Twistor variables of relativistic mechanics

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(Received 18 November 1974)

This paper is concerned with the description of noninteracting dynamical systems in Penrose's twistor theory. We begin with an introduction to the formal treatment of massive systems relying on the angular momentum twistor $A^{\alpha\beta}$. Expressions for dynamical data in terms of $A^{\alpha\beta}$ are given. The internal-symmetry transformations involved in the parametrization of $A^{\alpha\beta}$ by one-index twistors are shown to be of the canonical type. The case of two supporting twistors is discussed in detail. Relative to the frame of these twistors (a rest frame of the dynamical system) the classical spin has the remarkable structure $\vec{s} = \overline{\psi}\vec{\sigma}\psi$. When decomposing $A^{\alpha\beta}$ into three twistors, the minimal symmetry transformations define a 14-parameter group. We show that this group of symmetries is locally isomorphic to the inhomogeneous SU(3).

I. INTRODUCTION

It has been stressed by a number of authors[1-7] that a profound reconsideration of our current concepts of space-time structure is a prerequisite of any theoretical approach to gravity quantization. Particularly lucid arguments due to Wigner and Salecker[6,7] show that the Riemannian point continuum description of space-time geometry ceases to be meaningfully implementable in the quantum domain. The majority of quantum gravitation theories put forward in the past[8] cannot be reconciled with this observation. However, an attempt has been made by Penrose[9-12] to incorporate it, at an elementary conceptual level, in physical theory.

Penrose's twistor theory is centered around two propositions. According to the first of these, the *points* of space-time are composite objects which may have structure in terms of more fundamental entities, called the twistors. The second proposition is that rest mass should also be considered a derived property of matter. So it emerges that the underlying objects (i.e., twistors) are closely related to massless particles.

Under circumstances when the quantum properties of space-time are suppressed, twistor theory offers just an alternative picture of physical phenomena. In this picture, the role of space-time points is taken over by points of the twistor space T. *A fortiori*, then, it is possible to translate all statements of the conventional theory into twistor relations and conversely. Already at this stage the twistor picture may (and does in fact) yield a simpler treatment of some parts of physics, thereby leading to new insights.[13,14]

No such correspondence is available whenever quantum properties of gravity prevail (essentially the reason is that a space-time point should, by then, be thought of as a kind of a "fuzzy" probabilistic structure). The ultimate goal of the twistor approach is the theoretical description of this situation.[5]

The present paper has the modest aim to contribute to a better understanding of the treatment of dynamical systems in the framework of twistor theory. We shall restrict our attention to the classical relativistic mechanics of noninteracting systems. No attempt will be made at the description of quantum properties at this stage.

Characterization of the (conformally invariant) zero-mass systems becomes straightforward in the twistor picture.[11] These systems may be represented simply as points of T (cf. also Sec. II). Conformal invariance must be broken in order to be able to incorporate rest mass into the theory. A formal way of doing this (formal from the viewpoint of twistor theory, though still equivalent to the conventional approach), following Penrose's suggestion, is to include the Poincaré-invariant concept of an *infinity twistor*.[9,11] In Sec. II an introduction to this approach will be given as well as some explicit expressions for dynamical parameters of the system in terms of its *angular momentum twistor.*

Section III is concerned with the parametrization of an angular momentum twistor by one-index twistors. This can be done in many ways.[11,12] We shall show here that the symmetry transformations involved in the parametrization are of the canonical type. The corresponding generating functions will be explicitly given. This result points toward the general twistor quantization rules for massive systems (cf. Sec. VII).

In Sec. IV we discuss some aspects of the two-twistor description of massive systems. Particular attention will be paid to the twistor structure of internal spin. Section V is mainly devoted to the internal-symmetry group of the three-twistor description. We shall show that the number of generators of this group can be reduced to 14, without destroying its transitivity over the angular momentum surfaces. In the next section it will be proved that this minimal symmetry group is locally isomorphic to the inhomogeneous SU(3) group. Section VII concludes the paper with a discussion of possible implications of our results for the general treatment of interactions and quantization in the framework of twistor theory.

Some notational innovations will be encountered in Appendix A. Appendix B contains the derivation of an identity for the twistor δ^α_β by use of a naturally chosen basis.

II. ELEMENTS OF TWISTOR MECHANICS

A Poincaré-covariant characterization of some material system in the Minkowski space-time M is given by the total four-momentum P^a and total angular momentum M^{ab}. Using these data, we obtain the Pauli-Lubanski vector S_a proportional to the total spin,[15]

$$S_a = -\tfrac{1}{2}\eta_{abcd}M^{bc}P^d \quad . \tag{2.1}$$

For $m > 0$ the spin vector is S_a/m, where the invariant mass m satisfies $P_aP^a = m^2$. The magnitude of the spin is expressed $s^2 = -S_aS^a/m^2$. The condition $m = 0$ implies[11] that $S_a = sP_a$, with the factor of proportionality s as the helicity of the massless object.

When the system is isolated, (P^a, M^{ab}) are constants of the motion and for any positive m, the history of the center of mass consists of the points of M

$$X^a(\tau) = M^{ab}P_b/m^2 + \tau P^a/m \quad . \tag{2.2}$$

Denoting proper time (τ) derivatives by a dot, (2.2) yields the four-velocity

$$\dot{X}^a(\tau) = P^a/m \quad , \tag{2.3}$$

with $\dot{X}^a\dot{X}^b g_{ab} = 1$.

The behavior of these quantities under space-time translations (unlike Lorentz covariance) is not apparent in the tensor notation. The real point transformations

$$\tilde{x}^a = x^a + A^a \tag{2.4}$$

affect them as follows:

$$\tilde{P}^a = P^a, \quad \tilde{M}^{ab} = M^{ab} - 2A^{[a}P^{b]} \quad . \tag{2.5}$$

Hence it is seen that $\tilde{S}_a = S_a$ and that the coordinates X^a of the center of mass change according to the general rule (2.4).

When wishing to obtain a twistor formulation of these relations, one first finds their spinor structure. The real four-momentum is then expressed as the Hermitian spinor $P_{AA'} = P_a$ and the skew-symmetric M^{ab} uniquely corresponds to a symmetric spinor μ^{AB} satisfying[11]

$$M^{ab} = \mu^{AB}\epsilon^{A'B'} + \bar{\mu}^{A'B'}\epsilon^{AB} \quad . \tag{2.6}$$

(We are adopting here a slightly modified version of the Battelle convention. For details, see Appendix A.) Its dual, $M^*_{ab} = \frac{1}{2}\eta_{abcd}M^{cd}$, can be written[16]

$$M^*_{ab} = i(\bar{\mu}_{A'B'}\epsilon_{AB} - \mu_{AB}\epsilon_{A'B'}) \quad . \tag{2.7}$$

Equations (2.6) and (2.7) give us the spinor expressions for the center of mass ($m \neq 0$) and Pauli-Lubanski vector, respectively,

$$X^{AA'}(\tau) = \frac{1}{m^2}(\mu^{AB}P_B^{A'} + \bar{\mu}^{A'B'}P^A_{B'}) + \frac{\tau}{m}P^{AA'} \quad ,$$
$$S_{AA'} = i(\mu_{AB}P^B_{A'} - \bar{\mu}_{A'B'}P^{B'}_A) \quad . \tag{2.8}$$

Equations (2.8) suggest that $Z^{AA'}(\tau) = X^{AA'}(\tau) + iS^{AA'}/m^2$ may be considered as coordinates of the spinning system's world line in the complexified Minkowski space-time CM. In fact, for any (complex) value of τ, the total angular momentum can be made to vanish by shifting the origin to $Z^{AA'}(\tau)$. The need for a CM description frequently emerges in twistor theory[17] though, despite the resulting mathematical simplicity, thus far no motivation appears to arise from physics. Let us take the simple example of a one-index twistor Z^α with the components

$$(Z^\alpha) = (\omega^A, \pi_{A'}) \quad . \tag{2.9}$$

This is represented in the CM picture as the locus of all complex points $z^{AA'}$ for which

$$\omega^A = -iz^{AA'}\pi_{A'} \quad , \tag{2.10}$$

where this two-complex-dimensional locus is a totally null plane.[12] [When some solutions $z^{AA'}$ of (2.10) lie on a *real* null line, Z^α is a *null twistor.*]

The twistor Z^α may also be considered as a set of constants of motion for a free, massless particle. Such a particle will have the four-momentum

$$P_{AA'} = \bar{\pi}_A\pi_{A'} \tag{2.11}$$

and angular momentum spinor

$$\mu_{AB} = i\omega_{(A}\bar{\pi}_{B)} \quad . \tag{2.12}$$

This interpretation of Z^α shows that under a Lorentz rotation its components transform as

$$\begin{bmatrix} \hat{\omega}^A \\ \hat{\pi}_{A'} \end{bmatrix} = \begin{bmatrix} \Lambda^A{}_B & 0 \\ 0 & \bar{\Lambda}^{B'}{}_{A'} \end{bmatrix} \begin{bmatrix} \omega^B \\ \pi_{B'} \end{bmatrix} \quad , \tag{2.13}$$

where $\det[\Lambda^A{}_B] = 1$. Translations of the form (2.4) are represented by

$$\begin{bmatrix}\tilde{\omega}^A\\ \tilde{\pi}_{A'}\end{bmatrix}=\begin{bmatrix}\delta^A_B & -iA^{AB'}\\ 0 & \delta^{B'}_{A'}\end{bmatrix}\begin{bmatrix}\omega^B\\ \pi_{A'}\end{bmatrix}. \tag{2.14}$$

Since zero-mass particles are conformally invariant objects, the transformation properties of Z^α under dilatations, inversions, and reflections are also well defined,[9] even though their explicit representation will not be required for our present purposes. Dilatation and inversion invariances are destroyed when introducing the concept of null infinity. The Minkowski space-time M should be completed by adding points "at infinity" since the inversions $\tilde{x}^a=x^a/x_r x^r$ carry the light cone of the origin into the set of these points.[18] This already implies that null infinity has the topological properties of a light cone less its vertex point. Light cones are represented by skew twistors[9,11] (the reverse statement is not necessarily true: It applies only to twistors which are also *simple*[9]). In the present basis, the infinity twistor $I^{\alpha\beta}$ and its dual $I_{\alpha\beta}$ have the components

$$[I^{\alpha\beta}]=\begin{bmatrix}\epsilon^{AB} & 0\\ 0 & 0\end{bmatrix},\quad [I_{\alpha\beta}]=\begin{bmatrix}0 & 0\\ 0 & \epsilon^{A'B'}\end{bmatrix}. \tag{2.15}$$

The invariance group of (2.15) is just the Poincaré group: Fixing where infinity is amounts to destroying the conformal invariance.

Twistor indices are raised and lowered by complex conjugation. The Z^α twistor has the complex conjugate $\bar{Z}_\alpha$ of components

$$(\bar{Z}_\alpha)=(\bar{\pi}_A, \bar{\omega}^{A'}),$$

so that contractions (of which the simplest representative is $Z^\alpha\bar{Z}_\alpha$) are invariant under twistor transformations.

Real light cones correspond to self-dual skew twistors. In particular, the dual of the infinity twistor, $I^*_{\alpha\beta}=\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}I^{\gamma\delta}$, satisfies

$$I^*_{\alpha\beta}=\bar{I}_{\alpha\beta}. \tag{2.16}$$

We may omit the asterisk or the bar over $I_{\alpha\beta}$ without affecting the consistency of notation.

The transformation properties (2.5) show that momentum and angular momentum constitute the components of a single symmetric twistor $A^{\alpha\beta}$ according to the scheme

$$[A^{\alpha\beta}]=\begin{bmatrix}-2i\mu^{AB} & P^A_{B'}\\ P^{A'}_B & 0\end{bmatrix}. \tag{2.17}$$

$A^{\alpha\beta}$, the *angular momentum twistor*, satisfies for any given Z^α

$$\bar{Z}_\alpha A^{\alpha\beta}I_{\beta\gamma}Z^\gamma\geq 0. \tag{2.18}$$

This semidefiniteness property of $A^{\alpha\beta}$ is equivalent to the non-negative character of energy. With the four-momentum P^a pointing into the future, for any future-pointing null vector l^a we have

$$P^a l_a\geq 0. \tag{2.19}$$

We now see that (2.18) is the twistor version of this condition. $A^{\alpha\beta}$ and its conjugate $\bar{A}_{\alpha\beta}$ satisfy also the constraint equations

$$A^{\alpha\beta}I_{\beta\gamma}=I^{\alpha\beta}\bar{A}_{\beta\gamma}. \tag{2.20}$$

The twistor $A^{\alpha\beta}$ contains all Poincaré-covariant data about the system. Thus its determinant,

$$\det[A^{\alpha\beta}]\underset{\text{def}}{=}\Delta^2, \tag{2.21}$$

yields the rest mass,

$$\Delta=m^2/2. \tag{2.22}$$

The matrix $[A^{\alpha\beta}]$ is nonsingular for a massive system. There exists, then, a symmetric twistor $B_{\alpha\beta}$ as the inverse to $A^{\alpha\beta}$ such that

$$A^{\alpha\beta}B_{\beta\gamma}=\delta^\alpha_\gamma. \tag{2.23}$$

The solution of (2.23) for $B_{\alpha\beta}$ can be written in the explicit form

$$B_{\alpha\beta}=-\Delta^{-2}\bar{A}_{\alpha\rho}A^{\rho\sigma}\bar{A}_{\sigma\beta}-2\Delta^{-1}\bar{A}_{\alpha\beta}. \tag{2.24}$$

This expression becomes simpler when the system is spinless. Let us define

$$S_\alpha{}^\beta\underset{\text{def}}{=}\tfrac{1}{2}(\bar{A}_{\alpha\rho}A^{\rho\beta}+\Delta\delta^\beta_\alpha). \tag{2.25}$$

$S_\alpha{}^\beta$ is a Hermitian twistor [let us, however, recall here the indefiniteness property of the signature (+, +, −, −) inherent to twistor invariants]

$$\bar{S}^\alpha{}_\beta=S_\beta{}^\alpha, \tag{2.26}$$

and in the present twistor frame it has the components

$$[S_\alpha{}^\beta]=\begin{bmatrix}0 & 0\\ -S^{A'B} & 0\end{bmatrix}. \tag{2.27}$$

Hence we will call $S_\alpha{}^\beta$ the *spin twistor*. When the spin vanishes, we find, from (2.27) and (2.25),

$$\Delta_0{}^{-1}\bar{A}_{0\alpha\rho}A_0{}^{\rho\beta}=-\delta^\beta_\alpha \tag{2.28}$$

(a zero subscript indicates zero spin). By using this relation we get

$$B_{0\alpha\beta}=-\Delta_0{}^{-1}\bar{A}_{0\alpha\beta}. \tag{2.29}$$

It is a bit more complicated to extract the magnitude s of the spin from the spin twistor, the latter being singular and nilpotent [cf. Eq. (2.27)]. Penrose's method[19] is based on the existence of two expressions, each quadratic in the spin twistor and infinity twistor, respectively, which are proportional to each other:

$$S_\alpha{}^\rho S_\beta{}^\sigma - S_\beta{}^\rho S_\alpha{}^\sigma = \tfrac{1}{2}s^2 I_{\alpha\beta} I^{\rho\sigma} . \qquad (2.30)$$

This relationship can be easily verified by comparing components on both sides in a given twistor frame.

III. STRUCTURE OF THE ANGULAR MOMENTUM TWISTOR

We have seen in the previous section that two alternative twistor descriptions are available for zero-mass systems. One can either specify their (singular) angular momentum twistor $A^{\alpha\beta}$ having the components (2.11) and (2.12) or one can use the one-index twistor Z^α. By comparison of (2.9) and (2.17) we get[11]

$$A^{\alpha\beta} = 2Z^{(\alpha} I^{\beta)\gamma} \bar{Z}_\gamma . \qquad (3.1)$$

This structure suggests that Z^α will provide a more elementary description of the zero-mass system than does the angular momentum twistor which is already composed from the quantity Z^α. One should prefer a similarly fundamental object for the description of massive systems as well. This is facilitated by the observation[11] that $A^{\alpha\beta}$ can be decomposed into the sum of a finite number of zero-mass angular momentum twistors,

$$A^{\alpha\beta} = 2Z_{\underline{1}}^{(\alpha} I^{\beta)\gamma} \bar{Z}_{\underline{1}\gamma} + 2Z_{\underline{2}}^{(\alpha} I^{\beta)\gamma} \bar{Z}_{\underline{2}\gamma} + \cdots + 2Z_{\underline{p}}^{(\alpha} I^{\beta)\gamma} \bar{Z}_{\underline{p}\gamma} , \qquad (3.2)$$

where $Z_{\underline{k}}^\alpha$ ($\underline{k} = \underline{1}, \underline{2}, \ldots, \underline{p}$) represents the $\underline{k}$th massless system. Extending the summation convention to the labels $\underline{k}, \underline{l}, \ldots$ of massless subsystems, we concisely set (3.2) as

$$A^{\alpha\beta} = 2Z_{\underline{k}}^{(\alpha} I^{\beta)\gamma} \bar{Z}_{\underline{k}\gamma} \quad (\underline{k} = \underline{1}, \underline{2}, \ldots, \underline{p}) . \qquad (3.3)$$

This decomposition of the angular momentum twistor is not unique. If $Z_{\underline{k}}^\alpha$ with $\underline{k} = \underline{1}, \underline{2}, \ldots, \underline{p}$ is an appropriate set, so is the linear combination

$$Z_{\underline{i}}'^\alpha = U_{\underline{i}\underline{k}} Z_{\underline{k}}^\alpha \quad (\underline{i}, \underline{k}, \ldots = \underline{1}, \underline{2}, \ldots, \underline{p}) , \qquad (3.4a)$$

with the complex coefficients $U_{\underline{i}\underline{k}}$ satisfying

$$U_{\underline{i}\underline{k}} \bar{U}_{\underline{j}\underline{k}} = \delta_{\underline{i}\underline{j}} . \qquad (3.4b)$$

Thus, $[U_{\underline{i}\underline{k}}]$ is a unitary matrix of dimension $\underline{p}$. The expressions

$$Z_{\underline{i}}'^\alpha = Z_{\underline{i}}^\alpha + \Lambda_{\underline{i}\underline{k}} I^{\alpha\beta} \bar{Z}_{\underline{k}\beta} \qquad (3.5a)$$

define another symmetry operation with

$$\Lambda_{\underline{i}\underline{k}} = -\Lambda_{\underline{k}\underline{i}} . \qquad (3.5b)$$

Equations (3.4) and (3.5) are the invariance transformations of the twistor description of isolated systems. In gravitational interactions, the angular momentum twistor provides again a complete characterization of the dynamical system. The gravitational interaction Hamiltonian can depend only on the combinations $A^{\alpha\beta}$ and $\bar{A}_{\alpha\beta}$ of twistor variables:

$$H_{\text{grav}} = H(A^{\alpha\beta}(Z_{\underline{k}}^\alpha, \bar{Z}_{\underline{k}\alpha}); \bar{A}_{\alpha\beta}(Z_{\underline{k}}^\alpha, \bar{Z}_{\underline{k}\alpha})) . \qquad (3.6)$$

Condition (3.6) expresses the principle of equivalence in twistor theory.

Nongravitational interactions, on the other hand, are expected to violate the internal twistor symmetry. It may well be that this is the way that the twistor structure of dynamical systems becomes manifest in physics.

The Hamiltonian formulation of free particle dynamics is trivial in the present theory since the corresponding twistor variables are constants of the motion. Certain interactions of zero-mass particles have been studied by Penrose.[10] His results have been extended to the treatment of massive particle scattering by Tod and this author.[20] The canonical equations of interaction show that

$$p_{\underline{a}}^\alpha = -iZ_{\underline{a}}^\alpha \text{ and } q_{\underline{a}\alpha} = \bar{Z}_{\underline{a}\alpha} \qquad (3.7)$$

are canonically conjugate variables. A necessary condition for (3.7) to have an invariant meaning is that (3.4) and (3.5) be transformations of the canonical type. We now show that they, in fact, are. We achieve this by finding the corresponding generating function.

Given a function $S(\vec{q}, \vec{p}')$ of the old coordinates $\vec{q}$ and new momenta $\vec{p}'$, the canonical transformation $(\vec{q}, \vec{p}) \to (\vec{q}', \vec{p}')$ generated by S has the form

$$\vec{p} = \frac{\partial S(\vec{q}, \vec{p}')}{\partial \vec{q}} , \quad \vec{q}' = \frac{\partial S(\vec{q}, \vec{p}')}{\partial \vec{p}'} . \qquad (3.8)$$

We choose S so that it does not depend explicitly on time, hence $H' = H$. The reality condition $H = \bar{H}$ imposed upon the Hamiltonian will be unaffected even if we allow S to take complex values.

Equations (3.7) tell us that $S = S(\bar{Z}_{\underline{a}\alpha}, -iZ_{\underline{a}}'^\alpha)$ and

$$-iZ_{\underline{a}}^\alpha = \frac{\partial S(\bar{Z}_{\underline{a}\alpha}, -iZ_{\underline{a}}'^\alpha)}{\partial \bar{Z}_{\underline{a}\alpha}} , \qquad (3.9a)$$

$$\bar{Z}_{\underline{a}\alpha}' = \frac{\partial S(\bar{Z}_{\underline{a}\alpha}, -iZ_{\underline{a}}'^\alpha)}{\partial(-iZ_{\underline{a}}'^\alpha)} . \qquad (3.9b)$$

We first choose $S = -i\bar{Z}_{\underline{a}\alpha} U_{\underline{a}\underline{b}}^\dagger Z_{\underline{b}}'^\alpha$ with $[U_{\underline{a}\underline{b}}^\dagger]$ as a $\underline{p} \times \underline{p}$ unitary matrix. This gives

$$\begin{aligned} -iZ_{\underline{a}}^\alpha &= -iU_{\underline{a}\underline{b}}^\dagger Z_{\underline{b}}'^\alpha , \\ \bar{Z}_{\underline{a}\alpha}' &= \bar{U}_{\underline{a}\underline{b}} \bar{Z}_{\underline{b}\alpha} . \end{aligned} \qquad (3.10)$$

What we obtained is the inverse and the complex conjugate to transformations (3.4a). Next we take $S = -i(Z_{\underline{a}}'^\alpha - \tfrac{1}{2}\Lambda_{\underline{a}\underline{b}} I^{\alpha\beta} \bar{Z}_{\underline{b}\beta})(\bar{Z}_{\underline{a}\alpha} + \tfrac{1}{2}\bar{\Lambda}_{\underline{a}\underline{c}} I_{\alpha\gamma} Z_{\underline{c}}'^\gamma)$ and get

$$-iZ^{\alpha}_{\underline{a}} = -i(Z'^{\alpha}_{\underline{a}} - \Lambda_{\underline{a}\,\underline{b}} I^{\alpha\beta}\bar{Z}_{\underline{b}\beta}) ,$$
$$\bar{Z}'_{\underline{a}\alpha} = \bar{Z}_{\underline{a}\alpha} + \bar{\Lambda}_{\underline{a}\,\underline{c}} I_{\alpha\gamma} Z'^{\gamma}_{\underline{c}} . \tag{3.11}$$

From (3.11) it follows that $I_{\alpha\beta}Z^{\beta}_{\underline{a}} = I_{\alpha\beta}Z'^{\beta}_{\underline{a}}$. Using this result in (3.11) we obtain transformations (3.5).

IV. DECOMPOSITION INTO TWO-TWISTORS

Already two-twistors ($\underline{p}=2$) suffice[11] for decomposing an arbitrary $A^{\alpha\beta}$. These represent 16 real data, whereas the angular momentum twistor is given by 10 real parameters. The remaining 6-parameter freedom in choosing $\{Z^{\alpha}_{\underline{a}}\}$ gives rise to the internal-symmetry transformations (3.4) and (3.5) with $\underline{p}=2$. Indeed, the (2×2) unitary matrix $[U_{\underline{i}\,\underline{k}}]$ depends on four real parameters and $[\Lambda_{\underline{i}\,\underline{k}}]$ depends on two real ones. It has been shown by Penrose[12] that the group structure of the two-twistor transformations is locally isomorphic to that of $SU(2)\times\tilde{E}_2$, where $\tilde{E}_2$ indicates the double covering group of the Euclidean motions in the two-dimensional plane. His result follows from the fact that the SU(2) part of $[U_{\underline{i}\,\underline{k}}]$ commutes with all the rest of the transformations. The U(1) part of $[U_{\underline{i}\,\underline{k}}]$ represents the rotations in the plane and $[\Lambda_{\underline{i}\,\underline{k}}]$ gives the two translations. A peculiar feature of the group manifold is that two elements, $\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}\right]$, are contained both in the SU(2) and the $\tilde{E}_2$ factor groups.

We wish to express the physical properties of the system directly in terms of the twistors $Z^{\alpha}_{\underline{1}}$ and $Z^{\alpha}_{\underline{2}}$. Let us write

$$Z^{\alpha}_{\underline{1}} = Z^{\alpha}, \quad Z^{\alpha}_{\underline{2}} = W^{\alpha} . \tag{4.1}$$

The conjugate twistors are

$$\bar{Z}_{\underline{1}\alpha} = \bar{Z}_{\alpha}, \quad \bar{Z}_{\underline{2}\alpha} = \bar{W}_{\alpha} . \tag{4.2}$$

Decomposition (3.3) of the angular momentum twistor takes the form

$$A^{\alpha\beta} = 2Z^{(\alpha}I^{\beta)\gamma}\bar{Z}_{\gamma} + 2W^{(\alpha}I^{\beta)\gamma}\bar{W}_{\gamma} . \tag{4.3}$$

The determinant yields

$$m^2 = 2|Z^{\alpha}I_{\alpha\beta}W^{\beta}|^2 . \tag{4.4}$$

Spin twistor (2.25) is next expressed as

$$S_{\alpha}{}^{\beta} = \tfrac{1}{2}(Z^{\mu}\bar{Z}_{\mu} - W^{\mu}\bar{W}_{\mu})(I_{\alpha\rho}Z^{\rho}I^{\beta\sigma}\bar{Z}_{\sigma} - I_{\alpha\rho}W^{\rho}I^{\beta\sigma}\bar{W}_{\sigma}) + (Z^{\mu}\bar{W}_{\mu})I_{\alpha\rho}W^{\rho}I^{\beta\sigma}\bar{Z}_{\sigma} + (W^{\mu}\bar{Z}_{\mu})I_{\alpha\rho}Z^{\rho}I^{\beta\sigma}\bar{W}_{\sigma} . \tag{4.5}$$

The structure of relations of this kind can be made particularly transparent in Penrose's "blob notation" (Appendix A). In fact, the expression (4.5) has been obtained by blob manipulations and by using the completeness relation derived in Appendix B. Figure 1(a) shows the resulting blob expression for the spin twistor which is readily converted to the conventional form (4.5).

In order to get an insight into the invariant structure of the spin, we first note that $I_{\alpha\rho}Z^{\rho}$ and $I_{\beta\rho}W^{\rho}$ uniquely define a real orthonormal basis in the rest frame of the dynamical system. Let us denote their components

$$(I_{\alpha\rho}Z^{\rho}) = (0, \pi_{A'}), \quad (I_{\alpha\rho}W^{\rho}) = (0, \rho_{A'}) . \tag{4.6}$$

Equation (4.4) provides the normalization $|\pi_{A'}\rho^{A'}|^2 = m^2/2$ for the momentum spinors. Using (4.6), the four-momentum takes the form

$$P^{a} = \bar{\pi}^{A}\pi^{A'} + \bar{\rho}^{A}\rho^{A'} . \tag{4.7}$$

We introduce the spinor dyad of Newman and Penrose[21]

$$o_A = \left(\frac{2}{m^2}\right)^{1/4}\bar{\pi}_A, \quad \iota_A = \left(\frac{2}{m^2}\right)^{1/4}\bar{\rho}_A \tag{4.8}$$

normalized to $o_A\iota^A = 1$. This defines the null tetrad

$$l^a = o^A\bar{o}^{A'}, \quad n^a = \iota^A\bar{\iota}^{A'}, \quad m^a = o^A\bar{\iota}^{A'}, \quad \bar{m}^a = \iota^A\bar{o}^{A'} . \tag{4.9}$$

In terms of (4.9), a real orthonormal basis is given by

$$T^a = 2^{-1/2}(l^a + n^a), \quad X^a = 2^{-1/2}(m^a + \bar{m}^a),$$
$$Y^a = 2^{-1/2}i(\bar{m}^a - m^a), \quad Z^a = 2^{-1/2}(l^a - n^a) . \tag{4.10}$$

Four-momentum (4.7) has the components in the (4.10) frame

$$(P^a) = (m, 0, 0, 0) . \tag{4.11}$$

Equations (4.10) define, indeed, a rest frame of the massive system.

We obtain the rest-frame components of the Pauli-Lubanski vector by comparing Eq. (4.5) with (2.27)[22]:

$$(S^a) = -\tfrac{1}{2}m(0, Z^{\mu}\bar{W}_{\mu} + W^{\mu}\bar{Z}_{\mu}, i(Z^{\mu}\bar{W}_{\mu} - W^{\mu}\bar{Z}_{\mu}), Z^{\mu}\bar{Z}_{\mu} - W^{\mu}\bar{W}_{\mu}) . \tag{4.12}$$

The spacelike projection of Eq. (4.12) gives us the

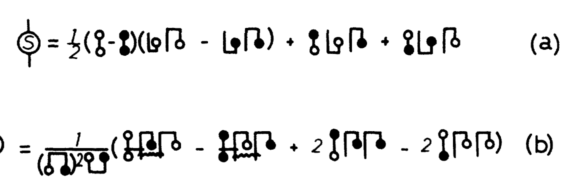

FIG. 1. Structure of (a) the spin twistor and (b) the inverse of the angular momentum twistor in terms of the zero-mass constituents Z^{α} (hollow blobs) and W^{α} (filled blobs).

components of the spin three-vector

$$(\vec{s})=\tfrac{1}{2}(Z^{\mu}\overline{W}_{\mu}+W^{\mu}\overline{Z}_{\mu},\, i(Z^{\mu}\overline{W}_{\mu}-W^{\mu}\overline{Z}_{\mu}),\, Z^{\mu}\overline{Z}_{\mu}-W^{\mu}\overline{W}_{\mu})\ . \tag{4.13}$$

Rather surprisingly, we can write this as

$$(\vec{s})=\tfrac{1}{2}(\overline{Z}_{\mu},\overline{W}_{\mu})(\vec{\sigma})\begin{pmatrix}Z^{\mu}\\ W^{\mu}\end{pmatrix}\ , \tag{4.14}$$

where $(\vec{\sigma})$ stands for the set of Pauli matrices

$$(\vec{\sigma})=\left[\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}0&-i\\i&0\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]. \tag{4.15}$$

In a somewhat symbolic notation,[23]

$$\vec{s}=\overline{\psi}\vec{\sigma}\psi\ , \tag{4.16}$$

where we have set $\psi=2^{-1/2}\binom{Z^{\mu}}{W^{\mu}}$. Eventually, (4.16) relates the structure of the spin with the factor group SU(2) of internal symmetries.

The quadratic invariant

$$S_a S^a=-\tfrac{1}{2}|Z^{\alpha}I_{\alpha\beta}W^{\beta}|^2[(Z^{\mu}\overline{Z}_{\mu}-W^{\mu}\overline{W}_{\mu})^2+4|Z^{\mu}\overline{W}_{\mu}|^2] \tag{4.17}$$

gives for the square of the spin

$$s^2=\tfrac{1}{4}(Z^{\mu}\overline{Z}_{\mu}-W^{\mu}\overline{W}_{\mu})^2+|Z^{\mu}\overline{W}_{\mu}|^2\ . \tag{4.18}$$

From the structure of (4.18) we see that the spin magnitude is conformally invariant.

We complete this section with the expression shown in Fig. 1(b) for the two-twistor decomposition of the inverse $B_{\alpha\beta}$ of the angular momentum twistor. The defining property of the inverse twistor $A^{\alpha\beta}B_{\beta\gamma}=\delta^{\alpha}_{\gamma}$ amounts to the identity of Fig. 6.

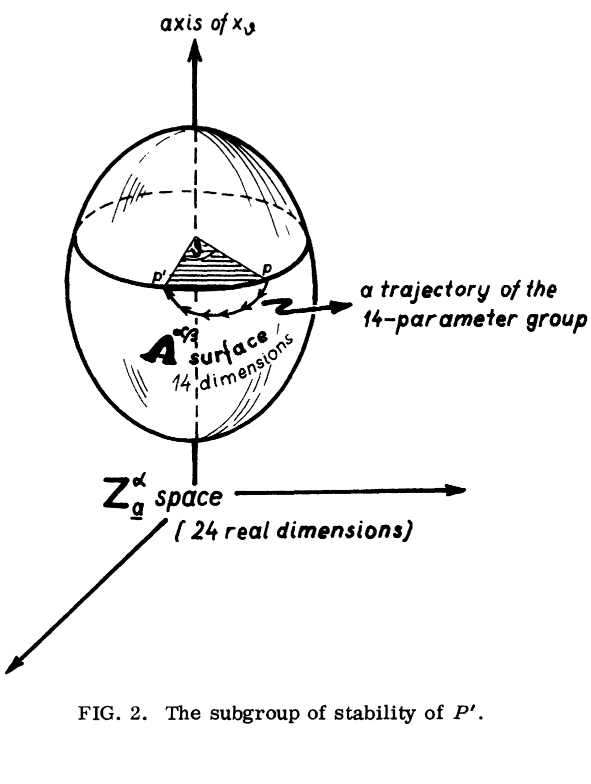

FIG. 2. The subgroup of stability of P'.

V. THREE-TWISTORS

In the space $\mathrm{T}\otimes\mathrm{T}\otimes\mathrm{T}$ of 24 dimensions of the three-twistors $Z^{\alpha}_{\underline{a}}$ ($\underline{a}=\underline{1},\underline{2},\underline{3}$), the angular momentum twistor represents a 14-dimensional quadratic surface. A generic point $P(Z^{\alpha}_{\underline{a}})$ of this surface defines (in more ways than one) a basis in the twistor space T. For example, $\{Z^{\alpha}_{\underline{a}}, I^{\alpha\beta}\overline{Z}_{\underline{a}\beta}\}$ is an overcomplete basis. Therefore, the coordinates of an arbitrary point $P'(Z'^{\alpha}_{\underline{a}})$ are expressible as linear combinations of the form

$$Z'^{\alpha}_{\underline{a}}=a_{\underline{a}\underline{b}}Z^{\alpha}_{\underline{b}}+b_{\underline{a}\underline{b}}I^{\alpha\beta}\overline{Z}_{\underline{b}\beta}\quad(\underline{a},\underline{b},\ldots=\underline{1},\underline{2},\underline{3})\ . \tag{5.1}$$

When P' lies on the given $A^{\alpha\beta}$ surface we must have

$$Z'^{\alpha}_{\underline{a}}=U_{\underline{a}\underline{b}}(Z^{\alpha}_{\underline{b}}+\Lambda_{\underline{b}\underline{c}}I^{\alpha\beta}\overline{Z}_{\underline{c}\beta})\ . \tag{5.2}$$

with $U_{\underline{a}\underline{b}}$ and $\Lambda_{\underline{a}\underline{b}}$ satisfying (3.4b) and (3.5b), respectively.

Transformations $\{\underline{U},\underline{\Lambda}\}$ of the form (5.2), such that $\underline{U}\in\mathrm{U}(3)$, $\underline{\Lambda}=-\underline{\Lambda}^T$, are elements of a nonsemisimple group. The group operation may be written

$$\{\underline{U}_2,\underline{\Lambda}_2\}\{\underline{U}_1,\underline{\Lambda}_1\}=\{\underline{U}_2\underline{U}_1,\ \underline{\Lambda}_1+\underline{U}_1^{\dagger}\underline{\Lambda}_2\overline{\underline{U}}_1\}\ . \tag{5.3}$$

This 15-parameter group has minimal transitivity surfaces of 14 dimensions, each given by a fixed value of $[A^{\alpha\beta}]$ (10 real data). Hence the question arises whether the group (5.2) itself is a minimal group in the sense that it has no proper subgroup transitive anywhere in the $A^{\alpha\beta}$ surfaces.

It is clear that the minimal symmetry group must have at least 14 parameters. So we come to considering the following possibility: In an appropriate basis for the Lie algebra we drop one of the generators (X_θ, say) and still then we may be left with a group which can carry an arbitrary point P on $A^{\alpha\beta}$ into any other point on it. (See Fig. 2. This is essentially a three-dimensional slice of the full picture. The $A^{\alpha\beta}$ surface may extend to infinity in some directions which are not shown on Fig. 2.)

Let the point P be carried into P' under the action of a group element $g(\theta)$ generated by X_θ. A necessary condition for the existence of the sought-for transitive subgroup is that the same point P' be attainable from P by the action of a group element $g'(P,\theta)$ generated by the remaining 14 generators. This is equivalent to saying that the elements of the subgroup of stability of P (now a one-parameter group) must be separable in the form

$$g^{-1}(\theta)g'(P,\theta)\ . \tag{5.4}$$

When this condition holds, it remains to be seen if the 14-parameter set of transformations (with X_θ ignored) has a group structure.

Let us first consider whether (5.4) can be satisfied. The stability transformations of $P(Z^\alpha_{\underline{a}})$ arise from (5.2) by setting $Z'^\alpha_{\underline{a}} = Z^\alpha_{\underline{a}}$. Contracting both sides with $\bar U_{\underline{x}\underline{a}}$, the equations to be solved, given $Z^\alpha_{\underline{a}}$, are

$$Z^\alpha_{\underline{x}} = \bar U_{\underline{x}\underline{a}} Z^\alpha_{\underline{a}} - \Lambda_{\underline{x}\underline{c}} I^{\alpha\beta}\bar Z_{\underline{c}\beta}\ . \tag{5.5}$$

We introduce the detailed notation $Z^\alpha_{\underline{1}} = Z^\alpha$, $Z^\alpha_{\underline{2}} = W^\alpha$, $Z^\alpha_{\underline{3}} = V^\alpha$,

$$[\bar U_{\underline{a}\underline{b}}] = \begin{pmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{pmatrix},\quad [\Lambda_{\underline{a}\underline{b}}] = \begin{pmatrix} 0 & -c & -b \\ c & 0 & -a \\ b & a & 0 \end{pmatrix}\ . \tag{5.6}$$

Equations (5.5) take the form

$$\begin{pmatrix} Z^\alpha \\ W^\alpha \\ V^\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta & \gamma \\ \lambda & \mu & \nu \\ \rho & \sigma & \tau \end{pmatrix} \begin{pmatrix} Z^\alpha \\ W^\alpha \\ V^\alpha \end{pmatrix} + \begin{pmatrix} 0 & c & b \\ -c & 0 & a \\ -b & -a & 0 \end{pmatrix} \begin{pmatrix} I^{\alpha\beta}\bar Z_\beta \\ I^{\alpha\beta}\bar W_\beta \\ I^{\alpha\beta}\bar V_\beta \end{pmatrix}\ . \tag{5.7}$$

A set of four linearly independent twistors $\{Z^\alpha, W^\alpha, I^{\alpha\beta}\bar Z_\beta, I^{\alpha\beta}\bar W_\beta\}$ is available whenever the decomposition of the angular momentum twistor

$$A^{\alpha\beta} = 2Z^{(\alpha}I^{\beta)\gamma}\bar Z_\gamma + 2W^{(\alpha}I^{\beta)\gamma}\bar W_\gamma + 2V^{(\alpha}I^{\beta)\gamma}\bar V_\gamma \tag{5.8}$$

is nontrivial (this implies that $Z^\alpha I_{\alpha\beta}W^\beta \neq 0$). The twistor V^α can be expressed, then, as the linear combination

$$V^\alpha = AZ^\alpha + BW^\alpha + CI^{\alpha\beta}\bar Z_\beta + DI^{\alpha\beta}\bar W_\beta\ , \tag{5.9a}$$

whence

$$I^{\alpha\beta}\bar V_\beta = \bar A I^{\alpha\beta}\bar Z_\beta + \bar B I^{\alpha\beta}\bar W_\beta\ . \tag{5.9b}$$

Contracting with elements of the conjugate basis $\{\bar Z_\alpha, \bar W_\alpha, I_{\alpha\beta}Z^\beta, I_{\alpha\beta}W^\beta\}$ we get

$$\begin{aligned} \bar Z_\alpha V^\alpha &= A\bar Z_\alpha Z^\alpha + B\bar Z_\alpha W^\alpha + D\bar Z_\alpha I^{\alpha\beta}\bar W_\beta\ , \\ \bar W_\alpha V^\alpha &= A\bar W_\alpha Z^\alpha + B\bar W_\alpha W^\alpha - C\bar Z_\alpha I^{\alpha\beta}\bar W_\beta\ , \\ Z^\alpha I_{\alpha\beta}V^\beta &= BZ^\alpha I_{\alpha\beta}W^\beta\ , \\ W^\alpha I_{\alpha\beta}V^\beta &= -AZ^\alpha I_{\alpha\beta}W^\beta\ . \end{aligned} \tag{5.10}$$

The coefficients in Eq. (5.9) are related to some twistor invariants. Let us introduce the blob notation shown in Fig. 3(a). Solution of Eqs. (5.10) provides the result in Fig. 3(b). Equations (5.6) are written

$$\begin{aligned} Z^\alpha &= (\alpha + A\gamma)Z^\alpha + (\beta + B\gamma)W^\alpha + (b\bar A + C\gamma)I^{\alpha\beta}\bar Z_\beta \\ &\quad + (b\bar B + D\gamma + c)I^{\alpha\beta}\bar W_\beta\ , \\ W^\alpha &= (\lambda + A\nu)Z^\alpha + (\mu + B\nu)W^\alpha + (a\bar A + C\nu - c)I^{\alpha\beta}\bar Z_\beta \\ &\quad + (a\bar B + D\nu)I^{\alpha\beta}\bar W_\beta\ , \\ V^\alpha &= (\rho + A\tau)Z^\alpha + (\sigma + B\tau)W^\alpha + (C\tau - b)I^{\alpha\beta}\bar Z_\beta \\ &\quad + (D\tau - a)I^{\alpha\beta}\bar W_\beta\ . \end{aligned} \tag{5.11}$$

Comparison of coefficients on both sides and use of the expressions in Fig. 3(b) yield the solution shown in Fig. 4, with y as an arbitrary real parameter and m the rest mass,

$$m^2 = 2(|Z^\alpha I_{\alpha\beta}W^\beta|^2 + |Z^\alpha I_{\alpha\beta}V^\beta|^2 + |W^\alpha I_{\alpha\beta}V^\beta|^2)\ . \tag{5.12}$$

From Fig. 4 we see that the phase θ of the determinant

$$\det \underline{U} = e^{i\theta} \tag{5.13}$$

is related to the parameter y by

$$\cot(\theta/2) = -y/m^2\ . \tag{5.14}$$

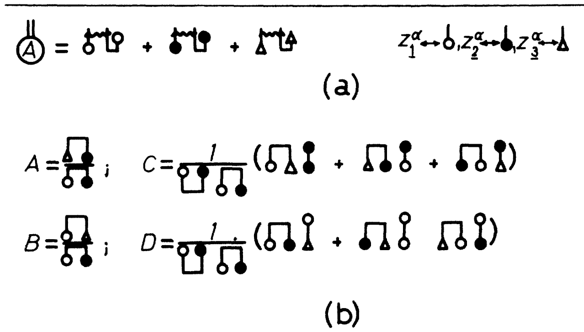

FIG. 3. (a) Decomposition of the angular momentum twistor into three null subsystems. (b) The expansion coefficients.

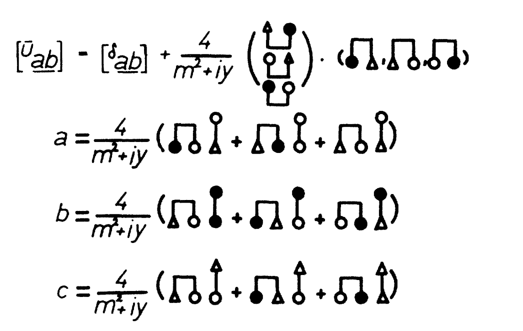

FIG. 4. The stability transformations.

Let us introduce the unimodular matrix $\underline{U}_1 = e^{-i\theta/3}\underline{U}$. In terms of $\underline{U}_1$, the stability transformations can be written

$$Z^{\alpha}_{\underline{a}} = e^{i\theta/3}U_{1\underline{a}\underline{b}}(Z^{\alpha}_{\underline{b}} + \Lambda_{\underline{b}\underline{c}}I^{\alpha\beta}\bar{Z}_{\underline{c}\beta}) . \tag{5.15}$$

Product rule (5.3) tells us that the group element appearing in Eq. (5.15) is of the form

$$\{e^{i\theta/3}\underline{U}_1, \underline{\Lambda}\} = \{e^{i\theta/3}, \underline{0}\}\{\underline{U}_1, \underline{\Lambda}\} . \tag{5.16}$$

Comparing with (5.4) we see that our necessary condition is satisfied and X_θ generates the phase rotations $e^{-i\theta/3}$.

Now we have proved that imposition of $\underline{U} \in SU(3)$ in transformations (5.4) does not destroy their transitivity in the angular momentum surfaces. Product rule (5.3) immediately warrants that this restriction still leaves us with a group.

VI. INHOMOGENEOUS SU(3)

We shall show that the minimal twistor symmetry group found in the previous section is locally isomorphic to the inhomogeneous generalization of the group SU(3). The generalization arises by taking the semidirect product of translations in a three-complex-dimensional Euclidean space with SU(3) rotations. Let us introduce complex coordinates $\{q_{\underline{a}}\}$ ($\underline{a} = \underline{1}, \underline{2}, \underline{3}$) in this space. We consider point transformations

$$q'_{\underline{a}} = U_{\underline{a}\underline{b}}\, q_{\underline{b}} + t_{\underline{a}} , \tag{6.1}$$

with $[U_{\underline{a}\underline{b}}] \in SU(3)$ and $t_{\underline{a}}$ as a complex three-vector. The quadratic form $dq_{\underline{a}}\, d\bar{q}_{\underline{a}}$ is invariant under transformations (6.1). From the unimodularity of $[U_{\underline{a}\underline{b}}]$ it ensues that the alternating tensor $\epsilon_{\underline{abc}}$ [with $(\epsilon_{\underline{123}}) = 1$] is another invariant.

We write (6.1) in the form

$$\begin{pmatrix} q'_{\underline{a}} \\ 1 \end{pmatrix} = \begin{pmatrix} U_{\underline{a}\underline{b}} & t_{\underline{a}} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q_{\underline{b}} \\ 1 \end{pmatrix} . \tag{6.2}$$

From (6.2) we immediately obtain the generators

$$[x_i] = \begin{pmatrix} \underline{\lambda}_i & 0 \\ 0 & 0 \end{pmatrix}, \quad [y^+_{\underline{a}}] = \begin{pmatrix} \underline{0} & \vec{\mu}_{\underline{a}} \\ 0 & 0 \end{pmatrix}, \quad [y^-_{\underline{a}}] = \begin{pmatrix} \underline{0} & i\vec{\mu}_{\underline{a}} \\ 0 & 0 \end{pmatrix}, \qquad i = 1, 2, \ldots, 8, \quad \underline{a} = 1, 2, 3 \tag{6.3}$$

where we set

$$[\vec{\mu}_1] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad (\vec{\mu}_2) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad (\vec{\mu}_3) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{6.4}$$

The SU(3) generators $\underline{\lambda}_i$ will be chosen the Gell-Mann matrices[24] except that we conveniently relabel them as follows:

$$\underline{\lambda}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \underline{\lambda}_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \underline{\lambda}_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{\lambda}_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$\underline{\lambda}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{\lambda}_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \underline{\lambda}_7 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{\lambda}_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{6.5}$$

$\underline{\lambda}_i$ matrices satisfy

$$[\underline{\lambda}_i, \underline{\lambda}_j] = 2i f_{ijk}\underline{\lambda}_k , \tag{6.6a}$$

$$\{\underline{\lambda}_i, \underline{\lambda}_j\} = 2d_{ijk}\underline{\lambda}_k + \tfrac{4}{3}\delta_{ij} , \qquad i, j, k = 1, 2, \ldots, 8 \tag{6.6b}$$

where the nonvanishing components of the completely skew f_{ijk} are given: $f_{123} = f_{147} = f_{156} = f_{245} = +f_{267} = -f_{346} = -\frac{1}{2}$; $f_{357} = 1$; $f_{168} = f_{248} = -\sqrt{3}/2$ and $d_{ijk} = d_{(ijk)}$ have the independent components $d_{338} = d_{558} = d_{778} = -d_{888} = 1/\sqrt{3}$; $d_{118} = d_{228} = d_{448} = d_{668} = -1/2\sqrt{3}$; $-d_{115} = d_{127} = -d_{134} = d_{225} = d_{236} = d_{445} = d_{467} = -d_{566} = \frac{1}{2}$. We note that $\{\underline{\lambda}_{\underline{a}} | \underline{a} = 1, 2, 3\}$ constitute a particular subalgebra of (6.6a) with

$$[\underline{\lambda}_{\underline{a}}, \underline{\lambda}_{\underline{b}}] = -i\epsilon_{\underline{abc}}\,\underline{\lambda}_{\underline{c}} , \qquad \underline{a}, \underline{b}, \underline{c} = 1, 2, 3. \tag{6.7}$$

Generators (6.3) of the inhomogeneous group provide us the Lie algebra

$$[\underline{x}_i, \underline{x}_j] = 2i f_{ijk} \underline{x}_k, \tag{6.8a}$$

$$[\underline{y}^{\pm}_{\underline{a}}, \underline{y}^{+}_{\underline{b}}] = 0, \quad [\underline{y}^{\pm}_{\underline{a}}, \underline{y}^{-}_{\underline{b}}] = 0, \tag{6.8b}$$

$$[\underline{x}_{\underline{a}}, \underline{y}^{\pm}_{\underline{b}}] = -i\epsilon_{\underline{abc}}\, \underline{y}^{\pm}_{\underline{c}}, \tag{6.8c}$$

$$[\underline{x}_{i'}, \underline{y}^{\pm}_{\underline{b}}] = \pm 2i d_{i'\underline{bc}}\, \underline{y}^{\mp}_{\underline{c}}, \quad i' = 4, 5, 6, 7, 8. \tag{6.8d}$$

Let us write the transformations of the minimal twistor group of Sec. V in the matrix form

$$\begin{pmatrix} \hat{Z}^{\alpha}_{\underline{a}} \\ \hat{\bar{Z}}_{\underline{a}\alpha} \end{pmatrix} = \begin{pmatrix} U_{\underline{ab}}\delta^{\alpha}_{\beta} & 0 \\ 0 & \bar{U}_{\underline{ab}}\delta^{\beta}_{\alpha} \end{pmatrix} \begin{pmatrix} Z^{\beta}_{\underline{b}} \\ \bar{Z}_{\underline{b}\beta} \end{pmatrix}, \quad \begin{pmatrix} \tilde{Z}^{\alpha}_{\underline{a}} \\ \tilde{\bar{Z}}_{\underline{a}\alpha} \end{pmatrix} = \begin{pmatrix} \delta_{\underline{ab}}\delta^{\alpha}_{\beta} & \Lambda_{\underline{ab}} I^{\alpha\beta} \\ \bar{\Lambda}_{\underline{ab}} I_{\alpha\beta} & \delta_{\underline{ab}}\delta^{\beta}_{\alpha} \end{pmatrix} \begin{pmatrix} Z^{\beta}_{\underline{b}} \\ \bar{Z}_{\underline{b}\beta} \end{pmatrix} \tag{6.9}$$

[here we have $\underline{U} \in \mathrm{SU}(3)$, $\underline{\Lambda} = -\underline{\Lambda}^T$]. The labeling of the $\underline{\lambda}_i$ matrices we are using here has the advantage that the corresponding generators may be written concisely:

$$[\underline{X}_i] = \begin{pmatrix} \underline{\lambda}_i \delta^{\alpha}_{\beta} & 0 \\ 0 & -\bar{\underline{\lambda}}_i \delta^{\beta}_{\alpha} \end{pmatrix}, \quad [\underline{Y}^{+}_{\underline{a}}] = \begin{pmatrix} 0 & i\underline{\lambda}_{\underline{a}} I^{\alpha\beta} \\ -i\underline{\lambda}_{\underline{a}} I_{\alpha\beta} & 0 \end{pmatrix}, \quad [\underline{Y}^{-}_{\underline{a}}] = \begin{pmatrix} 0 & \underline{\lambda}_{\underline{a}} I^{\alpha\beta} \\ \underline{\lambda}_{\underline{a}} I_{\alpha\beta} & 0 \end{pmatrix}. \tag{6.10}$$

Straightforward computation shows that the Lie algebra of generators (6.10) is isomorphic to (6.8) and that we have the correspondence in our basis $\underline{x}_i \to \underline{X}_i$, $\underline{y}^{\pm}_{\underline{a}} \to \underline{Y}^{\pm}_{\underline{a}}$. This result establishes the local isomorphism of the minimal symmetry group of three-twistors with the inhomogeneous SU(3).

VII. CONCLUSIONS AND ANTICIPATIONS

The fact that the internal-symmetry transformations have a canonical structure (Sec. III) provides further support to the view that the quantization rules $Z^{\alpha} \to Z^{\alpha}$, $\bar{Z}_{\alpha} \to -\partial/\partial Z^{\alpha}$ first obtained for zero-mass particles[10] hold, more generally, to massive systems (obviously, the exact proof assumes an exact twistor theory of interactions). Already at the present stage of development, the same rules can be attained by considering the equations of massive particle scattering on certain shock waves.[20] Assuming that these rules are the correct ones, the Hilbert space of massive particle states consists of analytic functions of several complex variables. Penrose's analytic bitwistor functions $F(Z^{\alpha}, W_{\alpha})$ for Dirac particles[11,12] may be recalled here as examples. However, the general scalar product between states remains to be found.

Another question intimately tied to the theory of interactions is whether in all cases three-twistors will suffice for describing massive systems. When dealing with free particles, two-twistors will always do. On the other hand, the approximate SU(3) structure of fundamental interactions together with the results of Sec. VI make hard to avoid the conclusion that systems with three constituent twistors actually will have to be considered.

ACKNOWLEDGMENTS

This work was partly done under a Leverhulme Visiting Fellowship at Birkbeck College, University of London. I feel the deepest indebtedness towards Professor R. Penrose, whose thoughts pervasively influenced the development of the results presented here. I thank Dr. M. Huszár and Dr. K. Szegő for helpful criticism and suggestions.

APPENDIX A: BATTELLE AND BLOB NOTATION

Here we are explaining some notational innovations which facilitate the discussion in the main text of the paper.

The representation of twistors in space-time is best achieved by resorting to the theory of two-component spinors. We shall be using a slight modification of Battelle convention[25] for exhibiting spinorial structure. The basic idea of this is to consider indexed quantities as representing *geometric objects,* as opposed to *components* in a given frame. Example: One puts T_{ab} for a second-rank tensor. The labels a and b cannot have numerical values since their only function is to display the structure of the entity.

One advantage of this convention is that symmetry operations can be explicitly carried out in a frame-independent way. Thus the symmetric part of the second-rank ($n=2$) tensor T_{ab} will be written as $T_{(ab)} = (1/n!)(T_{ab} + T_{ba})$ and the skew part is denoted $T_{[ab]} = (1/n!)(T_{ab} - T_{ba})$.

In certain cases one wishes to refer to components in some given frame. Normally, no extra index letter types will be needed for this purpose. We shall enclose components in matrix brackets. The Minkowski metric g_{ab}, say, has the components in standard real coordinates $[g_{ab}] = \mathrm{diag}(1,-1,-1,-1)$. Evidently, indices *inside* the bracket can have numerical values.[26] Thus the alternating tensor η_{abcd} is given by the components $[\eta_{0123}] = -1$, $[\eta^{0123}] = 1$. Also, for the fundamental spinor ϵ_{AB} we have $[\epsilon_{01}] = [\epsilon^{01}] = 1$.

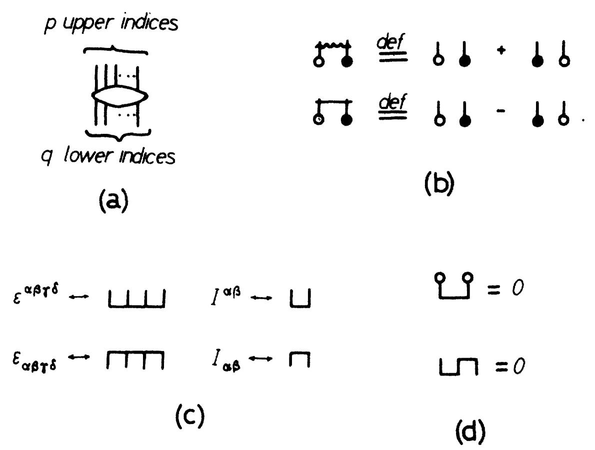

FIG. 5. Blob notation. (a) A $[^p_q]$ twistor. (b) Symmetry operations. (c) Skew-symmetric twistors. (d) Properties of the infinity twistor.

In the abstract index notation, a tensor index a is equivalent to a pair AA' of spinor indices.[25] One immediately obtains the spinor structure of an arbitrary tensor $X^{abc\cdots}{}_{rs\cdots}$ by writing

$$X^{abc\cdots}{}_{rs\cdots} = X^{AA'BB'CC'\cdots}{}_{RR'SS'\cdots}\,. \tag{A1}$$

An apparent drawback of using matrix brackets for components is that one can thus never consider "mixed" quantities, i.e., those which possess both abstract and component-type indices. Seldom occurs the need, however, for introducing such quantities[27] (never at least in this paper).

The structure of certain covariant expressions may be obscured by the abundance of indices. This difficulty is overcome by Penrose's "blob notation."[28] The use of blobs will be limited to twistorial expressions in the present paper. A $[^p_q]$ twistor may be drawn as a blob with p upward pointing and q downward oriented adjoining index lines [Fig. 5(a)]. The contraction of indices is

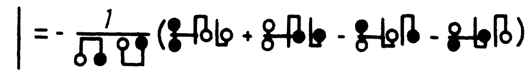

FIG. 6. Completeness relation for the $\{\zeta^{\alpha}_{\underline{a}}\}$ basis.

achieved by connecting the corresponding lines. In accordance with this, the twistor δ^{α}_{β} is represented by a vertical line segment.

The blob notation for symmetry operations is demonstrated in Fig. 5(b) by the simple example of the outer product of two $[^1_0]$ twistors. The generalization to more complicated cases is straightforward. Skew-symmetric twistors of importance are displayed in Fig. 5(c). Note that the skewness of the infinity twistor entails, for any twistor Z^{β},

$$Z^{\alpha} I_{\alpha\beta} Z^{\beta} = 0, \tag{A2}$$

and, from Eq. (2.15), we have

$$I^{\alpha\beta} I_{\beta\gamma} = 0. \tag{A3}$$

Properties (A2) and (A3) of the infinity twistor can be stated in terms of blobs as shown in Fig. 5(d).

APPENDIX B: THE ζ BASIS

Whenever the symmetric terms in the expression $Z^{(\alpha}I^{\beta)\gamma}\bar{Z}_{\gamma} + W^{(\alpha}I^{\beta)\gamma}\bar{W}_{\gamma}$ are both nonvanishing and it is *not* possible to find a twistor X^{α} for which

$$Z^{(\alpha}I^{\beta)\gamma}\bar{Z}_{\gamma} + W^{(\alpha}I^{\beta)\gamma}\bar{W}_{\gamma} = X^{(\alpha}I^{\beta)\gamma}\bar{X}_{\gamma}, \tag{B1}$$

the set $\{Z^{\alpha}, W^{\alpha}, I^{\alpha\beta}\bar{Z}_{\beta}, I^{\alpha\beta}\bar{W}_{\beta}\}$ consists of nonzero twistors satisfying $Z^{\alpha}I_{\alpha\beta}W^{\beta} \neq 0$. This set will be denoted also as $\{\zeta^{\alpha}_{\underline{a}}\}$ ($\underline{a} = \underline{1}, \underline{2}, \underline{3}, \underline{4}$). Its four twistors are linearly independent and they form a basis in the twistor space T. The corresponding basis in the dual space T* is given by $\{\bar{\zeta}_{\underline{a}\,\alpha}\} = \{\bar{Z}_{\alpha}, \bar{W}_{\alpha}, I_{\alpha\beta}Z^{\beta}, I_{\alpha\beta}W^{\beta}\}$.

In terms of this basis we define the Hermitian matrix

$$[G_{\underline{ab}}] \underset{\text{def}}{=} [\zeta^{\alpha}_{\underline{a}}\bar{\zeta}_{\underline{b}\,\alpha}] = \begin{bmatrix} \bar{Z}_{\alpha}Z^{\alpha} & \bar{W}_{\alpha}Z^{\alpha} & 0 & Z^{\alpha}I_{\alpha\beta}W^{\beta} \\ \bar{Z}_{\alpha}W^{\alpha} & \bar{W}_{\alpha}W^{\alpha} & -Z^{\alpha}I_{\alpha\beta}W^{\beta} & 0 \\ 0 & -\bar{Z}_{\alpha}I^{\alpha\beta}\bar{W}_{\beta} & 0 & 0 \\ Z^{\alpha}I_{\alpha\beta}W^{\beta} & 0 & 0 & 0 \end{bmatrix}, \quad \underline{a}, \underline{b}, \ldots = \underline{1}, \underline{2}, \underline{3}, \underline{4}. \tag{B2}$$

The elements of $[G_{\underline{ab}}]$ are invariant scalars and they have the properties $G_{\underline{ab}} = \bar{G}_{\underline{ba}}$ and

$$\det[G_{\underline{ab}}] = |Z^{\alpha}I_{\alpha\beta}W^{\beta}|^2. \tag{B3}$$

We shall refer to the nonsingular matrix $[G_{\underline{ab}}]$ as the *metric*. Its inverse $[G^{\underline{ab}}]$, satisfying $G_{\underline{ab}}G^{\underline{bc}} = \delta^{\underline{c}}_{\underline{a}}$, consists of the elements

$$[G^{\underline{ab}}]=|Z^{\mu}I_{\mu\nu}W^{\nu}|^{-2}\begin{bmatrix}0&0&0&Z^{\alpha}I_{\alpha\beta}W^{\beta}\\0&0&-Z^{\alpha}I_{\alpha\beta}W^{\beta}&0\\0&-\bar{Z}_{\alpha}I^{\alpha\beta}\bar{W}_{\beta}&-W^{\alpha}\bar{W}_{\alpha}&W^{\alpha}\bar{Z}_{\alpha}\\\bar{Z}_{\alpha}I^{\alpha\beta}\bar{W}_{\beta}&0&Z^{\alpha}\bar{W}_{\alpha}&-Z^{\alpha}\bar{Z}_{\alpha}\end{bmatrix}. \tag{B4}$$

An arbitrary twistor $X^{\alpha}\in \mathbb{T}$ may be expressed as

$$X^{\alpha}=X^{\underline{a}}\xi^{\alpha}_{\underline{a}}, \tag{B5}$$

with some complex coefficients $X^{\underline{a}}$. The $X^{\underline{a}}$'s are obtained from Eq. (B5) by contracting with $\bar{\xi}_{\underline{b}\,\alpha}G^{\underline{bc}}$

$$X^{\underline{c}}=X^{\alpha}\bar{\xi}_{\underline{b}\,\alpha}G^{\underline{bc}}. \tag{B6}$$

[On allowing basis indices to be lowered and raised by the metric $G_{\underline{ab}}$ and its inverse, it is possible to bring (B6) to the form $X^{\underline{a}}=X^{\alpha}\bar{\xi}^{\underline{a}}_{\alpha}$.] Using this result in equation (B5) we find $X^{\alpha}=X^{\beta}G^{\underline{bc}}Z^{\alpha}_{\underline{c}}\bar{Z}_{\underline{b}\,\beta}$ for all $X^{\alpha}\in\mathbb{T}$. Hence

$$\delta^{\alpha}_{\beta}=Z^{\alpha}_{\underline{a}}G^{\underline{ab}}\bar{Z}_{\underline{b}\,\beta}. \tag{B7}$$

Identity (B7) is a completeness relation for the $\{\xi^{\alpha}_{\underline{a}}\}$ basis. When written out in detail, it becomes rather unwieldy. A more convenient form, shown in Fig. 6, results in the blob formalism.

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[8] For a review of the "orthodox" theories, see D. R. Brill and R. H. Gowdy, Rep. Prog. Phys. 33, 413 (1970).

[9] R. Penrose, J. Math. Phys. 8, 345 (1967). Note that the representation of twistors used in this reference may be obtained by the rearrangement of components $(\omega^{A},\pi_{A'})\to(\bar{\pi}^{A},-\bar{\omega}_{A'})$.

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[15] For our notation, see Appendix A.

[16] F. A. E. Pirani, in *Lectures on General Relativity*, edited by S. Deser and K. W. Ford (Prentice-Hall, Englewood Cliffs, New Jersey, 1964).

[17] For a discussion of the CM representation of massive systems, see E. T. Newman and J. Winicour, J. Math. Phys. 15, 1113 (1974).

[18] R. Penrose, in *Relativity, Groups and Topology*, edited by B. DeWitt and C. M. DeWitt (Blackie, London, England, 1964).

[19] R. Penrose (private communication).

[20] P. Tod and Z. Perjés (unpublished).

[21] E. T. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962).

[22] The structure expressed in (4.13) has been considered in a geometrical context by R. Penrose (private communication).

[23] It appears, when physicists are persistent enough in matters of representation, that sometimes nature yields to them and lets them see something that more profoundly underlies their convention.

[24] See, e.g., S. Adler and R. F. Dashen, *Current Algebras* (Benjamin, New York, 1968). We exchange the indices of the standard Gell-Mann matrices according to the scheme $1\longleftrightarrow 7$ and $2\to 3\to 5\to 2$.

[25] R. Penrose, in *Battelle Rencontres*, edited by C. M. DeWitt and J. A. Wheeler (Benjamin, New York, 1968).

[26] When labeling components, space-time indices a, b, c,... and twistor indices α,β,γ,... range through 0, 1, 2, and 3. In the matrix notation the first suffix or superfix refers to the row and the second one refers to the column [cf. W. L. Bade and H. Jehle, Rev. Mod. Phys. 25, 714 (1953)].

[27] Eventually, the script and gothic fonts are always available when dealing with exceptional situations.

[28] R. Penrose, in *Combinatorial Mathematics*, edited by D. J. A. Welsh (Academic, New York, 1971).