases} \sum \frac{i\omega}{|\omega|}\Phi^{\rm up}(x,lm\omega)\Phi^{\rm out}(x',lm\omega)^*, & r(x)>r(x') \\ \sum \frac{i\omega}{|\omega|}\Phi^{\rm in}(x,lm\omega)\Phi^{\rm down}(x',lm\omega)^*, & r(x)<r(x') \end{cases} \tag{2.17} $$

can be shown[14] to be a retarded Green's function for test scalar fields in a Kerr background. Throughout this paper, the labels "in," "up," "out," and "down" refer to the global boundary conditions satisfied by the various scattering solutions. $\Phi^{\rm in}$ and $\Phi^{\rm up}$ are solutions to the homogeneous wave equation which, respectively, vanish on the past horizon and at past null infinity ($\mathscr{I}^-$). Similarly, $\Phi^{\rm out}$ and $\Phi^{\rm down}$ vanish, respectively, on the future horizon and at $\mathscr{I}^+$. (See Fig. 1 in Ref. 14.)

The homogeneous solutions $\Phi^{\rm up}$ and $\Phi^{\rm out}$ in (2.17) agree in amplitude and phase near $\mathscr{I}^+$ and satisfy the normalization condition

$$ \langle \Phi_{lm\omega}, \Phi_{l'm'\omega'}\rangle_{\mathscr{I}^+} = \frac{\omega}{|\omega|}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'), \tag{2.18} $$

where $\Phi_{lm\omega}$ is either the "up" or the "out" solution. An inner-product symbol subscripted with a surface (in this case $\mathscr{I}^+$) is not to be confused with the inner product to be defined by (2.21); this is the standard Klein-Gordon inner product. The fields $\Phi^{\rm in}$ and $\Phi^{\rm down}$ are normalized to give

$$ \langle \Phi^{\rm in}_{lm\omega}, \Phi^{\rm down}_{l'm'\omega'}\rangle_{\rm fh} = \frac{\omega}{|\omega|}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'). \tag{2.19} $$

Near the future horizon (fh) $\Phi^{\rm in}$ and $\Phi^{\rm down}$ agree in amplitude although they differ in phase [see (5.17)].

The above scalar Green's function may be used to find solutions to the inhomogeneous scalar wave equation $\Phi_{;\alpha}{}^{\alpha} = 4\pi T$, where T is a bounded source. Outside of the source radius, the field is

$$ \Phi(x) = \sum \frac{i\omega}{|\omega|}\Phi^{\rm up}(x,lm\omega)\langle \Phi^{\rm out}, T\rangle, \tag{2.20} $$

with

$$ \langle \Phi, T\rangle = \int \sqrt{-g}\, d^4x\, \Phi^* T. \tag{2.21} $$

Notice that $\Phi(x)$ is given by a sum over a complete set of scattering states $\Phi^{\rm up}_{lm\omega}$ satisfying the appropriate boundary conditions at spatial infinity; the amplitude of each state depends on the strength with which an associated state $\Phi^{\rm out}_{lm\omega}$ couples to the source of the waves.

With the knowledge that the above holds for scalar test fields in a Kerr background, Chrzanowski and Misner[14] conjecture that the most natural generalization of (2.17) holds for retarded Green's functions for test electromagnetic and gravitational potentials:

$$ G_{\mu\alpha}(x,x') = \begin{cases} \sum \frac{i\omega}{|\omega|} g^{PP'} A^{\rm up}_{\mu}(x,lm\omega P) A^{\rm out}_{\alpha}(x',lm\omega P')^*, & r(x)>r(x') \\ \sum \frac{i\omega}{|\omega|} g^{PP'} A^{\rm in}_{\mu}(x,lm\omega P) A^{\rm down}_{\alpha}(x',lm\omega P')^*, & r(x)<r(x') \end{cases} \tag{2.22} $$

$$ G_{\mu\nu\alpha\beta}(x,x') = \begin{cases} \sum \frac{i\omega}{|\omega|} g^{PP'} \bar{h}^{\rm up}_{\mu\nu}(x,lm\omega P)\bar{h}^{\rm out}_{\alpha\beta}(x',lm\omega P')^*, & r(x)>(x') \\ \sum \frac{i\omega}{|\omega|} g^{PP'} \bar{h}^{\rm in}_{\mu\nu}(x,lm\omega P)\bar{h}^{\rm down}_{\alpha\beta}(x',lm\omega P)^*, & r(x)<r(x') \end{cases} \tag{2.23} $$

with $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}h$. As before, the various scattering solutions are superscripted with the boundary condition satisfied and are normalized via Klein-Gordon inner products to give

$$\langle \psi_\alpha^{\rm up}(lm\omega P), \psi^{\alpha{\rm out}}(l'm'\omega'P')\rangle_{\mathfrak{g}^+} = \frac{\omega}{|\omega|} g_{PP'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'), \tag{2.24}$$

$$\langle \psi_\alpha^{\rm in}(lm\omega P), \psi^{\alpha{\rm down}}(l'm'\omega'P')\rangle_{\mathfrak{h}} = \frac{\omega}{|\omega|} g_{PP'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'),$$

where ψ_α denotes either of the homogeneous potentials A_μ or $\bar{h}_{\mu\nu}$. [As shown in Ref. 14, these normalization conditions follow from (2.22) and (2.23); they need not be conjectured separately.] Included in the symbol $\sum$ are summations of P and P' over the two linearly independent physical polarization states. The quantity $g_{PP'}$ is a two-dimensional metric for the polarization states proportional to $\delta_{PP'}$ should the states be chosen to be orthogonal.

The potentials generated by a bounded, arbitrary perturbative current J^α or stress-energy $T^{\alpha\beta}$, then, are given by

$$A_\mu = \sum \frac{i\omega}{|\omega|} g^{PP'} A_\mu^{\rm up}(lm\omega P)\langle A_\alpha^{\rm out}(lm\omega P'), J^\alpha\rangle,$$
$$\bar{h}_{\mu\nu} = \sum \frac{i\omega}{|\omega|} g^{PP'} \bar{h}_{\mu\nu}^{\rm up}(lm\omega P)\langle \bar{h}_{\alpha\beta}^{\rm out}(lm\omega P'), 4T^{\alpha\beta}\rangle \tag{2.25}$$

at radii $r(x)$ outside of which the source term vanishes. The factor of 4 in the gravitational case is due to the fact that $16\pi T^{\alpha\beta}$ is the source term in the gravitational perturbation equation, whereas $4\pi T$ and $4\pi J^\alpha$ appear in the other equations.

The above formulas, together with appropriate expressions for the stress-energy tensor of the generated field, may be used to find that[14]

$$E = \sum_{\omega>0} \omega\, |\langle A_\alpha^{\rm out}(lm\omega P), J^\alpha\rangle|^2, \tag{2.26}$$

$$E = \sum_{\omega>0} \omega\, |\langle h_{\alpha\beta}^{\rm out}(lm\omega P), T^{\alpha\beta}\rangle|^2 \tag{2.27}$$

are, respectively, the electromagnetic and gravitation radiation energy flux at infinity provided that the electromagnetic (gravitational) polarization states have been normalized to give $g_{PP'} = -\delta_{PP'}$ ($g_{PP'} = 2\delta_{PP'}$). The notation "$\omega > 0$" signifies that the integral over frequencies in the mode sum is restricted to positive frequencies.

In addition to the aforementioned fact that the Green's function takes a factorized form for scalar perturbations of Kerr, there is other evidence in support of the conjecture of factorized electromagnetic and gravitational potential Green's functions. The successful treatment,[14] employing (2.26) and (2.27), of radiation emission by test charges and test particles in ultrarelativistic circular orbits about a Kerr black hole indicates that the Green's functions are valid at least in the high-frequency limit. More importantly, the existence of these factorized Green's functions for electromagnetic and gravitational potentials can be established for perturbations of a nonrotating black hole. This is done in Appendix A. Hence, the factorized Green's functions are known to exist for the Schwarzschild geometry, for scalar fields in the Kerr case, and for high-frequency electromagnetic and gravitational test fields in the Kerr case.

That the Green's functions take the form indicated in (2.22) and (2.23) seems well justified but has not been proved at this stage. The ensuing derivation of the perturbed potentials relies on the existence of factorized Green's functions, so the validity of the resulting expressions for the potentials must be checked. Once the derived potentials have been verified as correct, (2.22) and (2.23) are established as valid Green's functions and can be used in future computations.

III. DERIVATION OF THE PERTURBED POTENTIALS

To obtain the expressions in Table I for the perturbed potentials, one first finds a formula for the perturbed field component Ω_s [see (2.1)] generated by an arbitrary perturbative source by using the factorized Green's functions. The inhomogeneous Teukolsky equations lead to an alternative expression for Ω_s, which, when compared with the first formula, yields the results displayed in Table I.

Consider, then, (2.25) which give at large radii the potentials generated by a bounded source. They follow from the assumption of factorized Green's functions and may be written in the compact form

$$P_\alpha(x) = \sum \frac{i\omega}{|\omega|} g^{PP'} P_\alpha^{\rm up}(x, lm\omega P)\langle P_\beta^{\rm out}(lm\omega P'), S^\beta\rangle, \tag{3.1}$$

where $P_\alpha = \Phi, A_\mu, \bar{h}_{\mu\nu}$ is one of the three perturbed potentials generated by the appropriate source term $S_\alpha = T, J_\mu, 4T_{\mu\nu}$. Of course, in the scalar case one has $g^{PP'} = 1$.

Now act on $P_\alpha(x)$ with the operator ${}_sD^\alpha$ which, by definition, extracts the Ω_s field component from the potential. The exact form of ${}_sD^\alpha$, although not needed, may be deduced from the results of Appendix B; the operator ${}_2D^{\mu\nu}$, for example, is given by

$$ {}_2\mathcal{D}^{\mu\nu} = -\tfrac{1}{2}\{(D-\rho^*-3\epsilon+\epsilon^*)(D-\rho^*-2\epsilon+2\epsilon^*)m^\mu m^\nu + (\delta+\pi^*-3\beta-\alpha^*)(\delta+\pi^*-2\beta-2\alpha^*)l^\mu l^\nu - [(D-\rho^*-3\epsilon+\epsilon^*)(\delta+2\pi^*-2\beta)+(\delta+\pi^*-3\beta-\alpha^*)(D-2\rho^*-2\epsilon)]l^{(\mu}m^{\nu)}\}. \quad (3.2) $$

(For the scalar field ${}_0D=1$.) The result of this operation is

$$ \Omega_s = \sum \frac{i\omega}{|\omega|} g^{PP'}\, {}_sD^\alpha P^{\rm up}_\alpha(lm\omega P)\langle P^{\rm out}_\beta(lm\omega P'), S^\beta\rangle. \quad (3.3) $$

The quantity ${}_sD^\alpha P^{\rm up}_\alpha(lm\omega P)$ appearing in the above equation is known to be a solution to the equation for Ω_s. In fact, it is convenient to define the $lm\omega$ mode of the potential to be that which obeys

$$ {}_sD^\alpha P^{\rm up}_\alpha(lm\omega P) = \lambda(P)\Omega^{\rm up}_s(lm\omega), \quad (3.4) $$

with $\lambda(P)$ an amplitude factor. As given in (2.2), $\Omega_s(lm\omega)$ is a separable component of Ω_s, here suitably normalized and labeled by the boundary condition satisfied by the field function. Combining (3.3) and (3.4) with the definition

$$ P^{\rm out}_\alpha(lm\omega) = \sum_{P,P'} \lambda(P)^* g^{PP'} P^{\rm out}_\alpha(lm\omega P'), \quad (3.5) $$

one obtains a formula

$$ \Omega_s = \sum \frac{i\omega}{|\omega|}\Omega^{\rm up}_s(lm\omega)\langle P^{\rm out}_\beta(lm\omega), S^\beta\rangle \quad (3.6) $$

to be compared with a second expression for Ω_s derived from the Teukolsky equations.

The decoupled Teukolsky equation for Ω_s, after a suitable choice of tetrad, reduces to radial equation (2.11), (2.12), or (2.13) depending on the coordinate system and angular equation (2.5). With the definitions

$$ \frac{dr^*}{dr} = \frac{r^2+a^2}{\Delta_K}, \quad u(r) = {}_sR(r)[(r^2+a^2)\Delta_K{}^s]^{1/2} \quad (3.7) $$

the radial equation becomes

$$ -\frac{d^2u}{dr^{*2}} + V_s(r^*)u = (r^2+a^2)^{-3/2}\Delta_K{}^{1+s/2}T, \quad (3.8) $$

where T is given by (2.14) and the precise form of $V_s(r^*)$ depends on the coordinate system and tetrad chosen.

Equation (3.8) may be solved by method of radial Green's functions, a procedure discussed at length in Refs. 14 and 22. The result is that the Green's function

$$ \left[-\frac{d^2}{dr^{*2}} + V_s(r^*)\right]G(r^*, r_0^*) = \delta(r^*, r_0^*) \quad (3.9) $$

is

$$ G(r^*, r_0^*) = \frac{i\omega}{2|\omega|} \times \begin{cases} u^{\rm in}(r_0^*)u^{\rm up}(r^*), & r^* > r_0^* \\ u^{\rm up}(r_0^*)u^{\rm in}(r^*), & r^* < r_0^*. \end{cases} \quad (3.10) $$

The labels "in" and "up" on the radial functions refer to the boundary conditions satisfied by the field quantities

$$ \Omega_s = \sum {}_s\tilde{R}(r)\,{}_sZ(\theta,\phi)e^{-i\omega\tau} \quad (3.11) $$

constructed out of source-free solutions to (3.8). The normalization condition imposed on these solutions to (3.8), with the numerical factors that appear in (3.10), is not given here since it is not needed; moreover, spurious overall numerical factors will be discarded in the remaining steps since they just affect the amplitude of the homogeneous potentials to be derived.

Equations (3.7) through (3.10) and (2.14) combine to give

$$ {}_s\tilde{R}(r) = \frac{i\omega}{|\omega|}\,{}_s\tilde{R}^{\rm up}(r)\int\sqrt{-g}\,d^4x\,{}_sT\Delta_K{}^s\,{}_s\tilde{R}^{\rm in}\,{}_sZ^*e^{i\omega\tau} \quad (3.12) $$

outside of a bounded source, so the generated field is

$$ \Omega_s = \sum \frac{i\omega}{|\omega|}\Omega^{\rm up}_s(lm\omega) \times\int\sqrt{-g}\,d^4x\,{}_sT\Delta_K{}^s\,{}_s\tilde{R}^{\rm in}\,{}_sZ^*e^{i\omega\tau}, \quad (3.13) $$

or, with the aid of (2.16), one has

$$ \Omega_s = \sum \frac{i\omega}{|\omega|}\Omega^{\rm up}_s(lm\omega) \times\int\sqrt{-g}\,d^4x[{}_{-s}\tilde{R}^{\rm out}\,{}_sZ\,e^{-i\omega\tau}]^*\,{}_sT. \quad (3.14) $$

To obtain from this an expression for Ω_s comparable with (3.6), the inner product between the source term and the test field must be integrated by parts; as written, derivatives of the source term appear rather than the source term itself.

Consider, for definiteness, the case $s=2$ for which the inner product in (3.14) is

$$ \int\sqrt{-g}\,d^4x[{}_{-2}\tilde{R}^{\rm out}\,{}_2Z\,e^{-i\omega\tau}]^*\,{}_2T = \langle {}_{-2}\tilde{R}^{\rm out}\,{}_2Z\,e^{-i\omega\tau}, {}_2T\rangle, \quad (3.15) $$

where, from Table II,

$$ {}_2T=(\delta+\pi^*-\alpha^*-3\beta-4\tau)[(D-2\epsilon-2\rho^*)T_{lm}-(\delta+\pi^*-2\alpha^*-2\beta)T_{ll}] \\ +(D-3\epsilon+\epsilon^*-4\rho-\rho^*)[(\delta+2\pi^*-2\beta)T_{lm}-(D-2\epsilon+2\epsilon^*-\rho^*)T_{mm}]. \tag{3.16}$$

The T_{ll} term in the inner product, denoted by $\langle\ ,\ \rangle_{ll}$, is given by

$$\langle\ ,\ \rangle_{ll}=-\int\sqrt{-g}\,d^4x[{}_{-2}\tilde{R}^{\rm out}{}_2Z\,e^{-i\omega\tau}]^*(\delta+\pi^*-\alpha^*-3\beta-4\tau)(\delta+\pi^*-2\alpha^*-2\beta)T_{ll}. \tag{3.17}$$

Integration by parts yields

$$\langle\ ,\ \rangle_{ll}=-\int\sqrt{-g}\,d^4x\,T_{ll}(-\delta+\pi^*-2\alpha^*-2\beta-m^\lambda{}_{;\lambda})(-\delta+\pi^*-\alpha^*-3\beta-4\tau-m^\lambda{}_{;\lambda})[{}_{-2}\tilde{R}^{\rm out}{}_2Z\,e^{-i\omega\tau}]^*, \tag{3.18}$$

and, using the fact that

$$m^\lambda{}_{;\lambda}=\pi^*+\beta-\alpha^*-\tau, \tag{3.19}$$

one finds

$$\langle\ ,\ \rangle_{ll}=-\int\sqrt{-g}\,d^4x\,T_{ll}[(\delta^*+\alpha+3\beta^*-\tau^*)(\delta^*+4\beta^*+3\tau^*)\,{}_{-2}\tilde{R}^{\rm out}{}_2Z\,e^{-i\omega\tau}]^*. \tag{3.20}$$

Similarly, with the aid of the result

$$l^\lambda{}_{;\lambda}=\epsilon+\epsilon^*-\rho-\rho^*, \tag{3.21}$$

the T_{lm} and T_{mm} terms become

$$\langle\ ,\ \rangle_{lm}=\int\sqrt{-g}\,d^4x\;T_{lm}\{[(D+\rho-\rho^*+\epsilon+3\epsilon^*)(\delta^*+4\beta^*+3\tau^*) \\ +(\delta^*+3\beta^*-\alpha-\pi-\tau^*)(D+3\rho^*+4\epsilon^*)]\,{}_{-2}\tilde{R}^{\rm out}{}_2Ze^{-i\omega\tau}\}^*, \tag{3.22}$$

$$\langle\ ,\ \rangle_{mm}=\int\sqrt{-g}\,d^4x\,T_{mm}\{-(D-\rho^*+3\epsilon^*-\epsilon)(D+3\rho^*+4\epsilon^*)\,{}_{-2}\tilde{R}^{\rm out}{}_2Z\,e^{-i\omega\tau}\}^*. \tag{3.23}$$

Hence, the inner product in the equation for Ω_2 is

$$\langle{}_2X^{\rm out}_{\alpha\beta},T^{\alpha\beta}\rangle=\int\sqrt{-g}\,d^4x({}_2X^*_{nn}T_{ll}-{}_2X^*_{(nm)}T_{lm}+{}_2X^*_{mm}T_{mm}), \tag{3.24}$$

with

$$ {}_2X^{\rm out}_{\alpha\beta}=\{-l_\alpha l_\beta(\delta^*+\alpha+3\beta^*-\tau^*)(\delta^*+4\beta^*+3\tau^*)-m^*_\alpha m^*_\beta(D-\rho^*+3\epsilon^*-\epsilon)(D+3\rho^*+4\epsilon^*) \\ +l_{(\alpha}m^*_{\beta)}[(D+\rho-\rho^*+\epsilon+3\epsilon^*)(\delta^*+4\beta^*+3\tau^*)+(\delta^*+3\beta^*-\alpha-\pi-\tau^*)(D+3\rho^*+4\epsilon^*)]\}\,{}_{-2}\tilde{R}^{\rm out}{}_2Z\,e^{-i\omega\tau}. \tag{3.25}$$

Exactly the same procedures, together with the formula

$$n^\lambda{}_{;\lambda}=\mu+\mu^*-\gamma-\gamma^*, \tag{3.26}$$

may be used to find similar expressions in the other three cases. The general result is

$$\Omega_s=\sum\frac{i\omega}{|\omega|}\Omega^{\rm up}_s(lm\omega)\langle{}_sX^{\rm out}_\alpha,S^\alpha\rangle, \tag{3.27}$$

with

$$ {}_{-2}X^{\rm out}_{\mu\nu}=\rho^{*-4}\{-n_\mu n_\nu(\delta-3\alpha^*-\beta+5\pi^*)(\delta-4\alpha^*+\pi^*) \\ +n_{(\mu}m_{\nu)}[(\delta+5\pi^*+\beta-3\alpha^*+\tau)(\Delta+\mu^*-4\gamma^*)+(\Delta+5\mu^*-\mu-3\gamma^*-\gamma)(\delta-4\alpha^*+\pi^*)] \\ -m_\mu m_\nu(\Delta+5\mu^*-3\gamma^*+\gamma)(\Delta+\mu^*-4\gamma^*)\}\,{}_2\tilde{R}^{\rm out}{}_{-2}Z\,e^{-i\omega\tau}, \tag{3.28}$$

$$ {}_{-1}X^{\rm out}_\mu=\rho^{*-2}[-m_\mu(\Delta+\mu^*-2\gamma^*)+n_\mu(\delta+\pi^*-2\alpha^*)]\,{}_1\tilde{R}^{\rm out}{}_{-1}Z\,e^{-i\omega\tau}, \tag{3.29}$$

$$ {}_1X^{\rm out}_\mu=[-l_\mu(\delta^*+2\beta^*+\tau^*)+m^*_\mu(D+\rho^*+2\epsilon^*)]\,{}_{-1}\tilde{R}^{\rm out}{}_1Z\,e^{-i\omega\tau}, \tag{3.30}$$

and ${}_2X^{\rm out}_{\mu\nu}$ given by (3.24).

Comparison of (3.6) and (3.27) reveals that

$$\langle P^{\rm out}_\alpha(lm\omega), S^\alpha\rangle = \langle {}_sX^{\rm out}_\alpha(lm\omega), S^\alpha\rangle, \qquad (3.31)$$

and since S^α is arbitrary except for the fact that it is divergence-free, one has

$$A^{\rm out}_\alpha(lm\omega) = {}_{\pm 1}X^{\rm out}(lm\omega) + \nabla_\alpha \phi^{\pm 1}, \qquad (3.32)$$

$$\bar{h}^{\rm out}_{\alpha\beta}(lm\omega) = {}_{\pm 2}X^{\rm out}_{\alpha\beta}(lm\omega) + \nabla_{(\alpha}\xi^{\pm 2}_{\beta)}, \qquad (3.33)$$

with $\phi^{\pm 1}$ and $\xi^{\pm 2}_\alpha$ arbitrary functions. That is to say, the result of identifying (3.6) and (3.27) is

$$A^{\rm out}_\alpha(lm\omega) = {}_{\pm 1}X^{\rm out}_\alpha(lm\omega), \quad \bar{h}^{\rm out}_{\alpha\beta}(lm\omega) = {}_{\pm 2}X^{\rm out}_{\alpha\beta}(lm\omega) \qquad (3.34)$$

modulo gauge transformations. Henceforth, the label "out" will be dropped. Since the entire argument carries through using, for example, the advanced rather than the retarded Green's function, the above derivation is completely independent of the imposed boundary condition.

What remains to be done is to decompose the potentials given by (3.34) into the two linearly independent solutions which constitute the perturbed field. [Recall that $P_\alpha(lm\omega)$ is a composite potential formed in (3.5) by summing over the polarization states.] Since the form of the Kerr metric is invariant under the parity operation $\mathsf{P} = (\theta \to \pi - \theta, \phi \to \phi + \pi)$, the definite parity states $P_\alpha(lm\omega) \pm \mathsf{P}P_\alpha(lm\omega)$ are the obvious choice for the two linearly independent solutions for each $lm\omega$ mode.

Specifically, consider again the $s=2$ case as an example. The effect of the parity operation is

$$\mathsf{P}X = X, \quad \mathsf{P}Y = Y^*, \quad \mathsf{P}Z = -Z^*, \qquad (3.35)$$

with

$$X = l_\mu, n_\mu, D, \Delta,$$
$$Y = \epsilon, \rho, \mu, \gamma + \text{c.c. terms}, \qquad (3.36)$$
$$Z = m_\mu, \delta, \tau, \pi, \alpha, \beta + \text{c.c. terms}.$$

These relations together with (2.6) and (3.28) yield the solutions

$$h_{\mu\nu}(x, lm\omega P = \pm) = \bar{h}_{\mu\nu}(x, lm\omega P = \pm) = \bar{h}_{\mu\nu}(lm\omega) \pm \mathsf{P}\bar{h}_{\mu\nu}(lm\omega) \qquad (3.37)$$

given by the $h_{\mu\nu}l^\nu = h_\mu{}^\mu = 0$ gauge expressions in Table I. The results in the $s=-1, 1, -2$ cases are listed in Table I as the potentials in, respectively, the $A_n = 0$, $A_l = 0$, and $h_{\mu\nu}n^\nu = h_\mu{}^\mu = 0$ gauges.

Verification that these potentials are correct is presented in Appendix C.

IV. THE HOMOGENEOUS POTENTIALS

The homogeneous potentials listed in Table I merit some discussion before the inhomogeneous equations and Green's functions for the potentials are reconsidered in detail. To appreciate fully the expressions in Table I, it is worthwhile to examine the question of gauge conditions, to establish the reality conditions obeyed by the potentials, and to understand the relationship in the Schwarzschild limit between the metric perturbation radial functions and the radial functions found by Bardeen and Press.[23]

It is noteworthy that the electromagnetic potentials are not presented in the Lorentz gauge; rather, the two gauge conditions specified in Table I are the gauges "naturally selected" by deriving the potentials from the field functions ϕ_0 and ϕ_2. The potential in the $A_l = 0$ gauge, for example, was found in the previous section by integrating by parts the source term in the ϕ_0 equation and reading off the coefficients to J^α in the inner product. Since no J_n source terms appear in the ϕ_0 equation source term (see ${}_1T$ in Table II), the quantity A_l, which is the coefficient of the J_n term in the invariant inner product $\langle A^{\rm out}_\alpha, J^\alpha\rangle$, must vanish. This result that $A_l = 0$ is then identified as a gauge condition.

As noted, the $A_l = 0$ expressions follow from the equation for ϕ_0, the "ingoing" field component. In this gauge the potential is transverse at the future horizon and past infinity. Hence, $A_l = 0$ will be referred to as the ingoing radiation gauge and will be used to do computations in these asymptotic regions. Alternate expressions for the potentials follow from the "outgoing" field function ϕ_2; this outgoing gauge condition $A_n = 0$ is useful near $\mathscr{I}^+$ and the past horizon, where it is transverse. Similarly, the gravitational potentials are presented in trace-free ingoing and outgoing gauges.

Now consider more closely the actual expressions for the potentials to see that, within overall phase factors, they are in fact real. Summing over modes, one finds that the vector potential in the outgoing gauge is

$$A_\alpha = \sum \rho^{*-2}[n_\alpha(\delta + \pi^* - 2\alpha^*) - m_\alpha(\Delta + \mu^* - 2\gamma^*)]\, {}_{-1}\tilde{R}\, {}_1Z e^{-i\omega\tau}$$
$$\pm \sum \rho^{-2}[n_\alpha(\delta^* + \pi - 2\alpha) - m^*_\alpha(\Delta + \mu - 2\gamma)]\, {}_{-1}\tilde{R}\, {}_{-1}Z e^{-i\omega\tau}, \qquad (4.1)$$

or, with the aid of (2.6) and (2.15),

$$A_\alpha = \sum \{\rho^{*-2}[n_\alpha(\delta+\pi^*-2\alpha^*) - m_\alpha(\Delta+\mu^*-2\gamma^*)]\, {}_{-1}\tilde{R}\, {}_1Z e^{-i\omega\tau} \pm \text{c.c.}\}. \tag{4.2}$$

Hence, the vector potential is simply

$$A_\alpha(P=+=M) = 2\,\text{Re}\sum \rho^{*-2}[n_\alpha(\delta+\pi^*-2\alpha^*) - m_\alpha(\Delta+\mu^*-2\gamma^*)]\, {}_{-1}\tilde{R}\, {}_1Z e^{-i\omega\tau}, \tag{4.3}$$

$$A_\alpha(P=-=E) = 2i\,\text{Im}\sum \rho^{*-2}[n_\alpha(\delta+\pi^*-2\alpha^*) - m_\alpha(\Delta+\mu^*-2\gamma^*)]\, {}_{-1}\tilde{R}\, {}_1Z e^{-i\omega\tau}. \tag{4.4}$$

The notation E, M anticipates the result that in the Schwarzschild limit the "E" modes are the electric parity potentials and the "M" modes are of magnetic parity. Analogous results hold in the other cases, e.g., in the ingoing trace-free gauge

$$h_{\mu\nu}(P=+=E) = 2\,\text{Re}\sum {}_2X_{\mu\nu}\,,$$
$$h_{\mu\nu}(P=-=M) = 2i\,\text{Im}\sum {}_2X_{\mu\nu}\,, \tag{4.5}$$

with ${}_2X_{\mu\nu}$ given by (3.24).

The Schwarzschild limit of the above equations is most interesting in that it reveals the connection between the $a=0$ Teukolsky functions, first found by Bardeen and Press,[23] and the radial functions which appear in studies[16–19] of metric perturbations of the Schwarzschild solution. Relationships amongst the Bardeen-Press, Regge-Wheeler, and Zerilli radial functions have been found recently by Chandrasekhar[24] using other methods; what is sketched out here is a simple alternative approach to the subject based on the expressions in Table I and work done by Moncrief.[25]

The idea is the following: The combinations of derivatives of spin-weighted spherical harmonics that appear in the definite-parity solutions $h_{\mu\nu}(E)$ and $h_{\mu\nu}(M)$ are the tensor spherical harmonics,[16,26,27] and the coefficients to these harmonics are the metric perturbation radial functions expressed in terms of derivatives of ${}_{\pm2}R(r)$. As shown by Moncrief, there exist gauge-invariant combinations of the metric functions which satisfy Schrödinger-type equations

$$\left[-\frac{d^2}{dr^{*2}} - \omega^2 + V^{(E)}\right] R^{(E)}(r) = 0\,, \tag{4.6}$$

$$\left[-\frac{d^2}{dr^{*2}} - \omega^2 + V^{(M)}\right] R^{(M)}(r) = 0 \tag{4.7}$$

for, respectively, the electric and magnetic parity cases. (The effective potentials $V^{(E)}$ and $V^{(M)}$ are given in Ref. 19.) One may then find these gauge-invariant quantities $R^{(E)}$ and $R^{(M)}$ as functions of ${}_{\pm2}R$ by using the ingoing or outgoing gauge metric radial functions read off from Table I.

In the notation of Regge and Wheeler,[16] the "0θ" component of the metric perturbation, for example, is

$$h_{0\theta} = h_0^{(E)}(r)\frac{\partial Y}{\partial\theta}e^{-i\omega t} - h_0(r)\frac{1}{\sin\theta}\frac{\partial Y}{\partial\varphi}e^{-i\omega t}, \tag{4.8}$$

and from (4.5), one has that in the ingoing radiation gauge

$$\left.\begin{matrix} h_{0\theta}(E) \\ h_{0\theta}(M)\end{matrix}\right\} = \left\{\begin{matrix}\text{Re} \\ i\,\text{Im}\end{matrix}\right\} l_0 m_\theta^* \frac{\sqrt{2}}{r}\left(D_\omega - \frac{2}{r}\right) \times\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi} - 2\cot\theta\right) {}_{-2}R\, {}_2Z e^{-i\omega t}, \tag{4.9}$$

with

$$D_\omega = \left(1-\frac{2M}{r}\right)^{-1}\left(-i\omega + \frac{d}{dr^*}\right). \tag{4.10}$$

Here and for the remainder of this section the Boyer-Lindquist coordinates and the Kinnersley tetrad are used. Equation (4.9) reduces to

$$\left.\begin{matrix} h_{0\theta}(E) \\ h_{0\theta}(M)\end{matrix}\right\} = \left\{\begin{matrix}\text{Re} \\ i\,\text{Im}\end{matrix}\right\} \left(D_\omega - \frac{2}{r}\right) {}_{-2}R \left[\frac{(l+2)(l-1)}{l(l+1)}\right]^{1/2} \times\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right) Y e^{-i\omega t}, \tag{4.11}$$

so the radial functions may be identified as being

$$h_0^{(E)} = \left[\frac{(l+2)(l-1)}{l(l+1)}\right]^{1/2}\left(D_\omega - \frac{2}{r}\right) {}_{-2}R\,, \tag{4.12}$$

$$h_0 = -i\left[\frac{(l+2)(l-1)}{l(l+1)}\right]^{1/2}\left(D_\omega - \frac{2}{r}\right) {}_{-2}R\,. \tag{4.13}$$

Consideration of the other components of the metric leads to the identification of the remaining metric radial functions with the results shown in Table III.

Moncrief has shown[25] that the electric parity function

$$R^{(E)}=[(l+2)(l-1)]^{1/2}\left[\frac{4r(1-2M/r)^2k_2+l(l+1)k_1}{(l+2)(l-1)+6M/r}\right], \quad (4.14)$$

with

$$k_1=K+r\frac{dG}{dr^*}-\frac{2}{r}(1-2M/r)h_1^{(E)},$$
$$k_2=\tfrac{1}{2}(1-2M/r)^{-1}\left[H_2-(1-M/(r-2M))K-r\frac{dK}{dr}\right], \quad (4.15)$$

is both gauge invariant and satisfies the Zerilli equation (4.6). In the ingoing radiation gauge this reduces to

$$R^{(E)}=[(l+2)(l-1)]^{1/2}\times\left[rK-2(1-2M/r)\left(\frac{rH_2-l(l+1)h_1^{(E)}}{(l+2)(l-1)+6M/r}\right)\right]. \quad (4.16)$$

Similarly, for the magnetic parity case, the quantity

$$R^{(M)}=-\frac{1}{r}(1-2M/r)\left(h_1+\frac{1}{2}\frac{dh_2}{dr}-\frac{1}{r}h_2\right) \quad (4.17)$$

is gauge invariant and is a solution to (4.7). Equations (4.16) and (4.17) together with Table III allow one to construct $R^{(M)}$ and $R^{(E)}$ from solutions to the Bardeen-Press equation.

In a more thorough discussion of the metric perturbation and the Bardeen-Press functions, Chandrasekhar[24] finds remarkably simple first-order differential relationships among $R^{(E)}$, $R^{(M)}$, and ${}_{-2}R$. Similarly, (4.16) and (4.17) may be reduced to first-order equations by using the Bardeen-Press equation to eliminate higher-order derivatives of ${}_{-2}R$.

Schrödinger-type equations with short-ranged effective potentials have yet to be found for perturbations of the Kerr solution except for the special case of axially symmetric perturbations which has been investigated by Chandrasekhar and Detweiler.[28]

V. THE INHOMOGENEOUS POTENTIALS

With the verification in Appendix C that the potentials in Table I are valid, the homogeneous potentials may be used in turn to construct the Green's functions postulated by Chrzanowski and Misner. That is to say, for the Kerr metric, the potential Green's functions introduced in Sec. II are valid. The form taken by these Green's functions was presented in that section, and the expressions for the homogeneous potentials which appear in the Green's functions have been derived. All that remains to be done is to use (2.24) to normalize the radial functions in the various scattering solutions. In this section these normalizations are fixed, and energy and angular momentum flux formulas for the asymptotic regions $r^*\to\pm\infty$ are derived together with Green's functions for the special case of stationary perturbations. As in the previous section, only the Kinnersley tetrad and the Boyer-Lindquist coordinates will be considered.

Examine first the potentials generated outside of a bounded source. As presented in Sec. II, the inhomogeneous potentials are

$$A_\alpha=\sum\frac{-i\omega}{|\omega|}A_\alpha^{\rm up}(lm\omega P)\langle A_\beta^{\rm out}(lm\omega P),J^\beta\rangle, \quad (5.1)$$

$$h_{\alpha\beta}=\sum\frac{2i\omega}{|\omega|}h_{\alpha\beta}^{\rm up}(lm\omega P)\langle h_{\mu\nu}^{\rm out}(lm\omega P),T^{\mu\nu}\rangle, \quad (5.2)$$

where, according to (2.24), the "out" and "up" solutions have a common asymptotic form near $\mathscr{I}^+$ and are normalized to give

TABLE III. Schwarzschild perturbation radial functions (in ingoing radiation gauge).

$$h_0^{(E)}(r)=-(1-2M/r)h_1^{(E)}(r)=\left[\frac{(l-1)(l+2)}{l(l+1)}\right]^{1/2}\left(D_\omega-\frac{2}{r}\right){}_{-2}R(r)$$

$$G(r)=\frac{2}{l(l+1)}K(r)=\frac{-2}{[l(l+1)(l+2)(l-1)]^{1/2}}\left(D_\omega+\frac{1}{r}\right)\left(D_\omega-\frac{3}{r}\right){}_{-2}R(r)$$

$$H_0(r)=H_2(r)=-H_1(r)=\frac{-1}{r^2-2Mr}\,\frac{1}{[l(l+1)(l+2)(l-1)]^{1/2}}\,{}_{-2}R(r)$$

$$h_0(r)=-(1-2M/r)h_1(r)=-\left[\frac{(l-1)(l+2)}{l(l+1)}\right]^{1/2}\left(D_\omega-\frac{2}{r}\right)[i\,{}_{-2}R(r)]$$

$$h_2(r)=\frac{-2r^2}{[l(l+1)(l+2)(l-1)]^{1/2}}\left(D_\omega+\frac{1}{r}\right)\left(D_\omega-\frac{3}{r}\right)[i\,{}_{-2}R(r)]$$

$$\langle A_\alpha(lm\omega P), A^\alpha(l'm'\omega'P')\rangle_{\mathscr{I}^+} = \frac{-\omega}{|\omega|}\delta_{PP'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'),$$
$$\langle h_{\alpha\beta}(lm\omega P), h^{\alpha\beta}(l'm'\omega'P')\rangle_{\mathscr{I}^+} = \frac{2\omega}{|\omega|}\delta_{PP'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'). \tag{5.3}$$

The polarization metric has been chosen[29] to satisfy $g_{PP'} = -\delta_{PP'}$ in the electromagnetic case. In the gravitational case one has $g_{PP'} = 2\delta_{PP'}$, and the "bars" have been dropped since the gauge conditions used here are trace-free.

In the above expressions it is understood that the sums over l do not include the modes $l < s$, for the Teukolsky functions (and hence these potentials) are not defined when $l < s$. From the work of Wald[11] and Fackerell and Ipser,[30] it is known that these modes cannot be radiative; they carry information as to the total charge, mass, and angular momentum of the system. (See Appendix A for an analysis of the $l < 2$ modes for the Schwarzschild solution.) The $l < s$ potentials have not been found by the procedures discussed here, basically because one does not obtain the constants of integration of the system by determining the potentials through differentiation.

The outgoing gauge potentials are transverse near $\mathscr{I}^+$, so they are used in the construction of the Green's function outside of the source radius. Equations (5.3) become

$$\frac{1}{4\pi i}\int dt\, d\Omega\, r^2[A^{\alpha\,\text{out}}(l'm'\omega'P')^*\overleftrightarrow{\partial}_r A^{\text{up}}_\alpha(lm\omega P)] = -\frac{\omega}{|\omega|}\delta_{PP'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'), \tag{5.4}$$

$$\frac{1}{4\pi i}\int dt d\Omega\, r^2[h^{\alpha\beta\,\text{out}}(l'm'\omega'P')^*\overleftrightarrow{\partial}_r h^{\text{up}}_{\alpha\beta}(lm\omega P)] = \frac{2\omega}{|\omega|}\delta_{PP'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'). \tag{5.5}$$

By substituting the outgoing gauge potentials into these equations one finds that the Table I expressions are properly normalized provided that near infinity the $e^{i\omega r^*}$ piece of the radial functions obeys

$${}_sR(r)\sim(-i/\omega)^s\frac{1}{(|\omega|)^{1/2}}\frac{e^{i\omega r^*}}{r^{1+2s}}\sqrt{2}^{\,s-2} + (e^{-i\omega r^*}\text{ term for "out"}) \tag{5.6}$$

for $s = 1, 2$. Hence, the outgoing gauge expressions in Table I together with (5.1), (5.2), and (5.6) constitute Green's functions (valid outside the source radius) for electromagnetic and gravitational perturbations of the Kerr solution.

These Green's functions may be used to calculate the energy and angular momentum flux at infinity. What one uses are

$$E = \lim_{r\to\infty} r^2\int dt d\Omega\, T^r{}_0, \tag{5.7}$$

$$J = -\lim_{r\to\infty} r^2\int dt d\Omega\, T^r{}_\varphi, \tag{5.8}$$

where $T^{\mu\nu}$, in the gravitational case, is the Isaacson[31] effective stress-energy tensor. Substitution of (5.1) and (5.2) into (5.7) and (5.8) gives the previously quoted results for the energy flux

$$E = \sum_{\omega>0}\omega|\langle A^{\text{out}}_\alpha(lm\omega P), J^\alpha\rangle|^2, \tag{5.9}$$

$$E = \sum_{\omega>0}\omega|\langle h^{\text{out}}_{\alpha\beta}(lm\omega P), T^{\alpha\beta}\rangle|^2, \tag{5.10}$$

and

$$J = \sum_{\omega>0} m|\langle A^{\text{out}}_\alpha(lm\omega P), J^\alpha\rangle|^2, \tag{5.11}$$

$$J = \sum_{\omega>0} m|\langle h^{\text{out}}_{\alpha\beta}(lm\omega P), T^{\alpha\beta}\rangle|^2 \tag{5.12}$$

for the angular momentum flux. These formulas may be shown to be equivalent to the corresponding formulas of Teukolsky.[1]

Now consider the generated potentials near the horizon, i.e., inside the source radius. The appropriate formulas are

$$A_\alpha = \sum\frac{-i\omega}{|\omega|}A^{\text{in}}_\alpha(lm\omega P)\langle A^{\text{down}}_\beta(lm\omega P), J^\beta\rangle, \tag{5.13}$$

$$h_{\alpha\beta} = \sum\frac{2i\omega}{|\omega|}h^{\text{in}}_{\alpha\beta}(lm\omega P)\langle h^{\text{down}}_{\mu\nu}(lm\omega P), T^{\mu\nu}\rangle. \tag{5.14}$$

Here one works in the ingoing gauge and normalizes the "in" and "down" solutions to obey

$$\langle A^{\text{in}}_\alpha(lm\omega P), A^{\alpha\,\text{down}}(l'm'\omega'P')\rangle_{\text{fh}} = -\frac{\omega}{|\omega|}\delta_{PP'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'), \tag{5.15}$$

$$\langle h^{\text{in}}_{\alpha\beta}(lm\omega P), h^{\alpha\beta\,\text{down}}(l'm'\omega'P')\rangle_{\text{fh}} = 2\frac{\omega}{|\omega|}\delta_{PP'}\delta_{ll'}\delta_{mm'}\delta(\omega-\omega'), \tag{5.16}$$

and to agree in amplitude near the future horizon, although the e^{-ikr^*} terms differ in phase by[14]

$${}_sR^{\text{down}}(r)\frac{k\omega}{|k\omega|}\sim{}_sR^{\text{in}}(r)+(e^{ikr^*}\text{ term}), \quad s=-1,-2 \tag{5.17}$$

near the horizon. Evaluation of these inner products yields the correct normalization of the Table I homogeneous solutions that appear in the Green's function. The result is

$$[D_{m\omega}]^{-s}{}_sR^{\text{in}} \sim \frac{k\omega}{|k\omega|}(2Mr+|k|)^{-1/2}e^{-ikr^*}\sqrt{2}^{\,-s-2}, \quad s=-1,-2 \quad (5.18)$$

or

$$_{-1}R^{\text{in}} \sim \frac{1}{\sqrt{2}}\frac{k\omega}{|k\omega|}\Delta_K e^{-ikr^*}(8Mr_+)^{-1/2} \times[(-2iMr_+k+(M^2-a^2)^{1/2})|k|^{1/2}]^{-1}, \quad (5.19)$$

$$_{-2}R^{\text{in}} \sim \frac{1}{\sqrt{2}}\Delta_K[-2iMr_+k+2(M^2-a^2)^{1/2}]^{-1}{}_{-1}R^{\text{in}} \quad (5.20)$$

near the horizon, where

$$D_{m\omega}=\frac{d}{dr}-\frac{iK}{\Delta_K}, \quad k=\omega-m\omega_+=\omega-\frac{ma}{2Mr_+}, \quad r_\pm=M\pm(M^2-a^2)^{1/2}. \quad (5.21)$$

Equations (5.13), (5.14), (5.17), (5.19), and (5.20) combine with the Table I expressions in the ingoing gauge to give a Green's function to be used inside the source radius.

It is interesting to note that these gravitational potentials may be used to calculate the gravitational energy and angular momentum flux across the horizon. Since the radiation is highly blueshifted near the horizon, the Isaacson effective stress-energy tensor is applicable in the formula[7]

$$E=\int dt\,d\Omega\,\frac{\omega}{k}\left(\frac{\Delta_K}{4Mr_+}\right)^2 2Mr_+T^{\mu\nu}l_\mu l_\nu. \quad (5.22)$$

Substitution of (5.14) into the above gives

$$E=\sum_{\omega>0}2Mr_+\frac{\omega}{k}\left[\frac{\Delta_K}{2(r^2+a^2)}\right]^2|D_{m\omega}{}^3\,{}_{-2}R^{\text{in}}|^2 \times|\langle h^{\text{down}}_{\alpha\beta},T^{\alpha\beta}\rangle|^2$$
$$=\sum_{\omega>0}2Mr_+\omega k|D_{m\omega}{}^2\,{}_{-2}R^{\text{in}}|^2 \times|\langle h^{\text{down}}_{\alpha\beta}(lm\omega P),T^{\alpha\beta}\rangle|^2, \quad (5.23)$$

which, by virtue of (5.18), simplifies to

$$E=\sum_{\omega>0}\frac{k}{|k|}\omega|\langle h^{\text{down}}_{\alpha\beta}(lm\omega P),T^{\alpha\beta}\rangle|^2. \quad (5.24)$$

Similarly, for the electromagnetic spectrum and the angular momentum flux one finds

$$E=\sum_{\omega>0}\frac{k}{|k|}\omega|\langle A^{\text{down}}_\alpha(lm\omega P),J^\alpha\rangle|^2, \quad (5.25)$$

$$J=\sum_{\omega>0}\frac{k}{|k|}m|\langle A^{\text{down}}_\alpha(lm\omega P),J^\alpha\rangle|^2, \quad (5.26)$$

$$J=\sum_{\omega>0}\frac{k}{|k|}m|\langle h^{\text{down}}_{\alpha\beta}(lm\omega P),T^{\alpha\beta}\rangle|^2. \quad (5.27)$$

Equation (5.24) may be shown to agree with the Hawking and Hartle[8] result

$$E=\int dt\,d\Omega\,\frac{\omega Mr_+}{2\pi k}|\sigma^{\text{HH}}|^2, \quad (5.28)$$

with the perturbed shear given by

$$\sigma^{\text{HH}}=-\left[\frac{\Delta_K}{2(r^2+a^2)}\right]^2\psi_0\left[ik+\frac{(M^2-a^2)^{1/2}}{2Mr_+}\right]^{-1}. \quad (5.29)$$

In fact, an alternate way of deriving (5.24) is to use the Hawking-Hartle formula together with

$$\sigma^{\text{HH}}=-\tfrac{1}{2}D^{\text{HH}}h_{mm}=\frac{ik}{2}h_{mm}, \quad (5.30)$$

an expression derived in Appendix B.

Finally, consider the special case of stationary perturbations, for which the spheroidal harmonics reduce to spherical harmonics and the Teukolsky radial functions become hypergeometric functions. Two radial solutions used to construct a Green's function are[6,9]

$$_sR^+=x^{-s+\gamma}\frac{(1+x)^{-s-\gamma}}{(r_+-r_-)^{s-l}}\frac{\Gamma(l-s+1)\Gamma(1+l+2\gamma)}{\Gamma(l-s+2\gamma)\Gamma(1+2l)}F(-l-s,l-s+1;1-s+2\gamma;-x), \quad (5.31)$$

$$_sR^\infty=(-)^s(r-r_+)^{-l-s-1}(1+x^{-1})^{-s-\gamma}F(l+1-s,l+1-2\gamma;2l+2;-x^{-1}), \quad (5.32)$$

with

$$x=(r-r_+)/(r_+-r_-), \quad \gamma=iam/(r_+-r_-). \quad (5.33)$$

These solutions $_sR^+$ and $_sR^\infty$ are, respectively, regular at the horizon and at infinity and have the asymptotic forms

$$_sR^\infty\sim\begin{cases}(-)^s(r_+-r_-)^{-l-s-1}\left[\dfrac{\Gamma(2l+2)\Gamma(2\gamma-s)}{\Gamma(l+1+2\gamma)\Gamma(l+1-s)}e^{-im\omega_+r^*}+\dfrac{\Gamma(2l+2)\Gamma(s-2\gamma)}{\Gamma(l+s+1)\Gamma(l+1-2\gamma)}\left(\dfrac{\Delta_K}{(r_+-r_-)^2}\right)^{-s}e^{im\omega_+r^*}\right], & r^*\to-\infty\\ (-)^s r^{-l-s-1}, & r^*\to\infty\end{cases} \quad (5.34)$$

$$
{}_sR^+ \sim \begin{cases} \dfrac{(1+x)^{-s-\gamma}}{(r_+-r_-)^{-l+s}}\left[\dfrac{\Delta_K}{(r_+-r_-)^2}\right]^{-s} e^{im\omega_+ r^*}\dfrac{\Gamma(l-s+1)\Gamma(1+l+2\gamma)}{\Gamma(1-s+2\gamma)\Gamma(1+2l)}\,, & r^*\to-\infty \\ r^{l-s}+(r_+-r_-)^{2l+1}r^{-l-s-1}\dfrac{\Gamma(1+l-s)\Gamma(1+l+2\gamma)\Gamma(-2\gamma-1)}{\Gamma(2l+1)\Gamma(-l-s)\Gamma(2\gamma-l)}\,, & r^*\to\infty\,. \end{cases} \tag{5.35}
$$

The normalization of these solutions has been chosen so that the ingoing gauge potentials in Table I lead to the Green's function

$$
A_\alpha = \begin{cases} \sum \dfrac{-2\pi}{l(l+1)(2l+1)} A_\alpha^+(lmP)\langle A_\beta^\infty(lmP), J^\beta\rangle, & r^*\to-\infty \\ \sum \dfrac{-2\pi}{l(l+1)(2l+1)} A_\alpha^\infty(lmP)\langle A_\beta^+(lmP), J^\beta\rangle\,, & r^*\to\infty \end{cases} \tag{5.36}
$$

for stationary electromagnetic perturbations and

$$
h_{\alpha\beta} = \begin{cases} \sum \dfrac{8\pi}{(l-1)(l+2)(2l+1)} h_{\alpha\beta}^+(lmP)\langle h_{\mu\nu}^\infty(lmP), T^{\mu\nu}\rangle\,, & r^*\to-\infty \\ \sum \dfrac{8\pi}{(l-1)(l+2)(2l+1)} h_{\alpha\beta}^\infty(lmP)\langle h_{\mu\nu}^+(lmP), T^{\mu\nu}\rangle\,, & r^*\to\infty \end{cases} \tag{5.37}
$$

in the gravitational case. The inner products for this special case of stationary perturbations are

$$
\langle\psi, T\rangle = \int_{t=\text{const}} \sqrt{-g}\; dr\, d\theta\, d\varphi\, \psi^* T\,. \tag{5.38}
$$

VI. DISCUSSION

Recent work by Cohen and Kegeles[15] is suggestive that the results obtained here may be generalized easily to perturbations of other vacuum solutions. What they find is that for an algebraically special spacetime, solutions to the complex equation

$$
[(\Delta-\gamma^*+\gamma+\mu^*)(D+2\epsilon+\rho)-(\delta^*+\alpha+\beta^*-\tau^*)(\delta+2\beta+\tau)]\phi=0 \tag{6.1}
$$

may be used to generate test electromagnetic fields as follows:

$$
\phi_0=-(D-\epsilon+\epsilon^*-\rho^*)(D+2\epsilon^*+\rho^*)\phi^*\,, \tag{6.2}
$$

$$
\phi_1=[-(D+\epsilon^*+\epsilon)(\delta^*+2\beta^*+\tau^*)+(\pi+\tau^*)(D+2\epsilon^*+\rho^*)]\phi^*\,, \tag{6.3}
$$

$$
\phi_2=[-(\delta^*+\alpha+\beta^*-\tau^*)(\delta^*+2\beta^*+\tau^*)+\lambda(D+2\epsilon^*+\rho^*)]\phi^*\,. \tag{6.4}
$$

For the Kerr case, these formulas are shown in this section to be equivalent to the equations in Table I for the vector potential in the ingoing gauge. In addition, here it is speculated that generalizations of (6.1) through (6.4) hold for metric perturbations of algebraically special spacetimes. The results of Cohen and Kegeles and the formulas for the metric perturbations of the Kerr background suggest the form such equations might take.

From the calculations in Sec. III, the ingoing gauge vector potential in the Kinnersley tetrad is given by

$$
A_\alpha=\sum[-l_\alpha(\delta^*+2\beta^*+\tau^*)+m_\alpha^*(D+2\epsilon^*+\rho^*)]\,{}_{-1}R\,{}_1Ze^{-i\omega\tau}\,, \tag{6.5}
$$

or, with the aid of (2.7) and (2.15),

$$
A_\alpha=\sum[-l_\alpha(\delta^*+2\beta^*+\tau^*)+m_\alpha^*(D+2\epsilon^*+\rho^*)]\,{}_{-1}R^*\,{}_{-1}Z^*e^{i\omega\tau}\,. \tag{6.6}
$$

This becomes

$$
A_\alpha=[-l_\alpha(\delta^*+2\beta^*+\tau^*)+m_\alpha^*(D+2\epsilon^*+\rho^*)]\phi^* \tag{6.7}
$$

if one defines a function $\phi=[\rho^{-2}\phi_2]$, which satisfies (6.1). Now use (B5) through (B7) to construct the electromagnetic field functions ϕ_0, ϕ_1, ϕ_2 from A_α. The result is identical to the Cohen and Kegeles formulas for the special case of Kerr perturbations, where $\lambda=0$.

It is interesting to note that there is a simple three-step prescription one can follow to obtain (6.2) through (6.4) for arbitrary algebraically special solutions:

(1) By choosing the unperturbed tetrad such that $\psi_0 = \psi_1 = \kappa = \sigma = 0$, obtain the decoupled equation for φ_0,

$$[(D - \epsilon + \epsilon^* - 2\rho + \rho^*)(\Delta + \mu + 2\gamma) - (\delta - \beta - \alpha^* - 2\tau + \pi^*)(\delta^* + \pi - 2\alpha)]\, \phi_0 = 2\pi\, {}_1T\,, \tag{6.8}$$

with ${}_1T$ given in Table II.

(2) Make a tetrad rotation to set the background $\psi_3 = 0$ and define a quantity $(\psi_2)^{2/3}\varphi$ as the function which satisfies the source-free version of (6.8) with the tetrad legs l and n, m and m^* interchanged. Then φ obeys (6.1) provided that the final tetrad freedom is used to set $\epsilon = 0$.

(3) Read off the perturbed vector potential from the equation

$$\langle \phi^*, {}_1T\rangle = \langle A_\alpha, J^\alpha\rangle\,, \tag{6.9}$$

as was done in Sec. III. The result, together with (B5) through (B7), is the Cohen and Kegeles equations.

One might speculate that a similar prescription holds for metric perturbations of algebraically special solutions. In a tetrad where the background ψ_0 and ψ_1 vanish, the perturbed ψ_0 can be shown to satisfy

$$\begin{aligned}[(D - 3\epsilon + \epsilon^* - 4\rho - \rho^*)(\Delta - 4\gamma + \mu) - (\delta + \pi^* - \alpha - 3\beta - 4\tau)(\delta^* + \pi - 4\alpha) - 3\psi_2]\psi_0 \\ = 2\pi[{}_2T - 2(D - 3\epsilon + \epsilon^* - 4\rho - \rho^*)\lambda^* T_{ll}] \\ = 2\pi S\,, \end{aligned} \tag{6.10}$$

with ${}_2T$ given in Table II. Rotate the tetrad to set $\psi_3 = 0$ and define $(\psi_2)^{4/3}\psi$ to be a solution to the source-free (6.10) with $l \leftrightarrow n$ and $m \leftrightarrow m^*$. After one fixes $\epsilon = 0$, the function ψ satisfies

$$[(\Delta + 3\gamma - \gamma^* + \mu^*)(D + 3\rho) - (\delta^* - \tau^* + \beta^* + 3\alpha)(\delta + 3\tau + 4\beta) - 3\psi_2]\,\psi = 0\,. \tag{6.11}$$

The speculation, then, is that, in complete analogy with the electromagnetic case, the perturbed metric is given by

$$\langle \psi^*, S\rangle = \langle h_{\alpha\beta}, T^{\alpha\beta}\rangle \tag{6.12}$$

or

$$\begin{aligned} h_{\alpha\beta} = \{ & - l_\alpha l_\beta(\delta^* + \alpha + 3\beta^* - \tau^*)(\delta^* + 4\beta^* + 3\tau^*) - m^*_\alpha m^*_\beta(D - \rho^* + 3\epsilon^* - \epsilon)(D + 3\rho^* + 4\epsilon^*) \\ & + l_{(\alpha} m^*_{\beta)}[(D + \rho - \rho^* + \epsilon + 3\epsilon^*)(\delta^* + 4\beta^* + 3\tau^*) + (\delta^* + 3\beta^* - \alpha - \pi - \tau^*)(D + 3\rho^* + 4\epsilon^*)] \\ & + l_\alpha l_\beta \lambda(D + 4\epsilon^* + 3\rho^*)\}\,\psi^* \end{aligned} \tag{6.13}$$

with $\epsilon = 0$. Equation (6.13) is valid for perturbations of the Kerr metric; that it holds for perturbations of other algebraically special solutions is presently being checked.

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APPENDIX A: GREEN'S FUNCTIONS FOR SCHWARZSCHILD PERTURBATIONS

The calculations presented in this appendix serve a threefold purpose. First and foremost, they demonstrate the existence of factorized Green's functions for metric perturbations of the Schwarzschild solution. Secondly, these computations illustrate that these Green's functions, in fact, are the most lucid way of presenting the inhomogeneous perturbation equations. The Green's function description clarifies the physics of the coupling of the source to the perturbed field and naturally gives the potentials in a radiation gauge. Finally, with the more general overview of the perturbation equations arising from this approach, one can extend the Zerilli[19] analysis of the $l < 2$ modes to an arbitrary perturbative source.

The magnetic parity equations are studied in some detail here, but for the electric parity and $l < 2$ modes only results are quoted. Even the discussion of the magnetic parity computations is very brief since the procedures followed are described at length in Ref. 14 and are used again in Sec. III. The notation is that of Zerilli[19]; his work is to be consulted for the definitions of the tensor harmonics and the radial functions. The only change made here is that the Schwarzschild metric is used in the inner products between the various harmonics and the perturbing stress-energy tensor, e.g., $Q_{lm\omega}$ is given by

$$Q_{lm\omega} = (1-2M/r)^{-1}\int d\Omega\, c_{\alpha\beta}(\theta,\varphi)^* T_{\mu\nu} g^{\alpha\mu} g^{\beta\nu}\,. \tag{A1}$$

The magnetic parity Schrödinger-type radial equation is

$$\left[-\frac{d^2}{dr^{*2}} - \omega^2 + (1-2M/r)\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right)\right] R^{(M)}(r)$$
$$= \frac{8\pi i(1-2M/r)}{2\pi r[l(l+1)(l+2)(l-1)]^{1/2}} \int dt\, e^{i\omega t}\left\{(r-2M)[(l-1)(l+2)]^{1/2} Q_{lm\omega} - \frac{d}{dr}\left((r^2-2Mr)D_{lm\omega}\right) + 2(r-2M)D_{lm\omega}\right\}, \tag{A2}$$

with

$$D_{lm\omega} = \int d\Omega\, d_{\alpha\beta}(\theta,\varphi)^* T^{\alpha\beta} \tag{A3}$$

and $Q_{lm\omega}$ given above. This equation may be solved by constructing a radial Green's function

$$\left[-\frac{d^2}{dr^{*2}} - \omega^2 + (1-2M/r)\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right)\right] G(r^*, r_0^*) = \delta(r^*, r_0^*)\,. \tag{A4}$$

The result is[22]

$$G(r^*, r_0^*) = \frac{i\omega}{2|\omega|}\times\begin{cases} u^{\rm up}(r_0^*)\,u^{\rm in}(r^*)\,, & r^* < r_0^* \\ u^{\rm in}(r_0^*)\,u^{\rm up}(r^*)\,, & r^* > r_0^* \end{cases} \tag{A5}$$

where $u^{\rm in}$ and $u^{\rm up}$ are homogeneous solutions to (A4) normalized to give

$$u^{\rm up} \sim \frac{1}{|\omega|^{1/2}}\times\begin{cases} e^{i\omega r^*}\,, & r^*\to\infty \\ T^{-1}e^{i\omega r^*} - (S/T)^* e^{-i\omega r^*}\,, & r^*\to-\infty \end{cases}$$
$$u^{\rm in} \sim \frac{1}{|\omega|^{1/2}}\times\begin{cases} e^{-i\omega r^*} + S e^{i\omega r^*}\,, & r^*\to\infty \\ T e^{-i\omega r^*}\,, & r^*\to-\infty\,. \end{cases} \tag{A6}$$

Using this Green's function, one can show that $R^{(M)}$ outside of the source radius is

$$R^{(M)} = \frac{\sqrt{2}\,i\omega^2}{|\omega|}\,\frac{u^{\rm up}}{[l(l+1)(l-1)(l+2)/2]^{1/2}}$$
$$\times\int \sqrt{-g}\; d^4x\, T^{\alpha\beta}\left[\frac{-i\sqrt{2}}{\omega}\left(\frac{1}{r}\frac{du^{\rm out}}{dr^*} d_{\alpha\beta} e^{-i\omega t} + [(l-1)(l+2)]^{1/2} r^{-2} u^{\rm out} c_{\alpha\beta} e^{-i\omega t}\right.\right.$$
$$\left.\left. + r^{-2}(1-2M/r) u^{\rm out} d_{\alpha\beta} e^{-i\omega t}\right)\right]^*\,. \tag{A7}$$

The quantity inside the large square brackets in (A7) is a solution to the homogeneous perturbation equation in a gauge which is transverse at infinity. To see that it is a solution, simply make the gauge transformation

$$\Lambda^{\rm out}_{lm\omega} = \frac{i\sqrt{2}}{\omega^3}\,\frac{d}{dr^*}[r u^{\rm out}]\,; \tag{A8}$$

the expression is then identified as being the perturbed metric in the Regge-Wheeler gauge. Henceforth the bracketed quantity in (A7) will be denoted by $h^{(R)\rm out}_{\alpha\beta}(lm\omega M)$; (R) signifies radiation gauge and M magnetic parity.

Now construct out of (A7) a solution to the perturbation equations in the Regge-Wheeler gauge

$$h_{\alpha\beta}(lm\omega M) = \frac{i}{r}[2l(l+1)]^{1/2}\left[-\frac{d}{dr^*}(rR^{(M)})c^{(0)}_{\alpha\beta} + \frac{r^2}{r-2M} R^{(M)} c_{\alpha\beta}\right] e^{-i\omega t} \tag{A9}$$

and use $-\Lambda^{\rm up}_{lm\omega}$ to transform this expression into the radiation gauge. The result is

$$h_{\alpha\beta}(lm\omega M)=\frac{2i\omega}{|\omega|}\,h^{(R)\mathrm{up}}_{\alpha\beta}(lm\omega M)\times\int\sqrt{-g}\,d^4x\,h^{(R)\mathrm{out}}_{\mu\nu}(lm\omega M)^*T^{\mu\nu} \qquad (\mathrm{A10})$$

or

$$h_{\alpha\beta}(M)=\sum\frac{2i\omega}{|\omega|}\,h^{(R)\mathrm{up}}_{\alpha\beta}(lm\omega M)\times\langle h^{(R)\mathrm{out}}_{\mu\nu}(lm\omega M),\,T^{\mu\nu}\rangle\ . \qquad (\mathrm{A11})$$

By virtue of (A6) and the definition of $h^{(R)}_{\alpha\beta}(lm\omega M)$, the "up" and "out" states are normalized to give

$$\langle h^{(R)\mathrm{up}}_{\alpha\beta}(lm\omega M),\,h^{\alpha\beta(R)\mathrm{out}}(l'm'\omega'M)\rangle_{g^+}=\frac{2\omega}{|\omega|}\,\delta_{ll'}\delta_{mm'}\delta(\omega-\omega')\ . \qquad (\mathrm{A12})$$

Similarly, for the electric parity modes, one finds

$$h_{\alpha\beta}(E)=\sum\frac{2i\omega}{|\omega|}\,h^{(R)\mathrm{up}}_{\alpha\beta}(lm\omega E)\langle h^{(R)\mathrm{out}}_{\mu\nu}(lm\omega E),\,T^{\mu\nu}\rangle\ , \qquad (\mathrm{A13})$$

with

$$h^{(R)}_{\alpha\beta}(lm\omega E)=\sqrt{2}\,\frac{v}{r}\,f_{\alpha\beta}e^{-i\omega t}+\left(\frac{1}{r^2}\text{ terms}\right) \qquad (\mathrm{A14})$$

and

$$\langle h^{(R)\mathrm{up}}_{\alpha\beta}(lm\omega E),\,h^{\alpha\beta(R)\mathrm{out}}(l'm'\omega'E)\rangle_{g^+}=\frac{2\omega}{|\omega|}\,\delta_{ll'}\delta_{mm'}\delta(\omega-\omega')\ . \qquad (\mathrm{A15})$$

The radial function $v(r^*)$ is a homogeneous solution to the Schrödinger-type electric parity equation and obeys the normalization condition (A6).

Equations (A11) and (A13) together constitute a factorized Green's function for perturbations of the Schwarzschild solution with $g_{PP'}=2\delta_{PP'}$ and $P=E,M$. That is, for each $lm\omega$ mode, the two polarization states in the Green's function are the solutions of definite parity.

The above analysis does not hold for the $l<2$ modes, each of which must be considered separately. Zerilli has given explicit solutions to these equations for the special case of a radially falling test particle.[19] Listed here are the results for the more general case of an arbitrary perturbation.

For the $l=0$ mode, the magnetic equation vanishes identically and the electric equation gives

$$\bar{h}_{00}=\frac{2}{r}\int_{2M}^{r}dr\,d\Omega\,r^2T^0{}_0\ ,$$
$$\bar{h}_{rr}=\frac{2}{r}\,(1-2M/r)^{-2}\int_{2M}^{r}dr\,d\Omega\,r^2T^0{}_0\ . \qquad (\mathrm{A16})$$

To linear order in the perturbation this corresponds to changing the mass in the Schwarzschild solution to

$$M\to M(r)=M+\int_{2M}^{r}dr\,d\Omega\,r^2T^0{}_0\ . \qquad (\mathrm{A17})$$

The $l=1$ magnetic parity mode gives the angular momentum of the perturbation. Specifically, one finds

$$\bar{h}_{\alpha\beta}=-\frac{16\pi}{3}\,\frac{c^{(0)}_{\alpha\beta}}{r}\int_{2M}^{r}dr\,d\Omega\,r^2c^{(0)}_{\mu\nu}{}^*T^{\mu\nu}\ , \qquad (\mathrm{A18})$$

which, for the special case of an axially symmetric perturbation, reduces to

$$\bar{h}_{0\varphi}=-\frac{2\sin^3\theta}{r}\int_{2M}^{r}dr\,d\Omega\,r^2T^0{}_\varphi\ . \qquad (\mathrm{A19})$$

Hence, the z angular momentum of an axially symmetric perturbation is

$$l_z(r)=\int_{2M}^{r}dr\,d\Omega\,r^2T^0{}_\varphi\ . \qquad (\mathrm{A20})$$

Finally, the $l=1$ electric parity perturbation is a gauge transformation that can be identified at large radii as being a coordinate transformation to the center-of-momentum frame. The gauge function is

$$\xi_\alpha=\nabla_\alpha\left[-\frac{F}{6M}\,rY_{l=1,m}(\theta,\varphi)\right]\ , \qquad (\mathrm{A21})$$

with

$$F=-8\pi\int_{2M}^{r}dr\,d\Omega\,r^3(1-2M/r)^2a^{(0)}_{\alpha\beta}{}^*T^{\alpha\beta}\ . \qquad (\mathrm{A22})$$

For an axially symmetric perturbation this corresponds at large radii to the coordinate transformation

$$z'-z=\frac{1}{M}\int_{2M}^{r}dr\,d\Omega\,r^2(1-2M/r)r\cos\theta T^0{}_0\ ,$$
$$t'-t=\frac{r\cos\theta}{M}\frac{\partial}{\partial t}\int_{2M}^{r}dr\,d\Omega\,r^2(1-2M/r)r\cos\theta T^0{}_0\ . \qquad (\mathrm{A23})$$

APPENDIX B: THE PERTURBED FIELD FUNCTIONS

In this appendix the field functions are reconstructed from the perturbed potentials. These expressions for the field functions are used in the next appendix to check the validity of the expressions given in Table I. One of these perturbed field functions, ψ_0, is needed to reconstruct the perturbed shear σ, which in turn may be used to compute the energy flux across the horizon.

First, consider the electromagnetic field $F_{\alpha\beta}=A_{\beta;\alpha}-A_{\alpha;\beta}$, where $F_{\alpha\beta}$ may be expanded in terms of three complex scalar functions:

$$\phi_0 = F^{\alpha\beta} l_\alpha m_\beta,$$
$$\phi_1 = F^{\alpha\beta}(l_\alpha n_\beta + m^*_\alpha n_\beta)/2, \tag{B1}$$
$$\phi_2 = F^{\alpha\beta} m^*_\alpha n_\beta .$$

The field quantity ϕ_0, for example, is given by

$$\phi_0 = A_{\alpha;\beta}(m^\alpha l^\beta - l^\alpha m^\beta) . \tag{B2}$$

Now expand the vector potential in terms of the null tetrad vectors

$$A_\alpha = A_l n_\alpha + A_n l_\alpha - A_m m^*_\alpha - A_{m^*} m_\alpha \tag{B3}$$

and substitute this expression into (B2). The result is

$$\begin{aligned}\phi_0 = {} & DA_m - \delta A_l + (l_{\alpha;\beta} m^\alpha l^\beta) A_n - (l_{\alpha;\beta} m^\alpha m^\beta) A_{m^*} \\ & + (n_{\alpha;\beta} m^\alpha l^\beta + l_{\alpha;\beta} n^\alpha m^\beta) A_l \\ & + (m_{\alpha;\beta} m^{*\alpha} l^\beta - l_{\alpha;\beta} m^{*\alpha} m^\beta) A_m \end{aligned} \tag{B4}$$

or

$$\begin{aligned}\phi_0 = {} & (D - \epsilon + \epsilon^* - \rho^*) A_m - (\delta + \pi^* - \beta - \alpha^*) A_l \\ & - \sigma A_{m^*} + \kappa A_n . \end{aligned} \tag{B5}$$

Similarly, the ϕ_1 and ϕ_2 electromagnetic field functions may be shown to be

$$\begin{aligned} 2\phi_1 = {} & (D + \epsilon + \epsilon^* + \rho - \rho^*) A_n - (\Delta - \gamma - \gamma^* + \mu^* - \mu) A_l \\ & + (\delta^* - \pi - \tau^* - \alpha + \beta^*) A_m \\ & - (\delta + \pi^* + \tau + \beta - \alpha^*) A_{m^*} , \end{aligned} \tag{B6}$$

$$\begin{aligned} \phi_2 = {} & (\delta^* + \alpha + \beta^* - \tau^*) A_n - (\Delta + \mu^* + \gamma - \gamma^*) A_{m^*} \\ & + \nu A_l - \lambda A_m . \end{aligned} \tag{B7}$$

For perturbations of the Kerr metric, several of the spin coefficients vanish, namely $\kappa = \sigma = \nu = \lambda = 0$.

To construct the perturbed Weyl tensor, one follows exactly the same procedures, only the algebra is more tedious. Start with the equation[32]

$$\begin{aligned} 2\dot{R}_{\alpha\beta\mu\nu} = {} & h_{(\alpha\mu);\beta\nu} + h_{(\beta\nu);\alpha\mu} - h_{(\beta\mu);\alpha\nu} - h_{(\alpha\nu);\beta\mu} \\ & + R_{\alpha\sigma\mu\nu} h^\sigma{}_\beta - R_{\beta\sigma\mu\nu} h^\sigma{}_\alpha , \end{aligned} \tag{B8}$$

with the dot denoting the perturbed value. (When the background quantity vanishes, the dot is omitted.) Projection of (B8) along the tetrad legs $l^\alpha m^\beta l^\mu m^\nu$ and the equation

$$\dot{\psi}_0 = -\dot{R}_{\alpha\beta\mu\nu} l^\alpha m^\beta l^\mu m^\nu \tag{B9}$$

leads to the general expression

$$\begin{aligned} -2\dot{\psi}_0 = {} & 2\{\kappa\kappa h_{nn} - 2\kappa\sigma h_{(nm^*)} + \sigma\sigma h_{m^*m^*} + [-\sigma(D - \rho^* + \rho) - (\delta + 2\pi^* - 3\beta - \alpha^*)\kappa] h_{(mm^*)} \\ & + [-\kappa(\delta + \pi^* + \tau) - (D - 2\rho^* - 3\epsilon + \epsilon^*)\sigma] h_{(ln)} + [(D - \rho^*)\kappa + \kappa(D - \rho^* + \rho - 3\epsilon + 3\epsilon^*) - \sigma\kappa^*] h_{(nm)} \\ & + [(\delta + \pi^*)\sigma + \sigma(\delta + \pi^* + \tau - 3\beta - 3\alpha^*) + \kappa\lambda^*] h_{(lm^*)}\} \\ & + [-(\delta + \pi^* - 3\beta - \alpha^*)\kappa^* + \kappa(\delta^* - 2\pi - \tau^* + 2\beta^* - 2\alpha) - \sigma\sigma^* + (D - \rho^* - 3\epsilon + \epsilon^*)(D - \rho^* - 2\epsilon + 2\epsilon^*)] h_{mm} \\ & + [(D - \rho^* - 3\epsilon + \epsilon^*)\lambda^* + \sigma(\Delta - 2\mu + \mu^* - 2\gamma - 2\gamma^*) + \kappa\nu^* + (\delta + \pi^* - 3\beta - \alpha^*)(\delta + \pi^* - 2\beta - 2\alpha^*)] h_{ll} \\ & - [\sigma(\delta^* - 2\pi - 2\tau^* - 2\alpha) + (D - \rho^* - 3\epsilon + \epsilon^*)(\delta + 2\pi^* - 2\beta) \\ & + \psi_1 + \kappa(\Delta - 2\mu + 2\mu^* - 2\gamma) + (\delta + \pi^* - 3\beta - \alpha^*)(D - 2\rho^* - 2\epsilon)] h_{(lm)} ; \end{aligned} \tag{B10}$$

an expression for $\dot{\psi}_4$ follows from the above by interchanging $l \leftrightarrow n$ and $m \leftrightarrow m^*$. For the Kerr metric ($\sigma = \kappa = \lambda = \nu = 0$), the formulas for ψ_0 and ψ_4 are

$$\begin{aligned} -2\psi_0 = {} & (\delta + \pi^* - 3\beta - \alpha^*)(\delta + \pi^* - 2\beta - 2\alpha^*) h_{ll} + (D - \rho^* - 3\epsilon + \epsilon^*)(D - \rho^* - 2\epsilon + 2\epsilon^*) h_{mm} \\ & - [(D - \rho^* - 3\epsilon + \epsilon^*)(\delta + 2\pi^* - 2\beta) + (\delta + \pi^* - 3\beta - \alpha^*)(D - 2\rho^* - 2\epsilon)] h_{(lm)} , \end{aligned} \tag{B11}$$

$$\begin{aligned} -2\psi_4 = {} & (\delta^* - \tau^* + 3\alpha + \beta^*)(\delta^* - \tau^* + 2\alpha + 2\beta^*) h_{nn} + (\Delta + \mu^* + 3\gamma - \gamma^*)(\Delta + \mu^* + 2\gamma - 2\gamma^*) h_{m^*m^*} \\ & - [(\Delta + \mu^* + 3\gamma - \gamma^*)(\delta^* - 2\tau^* + 2\alpha) + (\delta^* - \tau^* + 3\alpha + \beta^*)(\Delta + 2\mu^* + 2\gamma)] h_{(nm^*)} . \end{aligned} \tag{B12}$$

In a similar fashion, expressions for ψ_1, $\dot{\psi}_2$, and ψ_3 can be found.

Equation (B11) leads to a useful formula for the perturbed shear. In the Hartle-Hawking[7,8] tetrad, which is related to the Kinnersley tetrad by

$$l^{\mathrm{HH}} = \frac{\Delta_K}{2(r^2 + a^2)} l , \quad n^{\mathrm{HH}} = \frac{2(r^2 + a^2)}{\Delta_K} n , \tag{B13}$$

the shear is

$$\sigma^{\mathrm{HH}} = -\left[\frac{\Delta_K}{2(r^2 + a^2)}\right]^2 \frac{\psi_0}{ik + (M^2 - a^2)^{1/2}/2Mr_+} . \tag{B14}$$

Equation (B11) gives

$$\psi_0 = -\tfrac{1}{2} DDh_{mm} \tag{B15}$$

near the horizon, so the formula for the shear becomes

$$\sigma^{\rm IIH} = \frac{ik}{2} h_{mm} \, . \tag{B16}$$

APPENDIX C: VERIFICATION OF THE POTENTIALS IN TABLE I

Since the derivation of the potentials depends in a nontrivial way on an assumed form for the Green's functions, the expressions in Table I must be checked. Verification that the electromagnetic potentials are correct is presented here and in Sec. VI, and it is shown that the equations for ψ_0 and ψ_4 written in terms of derivatives of the metric potentials are satisfied. This guarantees that the physically measurable components of the perturbed Riemann tensor determined from the perturbed metric in Table I are correct.

Only the potentials in the ingoing gauge are considered in this appendix. The outgoing potentials also have been checked, but the details are not given here. The calculations are to be carried out in the Boyer-Lindquist coordinate system and the Kinnersley tetrad, where the only nonvanishing spin coefficients are

$$\begin{aligned} &\rho = -1/(r - ia\cos\theta), \quad \beta = -\rho^*\cot\theta/2\sqrt{2}\, , \\ &\pi = ia\rho^2\sin\theta/\sqrt{2}\, , \quad \alpha = \pi - \beta^*\, , \\ &\tau = -ia\rho\rho^*\sin\theta/\sqrt{2}\, , \quad \mu = \rho^2\rho^*\Delta_K/2, \\ &\gamma = \mu + \rho\rho^*(r - M)/2\, . \end{aligned} \tag{C1}$$

Consider the ingoing gauge vector potential

$$A_\alpha = \sum \left[-l_\alpha(\delta^* + 2\beta^* + \tau^*) + m_\alpha^*(D + \rho^*)\right] \times {}_{-1}R\, {}_1Z e^{-i\omega t} \, . \tag{C2}$$

This expression for A_α was decomposed at the end of Sec. III into the two linearly independent solution given in Table I. In Sec. IV these independent solutions were established as being the real and imaginary parts of (C2). For present purposes, it suffices to look at the composite potential given by (C2) and to examine any single $lm\omega$ mode.

First use (B5) to show that the ϕ_0 determined from this potential satisfies the appropriate equation. With $A_l = 0$, one finds from (B5) and (C2) that

$$\phi_0 = -(D - \rho^*)(D + \rho^*)\, {}_{-1}R\, {}_1Z e^{-i\omega t} \tag{C3}$$

or

$$\phi_0 = -DD\, {}_{-1}R\, {}_1Z e^{-i\omega t} \tag{C4}$$

since $D\rho = \rho^2$. The $lm\omega$ mode of the directional derivative D is just the operator $\mathfrak{D}$ defined by Teukolsky and Press[7]

$$\mathfrak{D} = \partial_r - i[(r^2 + a^2)\omega - am]\Delta_K^{-1} \, . \tag{C5}$$

Accordingly, (C4) becomes

$$\phi_0 = \mathfrak{D}\mathfrak{D}\, {}_{-1}R\, {}_1Z e^{-i\omega t} = -\tfrac{1}{2}\, {}_1R\, {}_1Z e^{-i\omega t} \tag{C6}$$

since[7]

$$\mathfrak{D}\mathfrak{D}\, {}_{-1}R = \tfrac{1}{2}\, {}_1R \, . \tag{C7}$$

Obviously, the quantity $-\frac{1}{2}\, {}_1R\, {}_1Z e^{-i\omega t}$ is a solution to the ϕ_0 equation.

Now consider the ϕ_2 equation. From (B7) and (C2), one has

$$\phi_2 = -(\delta^* + \alpha + \beta^* - \tau^*)(\delta^* + 2\beta^* + \tau^*)_{-1}R\, {}_1Z e^{-i\omega t} \, , \tag{C8}$$

which, with the aid of the Newman-Penrose equation

$$\delta^*\tau^* = \tau^*(\tau^* + \beta^* - \alpha)\, , \tag{C9}$$

reduces to

$$\phi_2 = -(\delta^* + \alpha + \beta^*)(\delta^* + 2\beta^*)\, {}_{-1}R\, {}_1Z e^{-i\omega t} \, . \tag{C10}$$

Equation (C1) and the Teukolsky and Press[7] definition

$$\mathfrak{L}_n = \partial_\theta + \frac{m}{\sin\theta} - a\omega\sin\theta + n\cot\theta \tag{C11}$$

simplify the above to

$$\phi_2 = -\frac{\rho^2}{2}\, \mathfrak{L}_0\mathfrak{L}_1\, {}_{-1}R\, {}_1Z e^{-i\omega t} \, . \tag{C12}$$

One then obtains

$$\phi_2 = -\frac{\rho^2}{2}\, B\, {}_{-1}R\, {}_{-1}Z e^{-i\omega t} \tag{C13}$$

when the identity[7]

$$\mathfrak{L}_0\mathfrak{L}_1\, {}_1Z = B\, {}_{-1}Z \tag{C14}$$

is used. The right-hand side of Eq. (C13) is clearly a solution to the ϕ_2 equation since B is a constant. It is also important to notice that the relative amplitudes of the expressions for ϕ_0 and ϕ_2 are in agreement with the results of Teukolsky and Press.

The demonstration that the expression for ϕ_1 determined from these potentials is correct is somewhat tedious and is not given in this appendix. Rather, refer to Sec. VI where it is shown that the ingoing radiation gauge potentials give rise to the equations derived by Cohen and Kegeles.

Since their expressions are known to be solutions to Maxwell's equations, the ϕ_1 field component found from the potentials must satisfy the appropriate equation.

Finally, consider the gauge-invariant, infinitesimal tetrad rotation invariant Weyl components ψ_0 and ψ_4. Expressions for ψ_0 and ψ_4 follow from (B11) and (B12) and the ingoing metric potentials

$$h_{\alpha\beta}=\sum\{-l_\alpha l_\beta(\delta^*+\alpha+3\beta^*-\tau^*)(\delta^*+4\beta^*+3\tau^*)-m^*_\alpha m^*_\beta(D-\rho^*)(D+3\rho^*)$$
$$+l_{(\alpha}m^*_{\beta)}[(D+\rho-\rho^*)(\delta^*+4\beta^*+3\tau^*)+(\delta^*+3\beta^*-\alpha-\pi-\tau^*)(D+3\rho^*)]\}\,{}_{-2}R\,{}_2Ze^{-i\omega t}\ . \tag{C15}$$

One has

$$-2\psi_0=-(D-\rho^*)(D-\rho^*)(D-\rho^*)(D+3\rho^*)\,{}_{-2}R\,{}_2Ze^{-i\omega t}\ , \tag{C16}$$

$$-2\psi_4=-(\delta^*-\tau^*+3\alpha+\beta^*)(\delta^*-\tau^*+2\alpha+2\beta^*)(\delta^*-\tau^*+\alpha+3\beta^*)(\delta^*+3\tau^*+4\beta^*)\,{}_{-2}R\,{}_2Ze^{-i\omega t}\ , \tag{C17}$$

which reduce to

$$-2\psi_0=-DDDD\,{}_{-2}R\,{}_2Ze^{-i\omega t}\ , \tag{C18}$$

$$-2\psi_4=-(\delta^*+3\alpha+\beta^*)(\delta^*+2\alpha+2\beta^*)(\delta^*+\alpha+3\beta^*)(\delta^*+4\beta^*)\,{}_{-2}R\,{}_2Ze^{-i\omega t}\ . \tag{C19}$$

Rewriting this as

$$-2\psi_0=-\mathfrak{D}\mathfrak{D}\mathfrak{D}\mathfrak{D}\,{}_{-2}R\,{}_2Ze^{-i\omega t}\ , \tag{C20}$$

$$-2\psi_4=-\frac{\rho^4}{4}\,\mathfrak{L}_{-1}\mathfrak{L}_0\mathfrak{L}_1\mathfrak{L}_2\,{}_{-2}R\,{}_2Ze^{-i\omega t}\ , \tag{C21}$$

one can use the equations[7]

$$\mathfrak{D}\mathfrak{D}\mathfrak{D}\mathfrak{D}\,{}_{-2}R=\tfrac{1}{4}\,{}_2R\ , \tag{C22}$$

$$\mathfrak{L}_{-1}\mathfrak{L}_0\mathfrak{L}_1\mathfrak{L}_2\,{}_2Z=\mathrm{Re}[C]\,{}_{-2}Z \tag{C23}$$

to obtain

$$-2\psi_0=-\tfrac{1}{4}\,{}_2R\,{}_2Ze^{-i\omega t}\ , \tag{C24}$$

$$-2\psi_4=-\frac{\rho^4}{4}\,\mathrm{Re}[C]\,{}_{-2}R\,{}_{-2}Ze^{-i\omega t}\ . \tag{C25}$$

Clearly, the right-hand side of each of the above two equations is a solution to the appropriate equation; in addition, the relative amplitudes of the two expressions are correct.

The gauge-dependent Weyl tensor components as yet have not been checked due to the complexities of the algebra. In principle, this should be done, but the fact that the potentials accurately give the invariant parts of the field is compelling evidence that they are correct as presented. The derivation of these potentials depends on the assumption of factorized Green's functions in exactly the same way that the electromagnetic derivation does, and in the electromagnetic case the gauge-dependent part of A_α is correctly determined.

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